Midterm 2 Solutions

(1) False. Take $z = -2i$, then $e^{iz} = e^{-i^2 2} = e^2$,

and $|e^2| > 1$.

In fact, e^{iz} can take every complex value other than zero, because iz can be any complex number.

(2) (a) Parameterize $|z-1| = 1$ as $z = 1 + e^{it}$ $t \in [0, 2\pi]$

So $dz = ie^{it}$

$$\oint_{|z|=1} \bar{z}^2 \, dz = \int_0^{2\pi} (\overline{1+e^{it}})^2 \, ie^{it} dt$$

$$= i \int_0^{2\pi} (1+e^{-it})^2 e^{it} dt = i \int_0^{2\pi} e^{it} + 2 + e^{-it} \, dt$$

$$= i \left[\frac{e^{it}}{i} \Big|_0^{2\pi} + 4\pi + \frac{e^{-it}}{-i} \Big|_0^{2\pi} \right]$$

$$= \underline{\underline{4\pi i}}$$

(b) $f(z) = z^4 e^{2/z^2}$ is analytic except for an isolated singularity at $z=0$. The Laurent series about zero is

$$f(z) = z^4\left(1 + \frac{2}{z^2} + \frac{1}{2!}\frac{4}{z^4} + \frac{1}{3!}\frac{8}{z^6}\cdots\right)$$

$$= z^4 + 2z^2 + \frac{1}{2!}4 + \frac{1}{3!}\frac{8}{z^2}\cdots$$

So $\text{Res}(0) = 0$ and $\oint_{|z|=1} f(z)\,dz = 2\pi i \,\text{Res}(0) = \underline{\underline{0}}$.

(3) $\sin(iz)$ is analytic everywhere, so f has singularities only at the zeros of $(z^3 - i - 1)^2$, which are the points where $z^3 = i+1 = \sqrt{2}\,e^{i\pi/4}$.

This has three solutions (the cube roots of $\sqrt{2}\,e^{i\pi/4}$), which are $z_1 = \sqrt[6]{2}\,e^{i\pi/12}$, $z_2 = \sqrt[6]{2}\,e^{i(\pi/12 + \frac{2\pi}{3})}$, $z_3 = \sqrt[6]{2}\,e^{i(\pi/12 + 4\pi/3)}$.

Thus we can factorize $f(z) = \dfrac{\sin(iz)}{(z-z_1)^2(z-z_2)^2(z-z_3)^2}$.

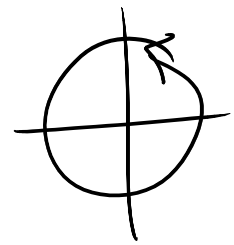
Since $\sin(iz) \neq 0$ at z_1, z_2, z_3, these are $\underline{\underline{\text{poles of order } 2}}$.

(4) Substitute $z = e^{i\theta}$ so $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$

$$dz = ie^{i\theta}\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz}\, dz.$$

The integral is now: $\displaystyle\oint_{|z|=1} \frac{dz}{iz\left(2 + \frac{1}{2i}\left(\frac{z^2-1}{z}\right)\right)}$



$$= \oint \frac{dz}{iz\left(\frac{4iz + z^2 - 1}{2iz}\right)} = 2\oint \frac{dz}{z^2 + 4iz - 1}$$

$$= 2\oint \frac{dz}{(z-(-2-\sqrt{3})i)(z-(-2+\sqrt{3})i)}$$

roots: $z = \dfrac{-4i \pm \sqrt{-16+4}}{2}$

$= \dfrac{-4i \pm 2\sqrt{3}i}{2}$

$= -2i \pm \sqrt{3}i$

$= 2 \cdot 2\pi i \operatorname{Res}\left((-2+\sqrt{3})i\right)$ since this is the only pole inside the contour

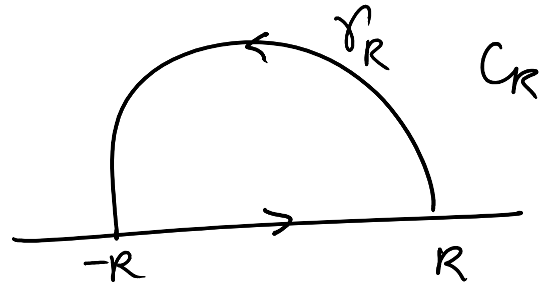
$$= 4\pi i \lim_{z \to (-2+\sqrt{3})i} \frac{1}{z - (-2-\sqrt{3})i} \quad \text{(simple pole)}$$

$$= \frac{4\pi i}{2\sqrt{3}i} = \underline{\underline{\frac{2\pi}{\sqrt{3}}}}$$

⑤ Since $\dfrac{x\sin(x/2)}{1+x^2}$ is even, we have

$$2I = \int_{-\infty}^{\infty} \frac{x\sin(x/2)}{1+x^2}\,dx = \operatorname{Im}\int_{-\infty}^{\infty} \underbrace{\frac{z\,e^{iz/2}}{1+z^2}}_{f(z)}\,dz$$

Consider the semicircle contour:



$\gamma_R(t) = Re^{it}$

$$\oint_{C_R} f(z)\,dz = \int_{\gamma_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

Since $f(z) = e^{iz/2}\dfrac{P(z)}{Q(z)}$ with $\tfrac{1}{2} > 0$ and $\deg(Q) > \deg(P)$,

Jordan's lemma says that $\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = 0$ so we get

$$\lim_{R\to\infty}\oint_{C_R} f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz.$$

$f(z)$ has simple poles at $z = \pm i$, so by the residue thm

$$\oint_{C_R} f(z)\,dz = 2\pi i\,\mathrm{Res}(i) = 2\pi i \lim_{z\to i} \frac{(z-i)\,z\,e^{iz/2}}{(z-i)(z+i)}$$

$$= \cancel{2}\pi i \cdot \frac{i e^{-1/2}}{\cancel{2i}} = \frac{i\pi}{\sqrt{e}}, \text{ so } I = \frac{\pi}{2\sqrt{e}}.$$