Practice Final Solutions

(1) Since f is odd, its sine-cosine Fourier coefficients are

given by $\quad a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(nx)\,dx$$

$$= \frac{2}{\pi} \left. \frac{-\cos(nx)}{n} \right|_0^{\pi}$$

$$= \frac{2}{\pi n}\left(1 - \cos(n\pi)\right)$$

$$= \frac{4}{\pi n} \text{ when } n \text{ is odd, zero otherwise.}$$

The corresponding exponential Fourier coefficients are

$$c_n = \frac{-i b_n}{2} \qquad c_{-n} = \frac{i b_n}{2} \qquad c_0 = 0$$

So by Parseval:

$$1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n\in\mathbb{Z}} |c_n|^2 = \sum_{n\geq 1} \left|\frac{-ib_n}{2}\right|^2 + \left|\frac{ib_n}{2}\right|^2$$

$$= \frac{1}{2}\sum_{n\geq 1} |b_n|^2 = \frac{1}{2}\sum_{n \text{ odd}} \left|\frac{4}{\pi n}\right|^2 = \frac{8}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}.$$

Thus, the sum of the series is $\underline{\underline{\frac{\pi^2}{8}}}$.

(2) $$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \cdot e^{-i\alpha x}\,dx$$

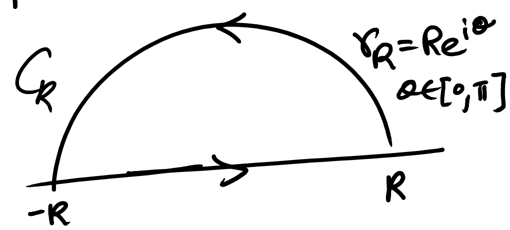
There are two ways to do this.

First Way: Direct integration using the Residue theorem.

$$\text{Let} \quad I = \int_{-\infty}^{\infty} \frac{e^{i(1-\alpha)x}}{1+x^2}\,dx$$

Case $1-\alpha \geq 0$ (i.e. $\alpha \leq 1$)

We use a semicircle contour in the upper halfplane:



$$\text{Let} \quad g(z) = \frac{e^{i(1-\alpha)z}}{1+z^2}.$$

This function has simple poles at $\pm i$.

By the Residue theorem $\quad 2\pi i \operatorname{Res}(i) = \lim_{R\to\infty} \oint_{C_R} g(z)\,dz$

$$= \lim_{R\to\infty} \int_{-R}^{R} g(z)\,dz + \lim_{R\to\infty} \int_{\gamma_R} g(z)\,dz$$

We estimate the latter integral using a triangle inequality argument:

$$\left| \int_{\gamma_R} \frac{e^{i(1-\alpha)z}}{1+z^2}\,dz \right| \leq \int_0^{\pi} \frac{|e^{i(1-\alpha)Re^{i\theta}}|}{|1+R^2e^{i2\theta}|} |Re^{i\theta}|\,d\theta$$

$$\leq \int_0^{\pi} \frac{1}{R^2-1} \cdot R\,d\theta$$

since $|e^{i(1-\alpha)Re^{i\theta}}| \leq e^{-\operatorname{Im}(i(1-\alpha)Re^{i\theta})} \leq e^0$ for $\theta \in [0,\pi]$

$\longrightarrow 0$ as $R \longrightarrow \infty$.

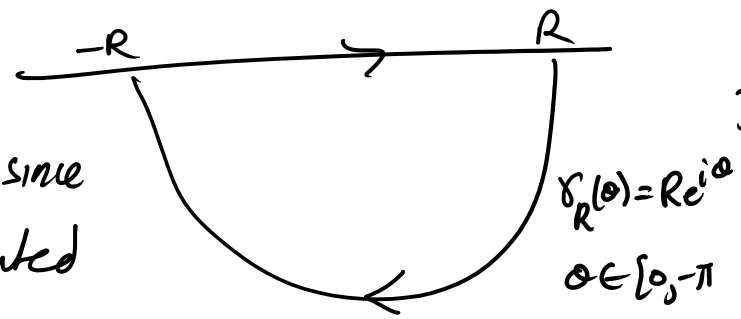
Thus, we have $I = 2\pi i \operatorname{Res}(i) = 2\pi i \lim\limits_{z \to i} \dfrac{e^{i(1-\alpha)z}}{(z-i)(z+i)}(z-i)$

