Math 54 Final Exam, Prof. Srivastava
December 15, 2016, 8:10am–11:00am, 155 Dwinelle Hall.

Name: ____________________

SID: ____________________

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. **Do not unstaple the exam.** Write your name and SID on every page. **Show your work** — numerical answers without justification will be considered suspicious and will not be given full credit. Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed. **If you are seen writing after time is up, you will lose 20 points.**

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Question	Points
1	20
2	15
3	6
4	12
5	8
6	11
7	11
8	6
9	7
Total:	96

Do not turn over this page until your instructor tells you to do so.

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1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) Every system of 3 linear equations in 4 variables has a solution. **T** **F**

   > **Solution:** False. Not every 3×4 matrix has a pivot in every row, so not every system is consistent.

   (b) If A is a square matrix and R is the reduced row echelon form of A then A and R must have the same eigenvalues. **T** **F**

   > **Solution:** False. For instance, the reduced row echelon form of every invertible matrix is the identity, but not all invertible matrices have eigenvalues equal to 1. More conceptually, row ops do not preserve eigenvalues.

   (c) Suppose A and B are 10×10 matrices and $v_1, \ldots, v_{10}$ is a basis of $\mathbb{R}^{10}$ such that $Av_i = Bv_i$ for all $i = 1, \ldots, 10$. Then it must be the case that $A = B$. **T** **F**

   > **Solution:** True. Every $x \in \mathbb{R}^{10}$ can be written as a linear combination of $v_1, \ldots, v_{10}$, so we have for this linear combination:
   >
   > $$Ax = A(c_1v_1 + \ldots c_{10}v_{10}) = c_1Av_1 + \ldots + c_{10}Av_{10} = c_1Bv_1 + \ldots = Bx.$$
   >
   > Thus $Ax = Bx$ for every x, so A and B must be the same. Another way to see this concretely is to take $x = e_1, \ldots, e_{10}$, which shows that A and B have the same columns.

   (d) Suppose A and B are 10×10 matrices and $v_1, \ldots, v_{10}$ is a basis of $\mathbb{R}^{10}$ such that each v_i is an eigenvector of **both** A and B for $i = 1, \ldots, 10$. Then it must be the case that $AB = BA$. **T** **F**

   > **Solution:** True. Let P be a matrix containing the v_i as columns. Since both A and B have a basis of eigenvectors, they must be diagonalizable and we can write $A = PDP^{-1}$ and $B = PCP^{-1}$ for some diagonal matrices D and C. But now
   >
   > $$AB = PDP^{-1}PCP^{-1} = PDCP^{-1} = PCDP^{-1} = PCP^{-1}PDP^{-1} = BA,$$
   >
   > since $DC = CD$.

   (e) If A and B are similar matrices then $\text{rank}(A) = \text{rank}(B)$. **T** **F**

   > **Solution:** True. Conceptual proof: Observe that A is the standard matrix of the linear transformation $T(x) = Ax$ and $B = PAP^{-1}$ is the matrix of T with

respect to the basis contained in the columns of P^{-1}. By the correspondence principle, the dimension of the Image of T is equal to the dimension of the column spaces of A and B. Thus, these dimensions must be equal, so $\text{rank}(A) = \text{rank}(B)$.

Algebraic Proof: Since $B = PAP^{-1}$ we have $BP = PA$ after multiplying on the right by P. Since multiplication by an invertible matrix does not change the rank of a matrix, we have $\text{rank}(B) = \text{rank}(BP)$ and $\text{rank}(PA) = \text{rank}(A)$. Thus, $\text{rank}(A) = \text{rank}(B)$.

(f) Suppose A is an $n \times n$ matrix and $B = [A \quad A]$ is the $2n \times n$ matrix containing two copies of A side by side. Then $\text{rank}(B) = 2\text{rank}(A)$. **T F**

Solution: False. Since the columns of B are contained in the span of the columns of A, we have $\text{rank}(B) = \text{rank}(A)$.

