Math 54 Fall 2016 Practice Final Solutions

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180 minutes, closed book, closed notes

1. (20 pts) True or False (no need for justification):

   (a) If $AB = 0$ for two square matrices A and B then either $A = 0$ or $B = 0$.
   Solution: False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

   (b) If A is a square invertible matrix then A and A^{-1} have the same rank.
   Solution: True. If A is invertible A^{-1} is also invertible, so they both have full rank (equal to n if both are $n \times n$).

   (c) If A and B are square and invertible then AB and BA have the same eigenvalues.
   Solution: True. Since B is invertible, we have

   $$B(AB)B^{-1} = BABB^{-1} = BA,$$

   so AB and BA are similar, and therefore have the same eigenvalues.

   (d) If every entry of a square matrix A is nonzero, then $\det(A) \neq 0$.
   Solution: False. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which has linearly dependent columns and is not invertible.

   (e) The sum of two diagonalizable matrices must be diagonalizable.
   Solution: False. For example, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ then A and B are diagonalizable, but $A + B$ is not diagonalizable.

   (f) If A is an $m \times n$ matrix then the rank of A^TA is equal to the rank of A.
   Solution: True. We will show that A^TA and A have the same null space. By the dimension theorem, which says that $n = \text{rank}(A) + \dim \text{Null}(A) = \text{rank}(A^TA) + \dim \text{Null}(A^TA)$, this will imply that they have the same rank.
   Suppose $x \in \text{Null}(A)$. Then $Ax = 0$ so $A^TAx = 0$, and x must be in $\text{Null}(A^TA)$.
   For the other direction, suppose $A^TAx = 0$. Let $y = Ax$; our goal is to show that $y = 0$, which would imply $x \in \text{Null}(A)$. Observe that $y \in \text{Col}(A)$. Suppose the columns of A are $a_1, \ldots, a_m$. Since these are the rows of A^T, our equation

$A^TAx = A^Ty = 0$ implies that $y \cdot a_i = 0$ for all i, which means $y \in \text{Col}(A)^\perp$. This means y is orthogonal to itself, so $y = 0$, as desired.

This was the hardest question on the exam. Understanding all the steps is a good way to make sure you are comfortable with dot products, transposes, and orthogonal complements.

(g) If $A = A^T$ and the only eigenvalue of A is $\lambda = 1$, then $A = I$.

Solution: True. By the spectral theorem A is diagonalizable since it is symmetric, i.e., $A = PDP^{-1}$ (actually the theorem says P is orthogonal, but we will not use this fact). Since 1 is the only eigenvalue we must have $D = I$. But now $A = PP^{-1} = I$.

(h) Any two orthogonal vectors in an inner product space must be linearly independent.

Solution: False. If both the vectors are equal to zero, then they are orthogonal but not linearly independent.

I did not mean this to be a trick question. I just missed an important word: I should have asked if this is true for two NONZERO vectors. In that case, it is true. For if $\langle v, w\rangle = 0$ and $v = cw$ the n $\langle v, cv\rangle = c\langle v, v\rangle = 0$, which implies that $v = 0$, which is not the case.

(i) Suppose W is a subspace of $\mathbb{R}^n$. If $v_1, \ldots, v_k$ is a basis for W and $u_1, \ldots, u_\ell$ is a basis for $W^\perp$ then $v_1, \ldots, v_k, u_1, \ldots, u_\ell$ must be a basis for $\mathbb{R}^n$.

Solution: True. Assume $y \in \mathbb{R}^n$. By the unique decomposition theorem we have $y = \hat{y} + z$ for *unique* vectors $\hat{y} \in W$ and $z \in W^\perp$. Since $v_1, \ldots, v_k$ and $u_1, \ldots, u_\ell$ are bases for W and $W^\perp$ respectively, there are *unique* coefficients $c_1, \ldots, c_k$ and $b_1, \ldots, b_\ell$ such that $\hat{y} = c_1v_1 + \ldots + c_kv_k$ and $z = b_1u_1 + \ldots, b_\ell u_\ell$. But now $y = c_1v_1 + \ldots + b_\ell u_\ell$ is a unique linear combination of $v_1, \ldots u_\ell$. Since this is true for all $y \in \mathbb{R}^n$, this set must be a basis for $\mathbb{R}^n$.

