Sample

Midterm 2 Solutions

(1) We have $(-2)^p = e^{p \log(-2)}$ for $p = -1, 1/4, i$.

Writing $-2 = 2e^{i\pi} = 2e^{i(\pi + 2\pi k)}$ $k \in \mathbb{Z}$

in polar form, we have

$$\log(-2) = \mathrm{Log}(2) + i(\pi + 2\pi k), \ k \in \mathbb{Z}.$$

$\underline{p = -1}$ $(-2)^{-1} = e^{-(\mathrm{Log}(2) + i(\pi + 2\pi k))} = e^{-\mathrm{Log}(2) - i\pi - i2\pi k}$

$$= e^{-\mathrm{Log}(2)} e^{-i\pi} e^{-i2\pi k}$$

$$= -\frac{1}{2} \text{ since } e^{-i2\pi k} = 1.$$ One value

(You can also just do this by using the defn $(-2)^{-1} = \frac{1}{-2}$ since -1 is an integer)

$\underline{p = 1/4}$ $(-2)^{1/4} = e^{\frac{1}{4}\mathrm{Log}(2) + i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)}$

$$= 2^{1/4} e^{i\pi/4 + \frac{\pi k}{2}} \quad k \in \mathbb{Z}$$

$$= 2^{1/4} e^{i\pi/4}, \ 2^{1/4} e^{i\frac{3\pi}{4}}, \ 2^{1/4} e^{i5\pi/4}, \ 2^{1/4} e^{i7\pi/4}.$$

only 4 values since $\frac{\pi k}{2} = 2\pi$ for $k = 4$.

(You can also do this without logs, since $p = 1/4$ is rational and in particular amounts to calculating 4th roots)

$\underline{p = i}$

$$(-2)^i = e^{i(\text{Log}2 + i(\pi + 2\pi k))}$$

$$= e^{i\text{Log}2} \cdot e^{i^2(\pi + 2\pi k)} \qquad k \in \mathbb{Z}$$

$$= e^{-\pi - 2\pi k} e^{i\text{Log}2} \qquad k \in \mathbb{Z}$$

infinitely many distinct values

since $\left| e^{-\pi-2\pi k} e^{i\text{Log}2} \right| = e^{-\pi - 2\pi k}$

are distinct for distinct k.

(2) $$f(z) = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0$$

Letting $z = x + iy$ we have

$$e^{i(x+iy)} + e^{-i(x+iy)}$$

$$= e^{ix} e^{-y} + e^{-ix} e^{y} = 0$$

So $$e^{ix} e^{-y} = -e^{-ix} e^{y} = e^{-ix + i\pi} e^{y}$$

Equating arguments and magnitudes:

$$x = -x + \pi + 2\pi k \Rightarrow 2x = \pi + 2\pi k$$

$$\Rightarrow x = \frac{\pi}{2} + \pi k, \quad k \in \mathbb{Z}$$

$$-y = y \Rightarrow \underline{\underline{y = 0}}$$

So the zeros are $\left\{ \pi/2 + \pi k \right\}_{k \in \mathbb{Z}}$

③

(a) Parameterize: $\gamma(t) = 2 + e^{it} \quad t \in [0, 2\pi]$.

$$\oint_{|z-2|=1} \bar{z}\, dz = \int_0^{2\pi} \overline{(2+e^{it})}\, ie^{it}\, dt$$

$$= \int_0^{2\pi} (2+e^{-it})\, ie^{it}\, dt = i\int_0^{2\pi} 2e^{it}\, dt + i\int_0^{2\pi} 1\, dt$$

$$= i\frac{2e^{it}}{i}\Big|_0^{2\pi} + it\Big|_0^{2\pi}$$

$$= \underline{\underline{2\pi i}}.$$

(b) Letting $f(z) = e^{z^2}$, analytic everywhere Cauchy's Integral Formula tells us that

$$\oint_{|z|=1} \frac{f(z)}{(2z-1)^2}\, dz = \frac{1}{4} \oint_{|z|=1} \frac{f(z)}{(z-1/2)^2}\, dz$$

$$= \frac{1}{4}\, 2\pi i\, f'(1/2) = \frac{2\pi i}{4} \cdot e^{z^2} \cdot 2z \Big|_{z=1/2}$$

$$= \underline{\underline{\frac{\pi i}{2} e^{1/4}}}$$

(4) $f(z) = \dfrac{e^{\pi/z}}{(z-\pi)^2}$ has :

- a pole of order 2 at $z=\pi$

since $f(z) = \dfrac{g(z)}{(z-\pi)^2}$ for $g(z) = e^{\pi/z}$

and $g(z)$ is analytic at $z=\pi$

(since $\frac{\pi}{z}$ is analytic at π

$e^{\pi/z}$ is too)

and $g(\pi) = e^{\pi/\pi} \neq 0$.