$$= 2\pi i \frac{e^{i^2(1-\alpha)}}{2i} = \pi e^{-(1-\alpha)}$$

So $\hat{f}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \pi e^{-(1-\alpha)}$ when $\alpha \leq 1$.

(Note that we could also have reached the same conclusion by appealing to Jordan's lemma since $g(z) = \dfrac{e^{iaz}}{P(z)}$ $a>0$, $\deg(P) > 1$)

Case $\alpha \geq 1$: we use a semicircle contour in the lower halfplane.



The same argument applies, but since the contour is negatively oriented we get

$$I = -2\pi i \operatorname{Res}(-i) = -2\pi i \lim_{z \to -i} \frac{e^{i(1-\alpha)z}}{(z-i)(z+i)}(z+i)$$

$$= -2\pi i \frac{e^{-i^2(1-a)}}{-2i} = \pi e^{-(\alpha-1)}$$

So $\hat{f}(\alpha) = \dfrac{1}{\sqrt{2\pi}} \pi e^{-(\alpha-1)}$. and in general,

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \pi e^{-|\alpha-1|}$$

Second Way:

Use the properties of the Fourier transform.

Recall that multiplication by an exponential corresponds to translating the fourier transform:

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{1+x^2} e^{-(\alpha-1)} dx$$

$$= \hat{g}(\alpha-1) \quad \text{where} \quad g(x) = \frac{1}{1+x^2}.$$

Now evaluate $\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \pi e^{-|\alpha|}$ using a single contour integral, which immediately gives $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \pi e^{-|\alpha-1|}$.

(3) Notice that $f(x)$ is a (scaling of) a Gaussian.

We will use the facts that

$$\mathcal{F}\left[e^{-\frac{x^2}{2\sigma^2}}\right] = \sigma e^{-\frac{\alpha^2\sigma^2}{2}} \quad \text{——— (1)}$$

and $$\mathcal{F}(f * g) = \sqrt{2\pi}\,\hat{f}(\alpha)\cdot\hat{g}(\alpha) \text{ for all } f, g \quad \text{——— (2)}$$

By (1), $\hat{f}(\alpha) = e^{-\frac{\alpha^2}{2}}$, taking $\sigma = 1$.

Then $$\widehat{f*f}(\alpha) = \sqrt{2\pi}\left(\hat{f}(\alpha)\right)^2 = \sqrt{2\pi}\, e^{-\frac{2\alpha^2}{2}} = \sqrt{2\pi}\, e^{-\frac{(\sqrt{2})^2\alpha^2}{2}}$$

By (2), with $\sigma = \sqrt{2}$: $$(f*f)(x) = \sqrt{2\pi}\,\frac{1}{\sqrt{2}}\, e^{-\frac{x^2}{2(\sqrt{2})^2}} = \sqrt{\pi}\, e^{-\frac{x^2}{4}}.$$

Thus $$(f*f)^2(x) = \left(\sqrt{\pi}\, e^{-\frac{x^2}{4}}\right)^2 = \pi e^{-\frac{x^2}{2}} = \underline{\underline{\pi f(x)}}$$

(4) Taking Laplace transforms of both sides, we have

$$p^2 Y + \omega^2 Y = \mathcal{L}[\delta(t-t_0)] + \mathcal{L}[\delta(t-2t_0)]$$
$$= e^{-pt_0} + e^{-2pt_0}$$

So $$Y(p) = \frac{e^{-pt_0}}{p^2+\omega^2} + \frac{e^{-2pt_0}}{p^2+\omega^2} = \frac{1}{\omega}\frac{\omega}{p^2+\omega^2}\cdot e^{-pt_0} + \frac{1}{\omega}\cdot\frac{\omega}{p^2+\omega^2} e^{-2pt_0}$$

which has inverse transform

$$y(t) = \frac{1}{\omega}\left[\sin(\omega(t-t_0))u(t-t_0) + \sin(\omega(t-2t_0))u(t-2t_0)\right]$$

where $u(t)$ is the step function.