(j) Two real-valued functions $y_1(t)$ and $y_2(t)$ are linearly independent if and only if their Wronskian determinant is nonzero everywhere.

Solution: False. This is only necessarily true when $y_1(t)$ and $y_2(t)$ are solutions of a differential equation.

2. (20 pts) For each of the following, either find an example (and explain why it has the property) or explain why no such example exists.

- A differential operator $T = a(d^2/dx^2) + b(d/dx) + cI$ with $a \neq 0$ on the vector space
$$V = \{f : \mathbb{R} \to \mathbb{R},\ f \text{ is infinitely differentiable}\}$$
which is one to one.

  Solution: There is no such T. Since a second order linear ODE with constant coefficients always has a nontrivial solution (by examining roots of the auxiliary equation), for every T there exists some nonzero function y such that $T(y) = 0$. But now $\ker(T) \neq \{0\}$, so T cannot be one to one.

- A second order linear differential equation with constant coefficients which has $y(t) = e^t$ and $y(t) = \sin(t)$ among its solutions.

  Solution: There is no such equation. Any such equation falls into one of three categories: the auxiliary equation has distinct real roots, a double real root, or complex conjugate roots, in which case the general solution is $c_1 e^{r_1 t} + c_2 e^{r_2 t}$, or $c_1 e^{rt} + c_2 t e^{rt}$, or $c_1 e^{\alpha t}\sin(\beta t) + c_2 e^{\alpha t}\cos(\beta t)$, respectively. So if $\sin(t)$ is a solution then e^{it} must be a root of the auxiliary polynomial. If e^t is a solution then 1 must be a root. But a real polynomial cannot have one real and one complex root, so this is impossible.

  I should have said second order homogeneous *linear diff eq. If you allow inhomogeneous equations, the answer is yes — just take a second order equation with homogeneous solution* $\sin(t)$ *and right hand side equal to* e^t.

- A 2×2 real matrix A such that the system of ODE $y'(t) = Ay(t)$, $y : \mathbb{R} \to \mathbb{R}^2$, has a fundamental matrix
$$\begin{bmatrix} -e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}.$$

  Recall that if A is diagonalizable, the solution space of $y'(t) = Ay(t)$ is spanned by $e^{\lambda t} v$ where $Av = \lambda v$ are eigenvalue-eigenvector pairs of A. This means that the fundamental matrix for a 2×2 diagonalizable system is $F(t) = [e^{\lambda_1 t} v_1 e^{\lambda_2 t} v_2]$. To obtain the matrix above, we take $\lambda_1 = 1, \lambda_2 = 2$ and $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The corresponding A is $A = PDP^{-1}$ where D has diagonal entries 1 and 2 and P has columns v_1 and v_2. The row reduction/matrix multiplication steps for calculating A are omitted (and it is fine to give an example without working out all the arithmetic — the important thing is the reasoning that brought you to the example).

- A square matrix A such that A is *not diagonal* and $A^2 = A$.

  Solution: There are many possibilities here. One is to take $A = [Proj_W]$ for some one dimensional subspace W of $\mathbb{R}^2$ (or more generally, any strict subspace of $\mathbb{R}^n$). Since $Proj_W \circ Proj_W = Proj_W$ we automatically have $A^2 = A$. For instance, taking $W = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ gives $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

  In general, it is good to remember that the matrix of the projection onto a one-dimensional subspace spanned by a vector v with is just $A = vv^T/\|v\|^2$, since $Proj_W(x) = \frac{(v^T x)}{\|v\|^2} v$

- Two linearly *independent* vector-valued functions $y_1 : \mathbb{R} \to \mathbb{R}^2$ and $y_2 : \mathbb{R} \to \mathbb{R}^2$ such that the vectors $y_1(0)$ and $y_2(0)$ are linearly *dependent* in $\mathbb{R}^2$.

Solution: Again there are many options. We are just looking for two vector valued functions which are not the same, but which have parallel values at zero. So consider:

$$y_1(t) = \begin{bmatrix} 1 \\ e^t \end{bmatrix} \qquad y_2(t) = \begin{bmatrix} 1 \\ e^{2t} \end{bmatrix}.$$

Since $y_1 \neq cy_2$ these are linearly independent, but

$$y_1(0) = y_2(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. (6 pts) Let

$$V = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} \right\}, W = span \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

be subspaces of $\mathbb{R}^4$. Find a nonzero vector in $V \cap W$ (i.e., which is in both subspaces).