(g) If A is a real symmetric matrix then A is similar to a real diagonal matrix. **T F**

Solution: True. By the spectral theorem, A is diagonalizable with real eigenvalues, so $A = PDP^{-1}$ for a real matrix D.

(h) If $v, w \in V$ are vectors in an inner product space with $\|v\| = \|w\| = 1$ and $\|v - w\| = \sqrt{2}$, then v and w must be orthogonal. **T F**

Solution: True. We have

$$2 = \|v - w\|^2 = \|v\|^2 + \|w\|^2 + 2\langle v, w\rangle = 2 + 2\langle v, w\rangle,$$

so it must be the case that $\langle v, w\rangle = 0$.

(i) If A is an invertible $n \times n$ matrix and v and w are orthogonal vectors in $\mathbb{R}^n$, then Av and Aw must also be orthogonal. **T F**

Solution: False. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is invertible and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are orthogonal, but this is not preserved under multiplication by A.

(j) The sum of any two solutions to

$$y''(t) + 3y'(t) - y(t) = e^t$$

is also a solution.

Solution: False. This is an inhomogeneous system. In particular, if y_1 and y_2 are solutions and we plug them into the left hand side, we get $2e^t$ on the right hand side, not e^t.

T F

2. For each of the following, either find an example (and explain why it has the property) or explain why no such example exists.

   (a) (3 points) A real 3×4 matrix A and vector $b \in \mathbb{R}^3$ such that $Ax = b$ has exactly 3 solutions.

   Solution: No such matrix exists. Every linear system $Ax = b$ has either $0, 1$ or infinitely many solutions, and three is not a possibility.

   (b) (4 points) A real 3×4 matrix A such that $\mathrm{Col}(A) = \{0\}$ and $\mathrm{Null}(A) = \{0\}$.

   Solution: No such matrix exists. By the dimension theorem, we must have $\dim \mathrm{Col}(A) + \dim \mathrm{Null}(A) = 4$, so both these subspaces cannot have dimension zero.

   (c) (4 points) A nonzero symmetric real 3×3 matrix A such that $\mathrm{Col}(A) = \mathrm{Null}(A)$.

   Solution: We are looking for a T such that $T(4te^{3t}) = 4T(te^{3t}) = 0$. This is equivalent to looking for a second order ODE $ay'' + by' + cy = 0$ with solution $y = te^{3t}$. Such a solution corresponds to an auxiliary equation with a double root at 3, i.e., $(r-3)^2 = r^2 - 6t + 9 = 0$. Thus, the corresponding T is given by $a = 1, b = -6, c = 9$.

   (d) (4 points) A positive real eigenvalue $\lambda > 0$ of the linear operator:

$$T(y) = \frac{d^2}{dt^2}y + \frac{d}{dt}y$$

   on the vector space

$$V = \{y : [0, \pi] \to \mathbb{R} \text{ infinitely differentiable, with } y(0) = y(\pi) = 0\}.$$

   Solution: No such eigenvalues exist. To see this, recall that a positive eigenvalue is a number $\lambda > 0$ such that $T(y) = \lambda y$ for a nonzero function $y \in V$, i.e.

$$y'' + y' = \lambda y.$$

This is a second order ODE with auxiliary equation $r^2 + r - \lambda = 0$, which has roots $r_1 = \frac{-1+\sqrt{1+4\lambda}}{2}$, $r_2 = \frac{-1-\sqrt{1+4\sqrt{\lambda}}}{2}$. Both roots are real and distinct, since $\lambda > 0$. Thus, the general solution to this ODE is given by

$$c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

A nonzero solution in V must additionally satisfy the boundary conditions:

$$c_1 e^0 + c_2 e^0 = 0 \qquad c_1 e^{r_1 \pi} + c_2 e^{r_2 \pi} = 0.$$

The first equation implies that $c_1 = -c_2$, which is inconsistent with the second equation since $r_1 \neq r_2$. Thus, there is no solution, and there are no such positive eigenvalues.

3. (6 points) For which values of $h \in \mathbb{R}$ is the following set of vectors linearly independent?

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}.$$

Show how you got your answer.