In other words,

$$y(t) = \begin{cases} 0 & t < t_0 \\ \frac{1}{\omega}\sin(\omega(t-t_0)) & t_0 < t < 2t_0 \\ \frac{1}{\omega}\left(\sin(\omega(t-t_0)) + \sin(\omega(t-2t_0))\right) & t > 2t_0. \end{cases}$$

I should have made it clearer what I meant by 'rest'.

→ If you chose to interpret this as $y(t) = 0$ for all t after some time, then observe that for this to happen we must have:

$$\sin(\omega t - \omega t_0) + \sin(\omega t - 2\omega t_0) = 0$$

when $\sin(\omega t - \omega t_0) = -\sin(\omega t - \omega t_0 - \omega t_0)$

which happens exactly when ωt_0 is an odd multiple of π, so if $t_0 = \frac{n\pi}{\omega}$ for odd n

then $y(t) = 0$ for $t > 2t_0$.

→ If you chose to interpret it as velocity, that also makes sense. In this case we want $\frac{d}{dt}\frac{1}{\omega}\left[\sin(\omega(t-t_0)) + \sin(\omega(t-2t_0))\right]$

$= \cos(\omega(t-t_0)) + \cos(\omega(t-2t_0)) = 0$

which also happens when ωt_0 is an odd multiple of π.

(5) $G(x,x')$ is the solution of

$$y''(x) + y(x) = \delta(x-x') \qquad \begin{array}{l} y(0)=0 \\ y'(\pi)=0 \end{array}$$

We find solutions to two homogeneous systems on either side of x', which we will then patch together by matching derivatives at x'.

$\underline{x < x'}$

$$y_<'' + y_< = 0$$

has the general solution

$$y_<(x) = C_1 \cos x + C_2 \sin x$$

Applying the boundary condition

$$y_<(0) = C_1 \cos(0) + C_2 \sin(0) = 0$$

$$\Rightarrow \quad C_1 = 0 .$$

So $y_<(x) = \underline{\underline{C_2 \sin(x)}}$

$\underline{x > x'}$

$$y_>'' + y_> = 0$$

$$y_>(x) = C_3 \cos x + C_4 \sin x .$$

$$y'_>(\pi) = -C_3 \sin\pi + C_4 \cos\pi$$

$$= -C_4 = 0$$

So $y_>(x) = \underline{\underline{C_3 \cos x}}$

We now match derivatives at x' to determine the constants C_2, C_3:

(1) $\underline{y(x) \text{ must be continuous at } x'}$:

$$\lim_{\epsilon \to 0^+} y_<(x'-\epsilon) = \lim_{\epsilon \to 0^+} y_>(x'+\epsilon)$$

So $\quad C_2 \sin(x') = C_3 \cos(x') \quad$ ——— (1)

(2) $\underline{y'(x) \text{ must jump by one at } x'}$: $\quad \lim_{\epsilon \to 0^+} y_<'(x'-\epsilon) + 1 = \lim_{y \to 0^+} y_>'(x'+\epsilon)$

So

$$1 + c_2 \cos(x') = -c_3 \sin(x') \quad \text{———} (2)$$

We now solve the linear equations (1) and (2):

$$(1) \Rightarrow \quad c_3 = c_2 \frac{\sin(x')}{\cos(x')}$$

$$(2) \rightarrow \quad c_2 \cos(x') + c_2 \frac{\sin^2(x')}{\cos(x')} = -1$$

$$\Rightarrow \quad c_2 \left(\cos^2(x') + \sin^2(x')\right) = -\cos(x')$$

$$\Rightarrow \quad c_2 = -\cos(x') \quad \text{and} \quad \underline{\underline{c_3 = -\sin(x')}}$$