Solution: Using the definition of span, we see that we are looking for coefficients c_1, c_2, c_3, c_4 (not all zero) such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} = c_3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ -6 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

By moving everything to one side, we find that this is the same as solving

$$\begin{bmatrix} 1 & 0 & -4 & -2 \\ 2 & 4 & -3 & -1 \\ 3 & 1 & -1 & -2 \\ 4 & -1 & 6 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0.$$

The RREF of A is

$$\begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix},$$

so by setting the free variable $c_4 = 3$ we have the solution $c_1 = 2, c_2 = -1, c_3 = -1, c_4 = 3$. To get an actual vector in $V \cap W$, we take the linear combination:

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 9 \end{bmatrix}.$$

To double check that this worked, we also calculate:

$$-\begin{bmatrix} 4 \\ 3 \\ 1 \\ -6 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 9. \end{bmatrix}$$

4. (8 pts) Consider the vector space of 2×2 real matrices with entrywise addition:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

and consider the function $T : V \to V$ defined by

$$T(X) = X + X^T.$$

(a) Show that T is a linear transformation.
(b) Find a basis for $\mathrm{Ker}(T)$.
(c) Find a basis for $\mathrm{Im}(T)$.
(d) Find an eigenvector of T, along with the corresponding eigenvalue.

Solution: (a) We check that

$$T(X+Y) = (X+Y)+(X+Y)^T = X+Y+X^T+Y^T = (X+X^T)+(Y+Y^T) = T(X)+T(Y)$$

and

$$T(cX) = cX + (cX)^T = c(X + X^T)$$

so T is linear.
(b) Let us find the matrix of T with respect to the basis $E =$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$T(e_1) = 2e_1, T(e_2) = e_2 + e_3, T(e_3) = e_3 + e_2 T(e_4) = 2e_4,$$

so

$$[T]_E = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Row reducing this matrix (do it!) reveals that the Null space of $[T]$ is spanned by the vector

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

so the Kernel of T is spanned by

$$e_2 - e_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This makes sense since $T(X) = X + X^T = 0$ implies that $X = -X^T$, which implies that the diagonal entries of X are zero and the off diagonals are negatives of each other. In fact, this is a perfectly valid alternative method to arrive at the same conclusion.

(c) Again by applying row reduction to $[T]$, we find that the pivot columns $1, 2, 4$ form a basis for Col($[T]$). The corresponding vectors in V are $2e_1, e_2 + e_3$, and $2e_4$, so these must form a basis for the image of T.

Again, this makes sense, since a moment's thought reveals that the span of the three vectors above is just the set of symmetric matrices $X = X^T$.

(d) The question just asks for one eigenvector, so before trying to compute all of them it is good to inspect $[T]$ to see if any eigenvectors are staring you in the face. Since the first and last columns are multiples of standard basis vectors, we immediately see that

$$[T]\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad [T]\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the corresponding vectors in V:

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are eigenvectors of T (either one is fine). There are other eigenvectors, but these are the easiest to find.

5. (6 pts) For which real values of a is the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

diagonalizable? For which a is it invertible?

Solution: Since the matrix is upper triangular, its eigenvalues are equal to the diagonal entries, which are just 1. The corresponding eigenspace is

$$E_1 = \text{Null}(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}).$$

If $a \neq 0$ then this matrix has one pivot, and so the null space has dimension one, and the matrix is not diagonalizable. If $a = 0$ then it is just zero, and the nullspace is all of $\mathbb{R}^2$, so it is diagonalizable. Thus, the matrix is diagonalizable if and only if $a = 0$.

Since the above matrix is in REF and has a pivot in every row, it is always invertible (for every value of a).

6. (7 pts) Let V be the vector space of all real valued continuous functions on the interval $[0,1]$, and consider the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

Find a nonzero function in V which is orthogonal to the functions x and x^2, with respect to this inner product.

