Math 54 Final Exam, Prof. Srivastava
December 15, 2016, 8:10am–11:00am, 155 Dwinelle Hall.

Name: ______________________________

SID: ______________________________

Instructions: Write all answers in the provided space. This exam includes two pages of scratch paper, which must be submitted but will not be graded. **Do not unstaple the exam.** Write your name and SID on every page. **Show your work** — numerical answers without justification will be considered suspicious and will not be given full credit. Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed. **If you are seen writing after time is up, you will lose 20 points.**

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Sign here: ______________________________

Name of the student to your left: ______________________________

Name of the student to your right: ______________________________

Question	Points
1	20
2	19
3	6
4	12
5	8
6	11
7	11
8	6
9	7
Total:	100

Do not turn over this page until your instructor tells you to do so.

[Scratch Paper 1]

Name and SID: ____________________

1. (20 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) Every system of 3 linear equations in 4 variables has a solution. **T** **F**

   (b) If A is a square matrix and R is the reduced row echelon form of A then A and R must have the same eigenvalues. **T** **F**

   (c) Suppose A and B are 10×10 matrices and $v_1, \ldots, v_{10}$ is a basis of $\mathbb{R}^{10}$ such that $Av_i = Bv_i$ for all $i = 1, \ldots, 10$. Then it must be the case that $A = B$. **T** **F**

   (d) Suppose A and B are 10×10 matrices and $v_1, \ldots, v_{10}$ is a basis of $\mathbb{R}^{10}$ such that each v_i is an eigenvector of **both** A and B for $i = 1, \ldots, 10$. Then it must be the case that $AB = BA$. **T** **F**

   (e) If A and B are similar matrices then $\text{rank}(A) = \text{rank}(B)$. **T** **F**

   (f) Suppose A is an $n \times n$ matrix and $B = [A \quad A]$ is the $2n \times n$ matrix containing two copies of A side by side. Then $\text{rank}(B) = 2\text{rank}(A)$. **T** **F**

   (g) If A is a real symmetric matrix then A is similar to a real diagonal matrix. **T** **F**

   (h) If $v, w \in V$ are vectors in an inner product space with $\|v\| = \|w\| = 1$ and $\|v - w\| = \sqrt{2}$, then v and w must be orthogonal. **T** **F**

   (i) If A is an invertible $n \times n$ matrix and v and w are orthogonal vectors in $\mathbb{R}^n$, then Av and Aw must also be orthogonal. **T** **F**

   (j) The sum of any two solutions to

   $$y''(t) + 3y'(t) - y(t) = e^t$$

   is also a solution. **T** **F**

2. For each of the following, either find an example (and explain why it has the property) or explain why no such example exists.

   (a) (3 points) A real 3×4 matrix A and vector $b \in \mathbb{R}^3$ such that $Ax = b$ has exactly 3 solutions.

   (b) (4 points) A real 3×4 matrix A such that $\mathrm{Col}(A) = \{0\}$ and $\mathrm{Null}(A) = \{0\}$.

(c) (4 points) A nonzero symmetric real 3×3 matrix A such that $\mathrm{Col}(A) = \mathrm{Null}(A)$.

(d) (4 points) A linear operator $T = a(d^2/dt^2) + b(d/dt) + cI$ with $a \neq 0, b, c \in \mathbb{R}$ on the vector space

$$V = \{y : \mathbb{R} \to \mathbb{R},\ y \text{ is infinitely differentiable}\}$$

such that the kernel of T contains the function $y(t) = 4te^{3t}$.

(e) (4 points) A positive real eigenvalue $\lambda > 0$ of the linear operator:

$$T(y) = \frac{d^2}{dt^2}y + \frac{d}{dt}y$$

on the vector space

$$V = \{y : [0, \pi] \to \mathbb{R} \text{ infinitely differentiable, with } y(0) = y(\pi) = 0\}.$$

3. (6 points) For which values of $h \in \mathbb{R}$ is the following set of vectors linearly independent?

$$v_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}.$$

Show how you got your answer.

4. Let $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the vector space of polynomials of degree at most 2 with coefficient-wise operations, and consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(q) = -\frac{d^2}{dt^2}q + 2t \cdot \frac{d}{dt}q + 3q.$$

(a) (4 points) Find a basis for the Image of T.

(b) (2 points) Is T onto? Explain why or why not.

(c) (6 points) Is there a basis $\mathcal{B}$ of $\mathbb{P}_2$ such that the matrix of T with respect to $\mathcal{B}$ is diagonal? If so, find such a basis as well as the corresponding matrix $[T]_{\mathcal{B}}$. If not, explain why.

5. Let

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(a) (6 points) Compute the projection onto $\text{Col}(A)^{\perp}$ of $v = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) (2 points) What is the distance between v and the closest vector to v in $\text{Col}(A)^{\perp}$?

6. (a) (4 points) Find a basis for the space of real solutions to the homogeneous differential equation:

$$y''(t) + 2y'(t) + 5y(t) = 0.$$

(b) (3 points) Let V the vector space of solutions you found in part (a), and consider the linear transformation

$$S : V \to \mathbb{R}^2$$

defined by

$$S(y) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

Is S an isomorphism? Explain why or why not.

(c) (4 points) Find the general solution to the inhomogeneous equation:

$$y''(t) + 2y'(t) + 5y(t) = 4e^{3t} + t.$$

7. Consider the second order homogeneous differential equation:

$$y''(t) + 2y'(t) - 8y(t) = 0.$$

(a) (4 points) Reduce the above equation to a system of first order differential equations, i.e., find a 2×2 matrix A such that a vector valued solution $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ of

$$\mathbf{y}'(t) = A\mathbf{y}(t)$$

contains a solution to the given second order equation in its first coordinate.

(b) (4 points) Using your matrix A from part (a), find a fundamental matrix for the system:

$$\mathbf{y}'(t) = A\mathbf{y}(t).$$

(c) (3 points) Without doing any matrix arithmetic, use your answer to (b) to find a fundamental matrix for:

$$\mathbf{y}'(t) = A^3\mathbf{y}(t).$$

Explain your reasoning.

8. (6 points) Find a solution to the heat equation on a rod of length $L = \pi$:

$$\frac{\partial}{\partial t}u(x,t) = 3\frac{\partial^2}{\partial x^2}u(x,t) \qquad u(0,t) = u(\pi,t) = 0,$$

for all $t > 0$, with the initial condition

$$u(x,0) = 2\sin(3x) + 3\sin(2x).$$

9. (7 points) Consider the function $f(x) = |x|$ defined on the interval $[-\pi, \pi]$. Draw a sketch of the function. Find coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_0 \cos(nx) + b_n \sin(nx).$$

Explain your reasoning.

[Scratch Paper 2]

Have a good break!