So the Green's function is

$$G(x,x') = \begin{cases} -\cos(x')\sin(x) & x < x' \\ -\sin(x')\cos(x) & x \geqslant x' \end{cases}$$

We now use this to find the solution to $y''(x) + y'(x) = f(x)$ with $f(x) = x$:

$$y(x) = \int_0^\pi G(x,x')\, x'\, dx' = -\int_0^x \sin(x')\cos(x)\, x'\, dx' \quad \Big\}\, x' \leq x$$

$$- \int_x^\pi \cos(x')\sin(x)\, x'\, dx' \quad \Big\}\, x' > x$$

$$= -\cos(x)\int_0^x x'\sin x'\,dx' - \sin(x)\int_x^\pi x'\cos(x')\,dx'$$

$$= -\cos(x)\left[-x'\cos x'\Big|_0^x + \int_0^x \cos x'\,dx'\right] - \sin x\left[x'\sin(x')\Big|_x^\pi - \int_x^\pi \sin(x')\,dx'\right]$$

$$= -\cos(x)\left[-x\cos x + \cancel{\sin x}^{\,0}\right] - \sin x\left[0 - x\sin x + \cos\pi \cancel{- \cos x}^{\,0}\right]$$

$$= x\cos^2 x + x\sin^2 x - \sin x\cos\pi = \underline{\underline{x+\sin x}}.$$

Check: $(x+\sin x)'' + x + \sin x = -\sin x + x + \sin x$ ✓

$= x$

$(x+\sin x)'(\pi) = \cos(\pi) + 1 = 0$ ✓

$(x+\sin x)(0) = 0$ ✓

(6) $z^3\exp(\frac{1}{z^2})$ has a singularity at zero.

The Laurent series about zero is:

$$z^3\left(1+\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z^4}\cdots\right) = z^3 + z + \frac{1}{2}\frac{1}{z} + \cdots$$

which is an essential singularity in the interior of $|z|=3$.

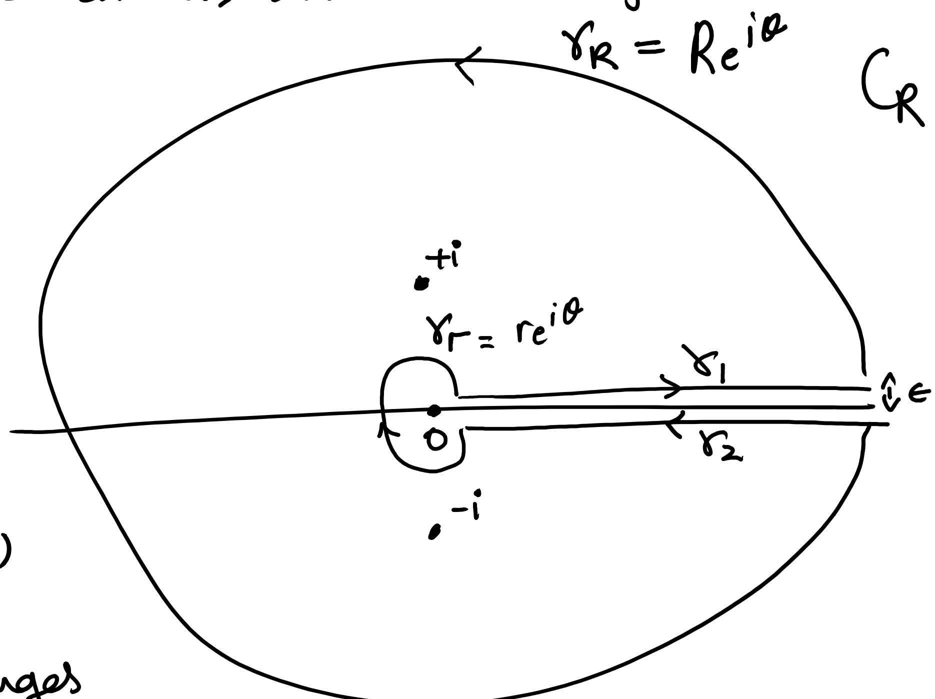
So, by the Residue theorem:
$$\oint_{|z|=3} z^3\exp\left(\frac{1}{z^2}\right)dz = 2\pi i\ \mathrm{Res}(0)$$
$$= 2\pi i\cdot\frac{1}{2} = \underline{\underline{i\pi}}.$$