Solution: We are looking for a function $f : [0,1] \to \mathbb{R}$ such that $\langle f, x\rangle = \langle f, x^2\rangle = 0$. Instead of guessing, let's start with any function that is not in $W = span\{x, x^2\}$.; the simplest is the constant function $g(x) = 1$, and you can check that this is not in W by writing down the linear combination and comparing coefficients. By the unique decomposition theorem, there are unique vectors f and h such that $g = h + f$ with $h \in W$ and $f \in W^\perp$. To find h, we first find an orthogonal basis for W: applying Gram-Schmidt to x, x^2 we obtain the basis

$$b_1 = x$$

$$b_2 = x^2 - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}x = x^2 - \frac{\int_0^1 x^3dx}{\int_0^1 x^2}x = x^2 - 3x/4.$$

The projection h is now given by:

$$h = \frac{\langle 1, b_1\rangle}{\langle b_1, b_1\rangle}b_1 + \frac{\langle 1, b_2\rangle}{\langle b_2, b_2\rangle}b_2.$$

We calculate the integrals:

$$\langle 1, b_1\rangle = \int_0^1 xdx = 1/2,$$

$$\langle b_1, b_1\rangle = \int_0^1 x^2dx = 1/3,$$

$$\langle 1, b_2\rangle = \int_0^1 x^2 - 3x/4dx = 1/3 - 3/8 = -1/24,$$

$$\langle b_2, b_2\rangle = \int_0^1 (x^2 - 3x/4)^2dx = \int_0^1 x^4 - 3x^3/2 + 9x^2/16dx = 1/5 - 3/8 + 3/16 = 1/80.$$

Plugging these in, we have

$$h = \frac{1/2}{1/3}x - \frac{1}{24}1/80(x^2 - 3x/4),$$

so

$$f = g - h = 1 - 4x10/3x^2$$

is orthogonal to both x and x^2, as desired.

Note that the answer to this question is not unique, and you will get different answers if you start with different g.

7. (10 pts) (a) Find a basis of real solutions to the homogeneous differential equation

$$y''(t) - 2y'(t) + 2y(t) = 0.$$

(b) Find the general solution to the inhommogeneous equation

$$y''(t) - 2y'(t) + 2y(t) = t^2 + e^t.$$

(c) Find a solution to (b) satisfying the initial conditions $y(0) = 1$ and $y'(0) = 2$.
(d) Write the equation in (a) as $T_1 \circ T_2(y) = 0$ for two *first order* differential operators T_1 and T_2.

Solution: (a) The auxiliary polynomial is $r^2 - 2r + 2 = (r - (1 + i))(r - (1 - i))$. Thus, $e^{(1+i)t}, e^{(1-i)t}$ form a basis for the space of complex solutions, and taking real and imaginary parts using Euler's identity we obtain the basis $e^t \sin(t), e^t \cos(t)$ of real solutions.
(b) First we find a particular solution to the inhomogeneous equation using the superposition principle. We write $f(t) = f_1(t) + f_2(t)$ for $f_1 = t^2$ and $f_2 = e^t$. For the inhomogeneity f_1 the method of undetermined coeffs tells us there is a solution of the form $y(t) = a_0 + a_1 t + a_2 t^2$; plugging this into the equation we obtain

$$2a_2 - 2(a_1 + 2ta_2) + 2(a_0 + a_1 t + a_2 t^2) = f_1(t) = t^2.$$

Matching coefficients gives:

$$2a_2 - 2a_1 + 2a_0 = 0$$
$$-4a_2 + 2a_1 = 0$$
$$2a_2 = 1.$$

which is easily solved by substitution to yield $a_2 = 1/2, a_2 = 1, a_0 = 1/2$, so $y_{p1}(t) = t^2/2 + t + 1/2$ is a particular solution to $T(y) = f_1$. For the other inhomgeneity $f_2(t) = e^t$ we observe since 1 is not a root of the auxiliary polynomial, there must be a particular solution of form $y_{p2}(t) = ce^t$. Substituting, we get:

$$ce^t - 2ce^t + 2ce^t = e^t \Rightarrow c = 1.$$

Thus, $y_{p2}(t) = e^t$ is a solution to $T(y) = f_2$.