Solution: See the attached exam.

4. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2 with coefficient-wise operations, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(q) = -\frac{d^2}{dt^2} q + 2t \cdot \frac{d}{dt} q + 3q.$$

(a) (4 points) Find a basis for the Image of T.

Solution: See the attached exam.

(b) (2 points) Is T onto? Explain why or why not.

Solution: See the attached exam.

(c) (6 points) Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T with respect to $\mathcal{B}$ is diagonal? If so, find such a basis as well as the corresponding matrix $[T]_{\mathcal{B}}$. If not, explain why.

Solution: See the attached exam.

5. Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) (6 points) Compute the projection onto $\text{Col}(A)^{\perp}$ of $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Solution: See the attached exam.

(b) (2 points) What is the distance between v and the closest vector to v in $\text{Col}(A)^{\perp}$?

Solution: See the attached exam.

6. (a) (4 points) Find a basis for the space of real solutions to the homogeneous differential equation:

$$y''(t) + 2y'(t) + 5y(t) = 0.$$

Solution: See the attached exam.

(b) (3 points) Let V the vector space of solutions you found in part (a), and consider the linear transformation

$$S : V \to \mathbb{R}^2$$

defined by

$$S(y) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

Is S an isomorphism? Explain why or why not.

Solution: See the attached exam.

(c) (4 points) Find the general solution to the inhomogeneous equation:

$$y''(t) + 2y'(t) + 5y(t) = 4e^{3t} + t.$$

Solution: See the attached exam.

7. Consider the second order homogeneous differential equation:

$$y''(t) + 2y'(t) - 8y(t) = 0.$$

(a) (4 points) Reduce the above equation to a system of first order differential equations, i.e., find a 2×2 matrix A such that a vector valued solution $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ of

$$\mathbf{y}'(t) = A\mathbf{y}(t)$$

contains a solution to the given second order equation in its first coordinate.

Solution: See the attached exam.

(b) (4 points) Using your matrix A from part (a), find a fundamental matrix for the system:

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

Solution: See the attached exam.

(c) (3 points) Without doing any matrix arithmetic, use your answer to (b) to find a fundamental matrix for:

$$\mathbf{y}'(t) = A^3\mathbf{y}(t).$$

Explain your reasoning.

Solution: See the attached exam.

8. (6 points) Find a solution to the heat equation on a rod of length $L = \pi$:

$$\frac{\partial}{\partial t}u(x,t) = 3\frac{\partial^2}{\partial x^2}u(x,t) \qquad u(0,t) = u(\pi,t) = 0,$$

for all $t > 0$, with the initial condition

$$u(x,0) = 2\sin(3x) + 3\sin(2x).$$

Solution: See the attached exam.

9. (7 points) Consider the function $f(x) = |x|$ defined on the interval $[-\pi, \pi]$. Draw a sketch of the function. Find coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_0 \cos(nx) + b_n \sin(nx).$$

Explain your reasoning.

Solution: See the attached exam.

Have a good break!

Name and SID: [redacted]

3. (6 points) For which values of $h \in \mathbb{R}$ is the following set of vectors linearly independent?

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}.$$

Show how you got your answer.

if v_1 v_2 v_3 are linearly independent,

then the matrix $[v_1 \; v_2 \; v_3]$ has a pivot in every

column. $[v_1 \; v_2 \; v_3] = \begin{bmatrix} 2 & 4 & -2 \\ -2 & -6 & 2 \\ 4 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -1 & h+4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & h+4 \end{bmatrix}$

There is a pivot in every column when $h+4 \neq 0$

so v_1 v_2 v_3 are linearly independent when

$\boxed{h \in \mathbb{R} \neq -4}$

This makes sense because if $h=-4$, $v_1 = -v_3$ so

$v_1 + 0v_2 + v_3 = 0$, so v_1 v_2 v_3 are not L.I.