(7) We will relate this integral to a contour integral of the function

$$g(z) = \frac{e^{\frac{1}{2}\mathrm{Log}(z)}}{(1+z)^2}$$

where $\mathrm{Log}(z) = \mathrm{Log}|z| + i\,\mathrm{Arg}(z)$, $\mathrm{Arg}(z) \in [0, 2\pi)$ is the principal branch of the logarithm. Thus, $g(z)$ is analytic everywhere except the nonnegative real axis.

We set up a keyhole contour as shown in the diagram:



Where the distance between the horizontal segments γ_1 and γ_2 and $\mathbb{R}$ is ϵ, the radius of the large circle is R, the radius of the small circle is r, and the ranges of θ for γ_R and γ_r are chosen to make them meet γ_1 and γ_2 as shown in the diagram.

For every fixed R, r, ϵ such that the contour contains -1, the sole pole of $g(z)$:

$$\oint_{C_R} g(z)\,dz = 2\pi i \left[\mathrm{Res}(-1)\right]$$

$$= 2\pi i \lim_{z \to -1} \frac{d}{dz} (z+1)^2 \frac{e^{\frac{1}{2}\mathrm{Log}(z)}}{(z+1)^2}$$

$$= 2\pi i \lim_{z \to -1} e^{\frac{1}{2}\mathrm{Log} z} \cdot \frac{1}{2z} = \pi i \frac{e^{\frac{1}{2}\mathrm{Log}(-1)}}{-1}$$

$$= - \pi i \, e^{\frac{1}{2}(i\pi)}$$

$$= - \pi i \, e^{\frac{i\pi}{2}} = - \pi i^2 = \underline{\underline{\pi}}$$

On the other hand, as $r \to 0$ we have

the angle for which γ_r meets γ_1 and γ_2.

$$\left| \int_{\gamma_r} g(z)\, dz \right| = \left| \int_{0+\theta_r(\epsilon)}^{2\pi+\theta_r(\epsilon)} \frac{e^{\frac{1}{2}\mathrm{Log}(re^{i\theta})}}{(1+ re^{i\theta})^2} re^{i\theta} \right|$$

$$\leq \int_0^{2\pi} \frac{|\sqrt{r}|\cdot|r|}{(1-|r|)^2}\, d\theta \longrightarrow 0 \quad \text{as } r \to 0$$

and

$$\left| \int_{\gamma_R} g(z)\, dz \right| \leq \int_{0+\theta_R(\epsilon)}^{2\pi-\theta_R(\epsilon)} \frac{|\sqrt{R}\cdot R|}{|R-1|^2}\, d\theta$$

$$\longrightarrow 0 \quad \text{as } R \to \infty$$

Thus, we must have $\displaystyle \lim_{\substack{R\to\infty \\ r\to 0 \\ \epsilon \to 0}} \oint_{C_R} g(z)\,dz = \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ r \to 0}} \int_{\gamma_1} g(z)\,dz + \lim_{\substack{R\to\infty \\ \epsilon\to 0 \\ r \to 0}} \int_{\gamma_2} g(z)\,dz$.