By the superposition principle, the sum

$$y_p(t) = t^2/2 + t + 1/2 + e^t$$

is a particular solution to $T(y) = f_1 + f_2$. Thus, a general solution to the inhomogeneous equation is given by

$$y(t) = c_1 e^t \sin(t) + c_2 e^t \cos(t) + t^2/2 + t + 1/2 + e^t.$$

(c) To solve the initial value problem, we plug in the initial data:

$$y(0) = c_1(0) + c_2(1) + 1/2 + 1 = 1,$$

$$y'(0) = c_1(e^0 \cos(0) + e^0 \sin(0)) + c_2(-e^0 \sin(0) + e^0 \cos(0)) + 1 + e^0 = 2.$$

The first equation is just $c_2 = -1/2$, and the second becomes

$$c_1 - 1/2 + 2 = 2 \to c_1 = 1/2.$$

So the solution to the IVP is

$$(1/2)e^t \sin(t) - (1/2)e^t \cos(t) + t^2/2 + t + 1/2 + e^t.$$

(d) Applying the same factorization as the auxiliary polynomial, we have

$$T = (d^2/dt^2) - 2(d/dt) + 2I = (d/dt - (1+i)I)(d/dt - (1-i)I),$$

and since multiplying operators corresponds to composition, we can take $T_1 = d/dt - (1+i)I$ and $T_2 = d/dt - (1-i)I$. (the order doesn't matter)

8. (8 pts) Find functions $y_1(t)$ and $y_2(t)$ such that

$$y_1' = -2y_1 + 2y_2 \qquad y_2' = 2y_1 + y_2$$

and $y_1(0) = -1, y_2(0) = 3$.

Solution: Writing this in normal form for $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, we have

$$y'(t) = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} y(t) = Ay(t).$$

First we find the general solution, which can be derived from the eigenvalues and eigenvectors of A. The characteristic polynomial is

$$\det(A - tI) = \det \begin{bmatrix} -2-t & 2 \\ 2 & 1-t \end{bmatrix} = -(1-t)(2+t) - 4 = t^2 + t - 6 = (t+3)(t-2),$$

and the corresponding eigenvectors are

$$E_{-3} = \text{Null} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}.$$

$$E_2 = \text{Null} \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

Let us call these vectors v_1 and v_2 respectively. Then, a basis of solutions is given by $e^{-3t}v_1$ and $e^{2t}v_2$, and the general solution is $c_1e^{-3t}v_1 + c_2e^{2t}v_2$. To solve the IVP we plug in $t = 0$:

$$c_1v_1 + c_2v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

which can be solved by row reducing

$$\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}$$

to obtain $c_1 = -1, c_2 = 1$, which yields the solution

$$-e^{-3t}v_1 + e^{2t}v_2 = \begin{bmatrix} -2e^{-3t} + e^{2t} \\ e^{-3t} + 2e^{2t} \end{bmatrix}.$$

9. (7 pts) Find a real-valued function $u : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ which satisfies the heat equation

$$\frac{\partial}{\partial t}u(x,t) = 2\frac{\partial^2}{\partial x^2}u(x,t) \qquad u(0,t) = u(\pi,t) = 0,$$

for all $t > 0$, as well as the initial condition

$$u(x,0) = \sin(3x) - \sin(5x).$$

Solution: We use the superposition principle. The solution for initial data $u(x,0) = \sin(3x)$ is $e^{-2(3)^2t}\sin(3x)$, and for $u(x,0) = -\sin(5x)$ it is $-e^{-2(5)^2t}\sin(5x)$. By linearity, the solution for the given initial data is

$$u(x,t) := e^{-18t}\sin(3t) - e^{-50t}\sin(5t).$$

10. (8 pts) Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = |\sin(x)|$. Draw a sketch of the function. Find coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_0\cos(nx) + b_n\sin(nx).$$

(hint: use the product to sum trig formulas)

Solution: The sketching is just so that you realize that the given function is even. This implies that the sine coefficients $b_n = 0$ for $n = 1, 2, \ldots$, since

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$$

because the integrand is a product of an even function and an odd function, and therefore odd.

We find the remaining coefficients by computing the integrals:

$$a_0/2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x)| dx = \frac{1}{\pi}\int_0^{\pi} \sin(x) dx = \frac{1}{\pi}(-\cos(\pi) + \cos(0)) = \frac{2}{\pi}.$$

For the other coefficients we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)| \cos(nx) dx = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx) dx.$$

We now apply the product to sum formula $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$:

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin((1+n)x) + \sin((1-n)x) dx$$

which integrates to

$$a_n = -\frac{2}{\pi}\frac{\cos(n\pi) + 1}{n^2 - 1}$$

when $n \neq 1$ and

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin(2x) dx = 0.$$

Note that the terms a_n are nonzero only when n is an even integer.