The Residue may be calculated as:

$$\text{Res}(\pi) = \lim_{z\to\pi} \frac{d}{dz} (z-\pi)^2 \frac{e^{\pi/z}}{(z-\pi)^2}$$

$$= \lim_{z\to\pi} e^{\pi/z} \cdot \left(-\frac{\pi}{z^2}\right)$$

$$= e\left(-\frac{\pi}{\pi^2}\right) = \underline{\underline{\frac{-e}{\pi}}}$$

- An essential singularity at $z=0$, since

$$\frac{e^{\pi/z}}{(z-\pi)^2} = \frac{1}{\pi^2\left(1-\frac{z}{\pi}\right)^2} e^{\pi/z}$$

$$= \frac{1}{\pi^2}\left(1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} \cdots\right)^2 \left(1 + \frac{\pi}{z} + \frac{1}{2!}\frac{\pi^2}{z^2} + \frac{1}{3!}\frac{\pi^3}{z^3} \cdots\right)$$

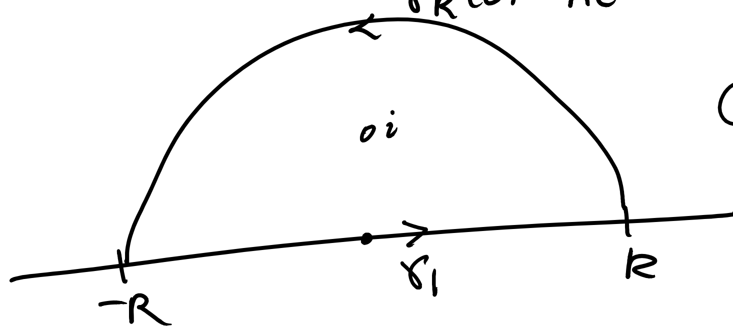
has infinitely many $\frac{1}{z^n}$ terms.

(5) $$I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(1+x^2)^2}\,dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i2z}}{(1+z^2)^2}\,dz.$$

$$= \operatorname{Re} \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{i2z}}{(1+z^2)^2}\,dz.$$

Since $1+z^2$ has no real zeros, we close this with a simple semicircle contour:

$\gamma_R(t) = Re^{it} \quad t\in[0,\pi]$



$C_R = \gamma_R + \gamma_1$

$$f(z) = \frac{e^{i2z}}{(z+i)^2(z-i)^2}$$ has poles of order 2 at $z = \pm i$.

for this contour, we have $\displaystyle\oint_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}(i)$

and $$\operatorname{Res}(i) = \lim_{z\to i} \frac{d}{dz}\, \cancel{(z-i)^2}\, \frac{e^{i2z}}{(z+i)^2\cancel{(z-i)^2}}$$

$$= \lim_{z\to i} \frac{2ie^{i2z}}{(z+i)^2} + \frac{e^{i2z}(-2)}{(z+i)^3}$$

$$= \frac{2ie^{-2}}{(2i)^2} - \frac{2e^{-2}}{(2i)^3} = \frac{2ie^{-2}}{-4} - \frac{2e^{-2}}{-8i}$$

$$= 2e^{-2}\left(\frac{-i}{4} - \frac{i}{8}\right) = 2e^{-2}\left(-\frac{3}{8}i\right).$$

So $$\oint_{C_R} f(z)\,dz = \cancel{2}\pi i \cdot \cancel{2}e^{-2}\left(-\frac{3}{\cancel{8}_2}i\right)$$

$$= \frac{3\pi}{2e^2}$$

We have $$\lim_{R\to\infty} \oint_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$$

$$+ \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz.$$

The latter integral is

$$\int_{\gamma_R} \frac{e^{i2z}}{(1+z^2)^2}\,dz \le \max_{z\in\gamma_R} \left|\frac{e^{i2z}}{(1+z^2)^2}\right| \cdot \pi R$$

for $z = x+iy$ on γ_R, we have $\left|e^{2i(x+iy)}\right| = \left|e^{2ix}\cdot e^{-2y}\right|$

≤ 1 since $y \ge 0$

and $$\left|\frac{1}{(1+z^2)^2}\right| = \frac{1}{|z^4+2z^2+1|}$$

$$\le \frac{1}{|z|^4 - 2|z|^2 - 1} = \frac{1}{R^4-2R^2-1}$$

$$= O\left(\frac{1}{R^4}\right).$$

Thus, $$\int_{\gamma_R} f(z)\,dz = O\left(\frac{1}{R^4}\right)\cdot \pi R = O\left(\frac{1}{R^3}\right) \to 0$$

as $R \to \infty$

and we have $I = \frac{3\pi}{2e^2}$ //.

(On the exam, it is fine to justify
$$\int_{\gamma_R} \frac{p(z)}{q(z)} e^{iaz}\,dz \to 0 \quad \text{as } R \to \infty$$
for polynomials p, q simply by citing Jordan's lemma.

But be sure to check that $\deg(q) \geq \deg(p) + 1$, and to be careful about the sign of a.)