$r \to 0$ C_R
$\epsilon \to 0$

$\epsilon \to 0$ γ_1
$r \to 0$

$\epsilon \to 0$ γ_2
$r \to 0$

We have $\displaystyle\int_{\gamma_1} g(z)\,dz$

$$= \int_{r+\epsilon i}^{R+\epsilon i} \frac{e^{\frac{1}{2}\mathrm{Log}(z)}}{1+z^2}\,dz$$

$$= \int_{r}^{R} \frac{e^{\frac{1}{2}\mathrm{Log}(x+i\epsilon)}}{(1+(x+i\epsilon))^2}\,dx$$

$$= \int_{r}^{R} \frac{\sqrt{x}\; e^{\mathrm{Log}(i\epsilon)}}{(1+(x+i\epsilon))^2}\,dx$$

$$\longrightarrow \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx \quad \text{as } \begin{array}{l} R\to\infty \\ r\to 0 \\ \epsilon\to 0 \end{array}$$

since $e^{\mathrm{Log}(i\epsilon)} \longrightarrow \underline{1}$.

On the other hand, by a similar argument

$$\int_{\gamma_2} g(z)\,dz = -\int_r^R \frac{\sqrt{x}\; e^{\frac{1}{2}\mathrm{Log}(-i\epsilon)}}{(1+(x-i\epsilon))^2}\,dx$$

because of reverse orientation

$$\xrightarrow[\substack{R\to\infty \\ r\to 0}]{\text{as}} \left(\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx\right)\left(-e^{\frac{1}{2}\mathrm{Log}(2\pi)}\right)$$

$$= \left(\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx\right)\left(-e^{i\pi}\right) = \int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$$

We conclude that $\pi = 2 \int_0^\infty \frac{\sqrt{x}}{(1+x)^2} dx$

So $\int_0^\infty \frac{\sqrt{x}}{(1+x)^2} dx = \frac{\pi}{2}$

(8) We apply the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)\, n\, x^{n-1}}{n(n-1)\, x^{n-2}} \right| = \lim_{n\to\infty} |x| \left| \frac{n+1}{n-1} \right|$$

$$= \lim_{n\to\infty} |x| \left| 1 + \frac{2}{n-1} \right| = |x| < 1 \text{ for } |x| < 1.$$

Thus, the radius of convergence is $R = 1$.

We observe that this series is is the second derivative, term by term, of the geometric series:

$$\sum_{n=0}^{\infty} n(n-1)\, x^{n-2} = \sum_{n=0}^{\infty} \frac{d^2}{dx^2} x^n$$

$$= \frac{d^2}{dx^2} \sum_{n=0}^{\infty} x^n$$ Since differentiating does not change the radius of convergence

$$= \frac{d^2}{dx^2} \frac{1}{1-x}$$ recognizing the geometric series

$$= \frac{d}{dx} \frac{1}{(1-x)^2} = \underline{\underline{\frac{2}{(1-x)^3}}}.$$

(9) Let A be the matrix in question.

If A was diagonalizable, we would have

$$A = BDB^{-1} \text{ for some diagonal } D \neq 0 \text{ and invertible } B.$$

Moreover, this would imply that for every integer n,

$$A^n = (BDB^{-1})^n = BD^nB^{-1} \neq 0.$$

However, in this case, $A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= 0,$$

So A cannot be diagonalizable.

(10) Taking total differentials, we obtain

$$dz = \frac{\partial}{\partial x}(xe^{-y})\,dx + \frac{\partial}{\partial y}(xe^{-y})\,dy$$

$$= e^{-y}dx - xe^{-y}dy.$$

and $dx = \sinh(t)\,dt, \quad dy = -\sin(s)\,ds.$

Substituting into dz:

$$dz = e^{-y}\sinh(t)\,dt + xe^{-y}\sin(s)\,ds$$

Since the coefficients in total differentials are partial derivatives:

$$\frac{\partial z}{\partial t} = e^{-y}\sinh(t) = \underline{\underline{e^{-\cos(s)}\sinh(t)}}$$

and $$\frac{\partial z}{\partial s} = xe^{-y}\sin(s) = \underline{\underline{\cosh(t)e^{-\cos(s)}\sin(s)}}$$

You could also just apply the chain rule:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial z}{\partial y}\cdot\frac{dy}{ds}$$

which gives the same thing.