Name and SID: [redacted]

4. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2 with coefficient-wise operations, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(q) = -\frac{d^2}{dt^2}q + 2t \cdot \frac{d}{dt}q + 3q.$$

(a) (4 points) Find a basis for the Image of T.

use basis $B = \{1, t, t^2\}$ for input and output of T

T with this basis is $A = \begin{bmatrix} [T(1)]_B & [T(t)]_B & [T(t^2)]_B \end{bmatrix}$

$= \begin{bmatrix} [3]_B & [2t + 3t]_B & [-2 + (2t)(2t) + 3t^2]_B \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 & -2 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ Image $T \cong$ Col A = Span $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix} \right\} = \mathbb{R}^3$

pivot every row $\rightarrow$

So a basis for the image of T is $\{3, 5t, 7t^2 - 2\}$

(b) (2 points) Is T onto? Explain why or why not.

T is onto; the matrix for T has a pivot in every row, so col A span $\mathbb{R}^3$, so ImT spans $\mathbb{P}_2$, so T is onto.

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(c) (6 points) Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T with respect to $\mathcal{B}$ is diagonal? If so, find such a basis as well as the corresponding matrix $[T]_{\mathcal{B}}$. If not, explain why.

when we used basis $\{1, t, t^2\}$ we got

Matrix $A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ We can now try to

diagonalize A to find a diagonal matrix for T

characteristic polynomial of A is $\det\left(\begin{bmatrix} 3-\lambda & 0 & -2 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 7-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow (3-\lambda)(5-\lambda)(7-\lambda) = 0 \quad \therefore \lambda = 3, 5, 7$

Eigvec for $\lambda = 3$ is in null $\begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Eigvec for $\lambda = 5$ is in null $\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Eigvec for $\lambda = 7$ is in null $\begin{bmatrix} -4 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

So $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}^{-1}$

So when we use the basis $\mathcal{B} = \{1, t, -2t^2 + 1\}$

$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ (diagonal)

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5. Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) (6 points) Compute the projection onto $\mathrm{Col}(A)^\perp$ of $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T) = \mathrm{Null}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$= \mathrm{Null}\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 3 & -2 \end{bmatrix} \sim \mathrm{Null}\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & -1 & 2/3 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \right\}$$

Find an orthogonal basis for span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \right\}$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \end{bmatrix} \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 3 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

proj $\mathrm{Col}(A)^\perp$ of $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is just $v' = \frac{v \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v \cdot v_2}{v_2 \cdot v_2} v_2$

$$= \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{-2}{12}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/6 \\ 1/6 \\ -1/6 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 4/6 \\ 2/6 \\ -3/6 \end{bmatrix}$$

(b) (2 points) What is the distance between v and the closest vector to v in $\mathrm{Col}(A)^\perp$?

closest vector to v in $\mathrm{Col}(A)^\perp$ is the projection we just found. $\mathrm{dist}(v, \mathrm{proj}\, v) = \|v - v'\| = \sqrt{(v-v')\cdot(v-v')}$

dist =

$v - v' = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/6 \\ 4/6 \\ 2/6 \\ -3/6 \end{bmatrix} = \begin{bmatrix} 5/6 \\ 2/6 \\ -2/6 \\ 3/6 \end{bmatrix} \Rightarrow \sqrt{\begin{bmatrix} 5/6 \\ 2/6 \\ -2/6 \\ 3/6 \end{bmatrix} \cdot \begin{bmatrix} 5/6 \\ 2/6 \\ -2/6 \\ 3/6 \end{bmatrix}} = \sqrt{\frac{42}{36}} = \boxed{\sqrt{\frac{7}{6}}}$

Name and SID: [redacted]

6. (a) (4 points) Find a basis for the space of real solutions to the homogeneous differential equation:

$$y''(t) + 2y'(t) + 5y(t) = 0.$$

Auxilary eqn is $r^2 + 2r + 5 = 0$

$r = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$

when $r = \alpha + ib$ is a root of aux eqn, a basis for real sols is $\{e^{at}\cos bt,\ e^{at}\sin bt\}$

So basis for space of real solns here is

$\{e^{-t}\cos 2t,\ e^{-t}\sin 2t\}$

(b) (3 points) Let V the vector space of solutions you found in part (a), and consider the linear transformation

$$S : V \to \mathbb{R}^2$$

defined by

$$S(y) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

Is S an isomorphism? Explain why or why not.

Using the basis we found above for V, and $\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$ for $\mathbb{R}^2$, the transformation matrix A for S is $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$y_1'(0) = -e^{0}\cos 2(0) - 2\sin 2(0)\, e^{-0} = -1$

$y_2'(0) = -e^{0}\sin(2)(0) + 2e^{0}\cos 2(0) = 2$

A has pivot in every row and column, so A is one-to-one, onto, and invertible, so S is an isomorphism

(so wronskian lemma holds (phew))

(c) (4 points) Find the general solution to the inhomogeneous equation:

$$y''(t) + 2y'(t) + 5y(t) = 4e^{3t} + t.$$

First find a particular solution to $y''(t) + 2y'(t) + 5y(t)$

→ find particulars for $4e^{3t}$ & t separately and add them

for $4e^{3t}$, try $y(t) = c_1 e^{3t}$: $y''(t) + 2y'(t) + 5y(t) = 9c_1e^{3t} + 6c_1e^{3t} + 5c_1e^{3t}$

$= 20c_1 e^{3t}$, $c_1 = \frac{1}{5}$, so $y(t) = \frac{1}{5}e^{3t}$

for t, try $y(t) = a_0 + a_1 t$

$\Rightarrow y''(t) + 2y'(t) + 5y(t) = 2a_1 + 5a_0 + 5a_1 t = t$

$a_1 = \frac{1}{5}$

$a_0 = \frac{-2}{25}$ so $y(t) = \frac{-2}{25} + \frac{1}{5}t$

so a particular solution is $y(t) = \frac{1}{5}e^{3t} + \frac{1}{5}t - \frac{2}{25}$

to find general, just add the particular to the general soln of the homogenous eqn, which we found in a). So gen soln is $y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{1}{5}e^{3t} + \frac{1}{5}t - \frac{2}{25}$

Name and SID: [redacted]

7. Consider the second order homogeneous differential equation:

$$y''(t) + 2y'(t) - 8y(t) = 0.$$

(a) (4 points) Reduce the above equation to a system of first order differential equations, i.e., find a 2×2 matrix A such that a vector valued solution $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ of

$$\mathbf{y}'(t) = A\mathbf{y}(t)$$

contains a solution to the given second order equation in its first coordinate.

let $y_1(t) = y$, $\quad y_1'(t) = y_2(t)$

then $y_2'(t) = y''(t) = 8(y)(t) - 2y'(t) = 8y_1(t) - 2y_2(t)$

so $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\left(A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix}\right)$

(b) (4 points) Using your matrix A from part (a), find a fundamental matrix for the system:

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

To do this find eigvecs of $A = \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \Rightarrow \det \begin{bmatrix} -\lambda & 1 \\ 8 & -2-\lambda \end{bmatrix} = 0$

$\Rightarrow \lambda^2 + 2\lambda - 8 = 0 \quad (\lambda+4)(\lambda-2) = 0 \quad \lambda = -4, 2$

eigvec for $\lambda = -4$ is in null $\begin{bmatrix} +4 & 1 \\ 8 & 2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

eigvec for $\lambda = 2$ is in null $\begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So fund soln is $c_1 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So a fund matrix is $\begin{bmatrix} e^{-4t} & e^{2t} \\ -4e^{-4t} & 2e^{2t} \end{bmatrix}$

$c_1 = c_2 = 1$

Name and SID: [redacted]

(c) (3 points) Without doing any matrix arithmetic, use your answer to (b) to find a fundamental matrix for:

$$\mathbf{y}'(t) = A^3\mathbf{y}(t).$$

Explain your reasoning.

for eigvec $\vec{v}$: $A^3\vec{v} = \lambda A^2\vec{v} = \lambda^2 A\vec{v} = \lambda^3\vec{v}$

A^3 has the same eigenvectors as A, and corresponding eigenvalues λ^3 for eigvals λ of A

so A^3 has eigvals -64, 8 with eigvecs $\begin{bmatrix}1\\-4\end{bmatrix}$ $\begin{bmatrix}1\\2\end{bmatrix}$

So fund soln is $c_1 e^{-64t}\begin{bmatrix}1\\-4\end{bmatrix} + c_2 e^{8t}\begin{bmatrix}1\\2\end{bmatrix}$

So a fund matrix is $\begin{bmatrix}e^{-64t} & e^{8t}\\ -4e^{-64t} & 2e^{8t}\end{bmatrix}$

let $c_1 = c_2 = 1$?

8. (6 points) Find a solution to the heat equation on a rod of length $L = \pi$:

check

$$\frac{\partial}{\partial t}u(x,t) = 3\frac{\partial^2}{\partial x^2}u(x,t) \qquad u(0,t) = u(\pi,t) = 0,$$

for all $t > 0$, with the initial condition

$$u(x,0) = 2\sin(3x) + 3\sin(2x).$$

We can use superposition and find solutions for $u(x,0) = 2\sin 3x$ and $u(x,0) = 3\sin(2x)$ seperately then add them

for $u(x,0) = c_1 \sin\left(\frac{n\pi x}{L}\right)$; a soln is $u(x,t) = c_1 e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$

here $L = \pi$, $\beta = 3$, so for $u(x,0) = 2\sin 3x$ a soln is $2e^{-27t}\sin 3x$

for $u(x,0)$ a soln is $3e^{-12t}\sin 2x$

So a solution for $u(x,0) = 2\sin(3x) + 3\sin(2x)$ is $u(x,t) = 2e^{-27t}\sin 3x + 3e^{-12t}\sin 2x$

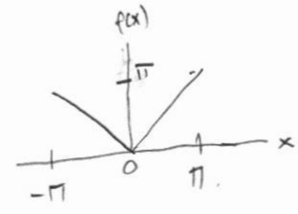
Name and SID: [redacted]

9. (7 points) Consider the function $f(x) = |x|$ defined on the interval $[-\pi, \pi]$. Draw a sketch of the function. Find coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \overset{a_n}{a_0} \cos(nx) + b_n \sin(nx).$$

Explain your reasoning.

A sketch of the function is



$$a_n = \frac{\langle |x|, \cos nx \rangle}{\langle \cos nx, \cos nx \rangle}$$

$$b_n = \frac{\langle |x|, \sin(nx) \rangle}{\langle \sin(nx), \sin(nx) \rangle}$$

using the inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

(in this inner prod space $\cos nx$, $\sin nx$ are orthonormal so coefs for projection are these simple eqns)

Note: $|x|$ is even, and $\sin nx$ is odd, so $(|x|)(\sin(nx))$ is odd, so $\int_{-\pi}^{\pi} (|x|)(\sin nx)\, dx = 0$ for all n, so

$\boxed{b_n = 0 \text{ for all } n}$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx \quad (|x|\cos nx \text{ is even})$$

tabular by parts

x	$\cos nx$
1	$\frac{1}{n} \sin nx$
0	$-\frac{1}{n^2} \cos nx$

$$= \frac{2}{\pi} \left(\frac{x}{n} \sin(nx) + \frac{1}{n^2} \cos(nx) \right) \Big|_0^{\pi}$$

$$\boxed{a_n = \frac{2}{\pi} \left(\frac{1}{n^2} \cos(n\pi) - \frac{1}{n^2} \right)}$$

which for $n \geq 1$ is $\frac{-4}{\pi n^2}$ if n is odd and 0 if n is even, if n is 0 then

$$\frac{a_0}{2} = \frac{\langle |x|, \cos(0) \rangle}{\langle \cos(0), \cos(0) \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{1}{\pi} \int_0^{\pi} x \, dx = \frac{1}{\pi} \left(\tfrac{1}{2} x^2 \right) \Big|_0^{\pi} = \frac{\pi}{2}$$

$\hookrightarrow a_0 = \pi$