Midterm 1 Solutions

① (a) Apply the comparison test:

$$\frac{(n!)^2}{n^2+(2n)!} \leq \frac{(n!)^2}{(2n)!} \quad \forall n.$$

For $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$, use the ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} \right| = \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{4n^2+4n+2} \right| = \frac{1}{4} < 1 \quad \text{so Convergent.}$$

(b) $\lim_{n\to\infty} \left| (-1)^n \cos(1/n) \right| = \cos(0) = 1 \neq 0$

So by the preliminary test the series diverges.

(c) We have $\log n \leq \sqrt{n} \quad n \geq 2$, so

$$\frac{\log n}{n^2} \leq \frac{1}{n^{3/2}} \text{ for } n \geq 2.$$

By the integral test $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges,

So by the comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^2} = \frac{\log(1)}{1} + \sum_{n=2}^{\infty} \frac{\log n}{n^2}$

converges as well.

(2) The binomial series is:

$$(1+x)^{1/5} = 1 + \frac{1}{5}x + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{2!}x^2 + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{3!}x^3 \ldots$$

for $x \in [0, 1/2]$, this is an alternating series from the third term onward, since the numerator is a product of negative numbers. Moreover, the ratio between consecutive terms is:

$$\left| \frac{\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\cdots\left(\frac{1}{5}-n\right)}{(n+1)!}x^{n+1}}{\frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\cdots\left(\frac{1}{5}-(n-1)\right)}{n!}x^n} \right| = \left| \frac{\left(\frac{1}{5}-n\right)x}{n+1} \right| < 1$$

for $x \in [0, 1/2]$,

So the terms are decreasing and tend to zero.

Thus, the series converges, and its value is at most the first term, which is $\left| \frac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2} x^2 \right| = \frac{4}{50}x^2$.

This is at most $\frac{4}{50}\left(\frac{1}{2}\right)^2 = \frac{1}{50} = \underline{\underline{0.02}}$ on $[0, 1/2]$.

③ We write

$$\begin{bmatrix} B(n) \\ S(n) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} S(n-1) \\ S(n-1) \end{bmatrix}.$$

Call this matrix A. Then, after n years, the populations are given by $A^n \begin{bmatrix} B(0) \\ S(0) \end{bmatrix} = A^n \begin{bmatrix} 100 \\ 50 \end{bmatrix}$.

We diagonalize A:

$$\begin{vmatrix} \lambda-2 & 1 \\ -1/2 & \lambda-1/2 \end{vmatrix} = (\lambda-2)(\lambda-1/2)+1/2$$
$$= \lambda^2 - \tfrac{5}{2}\lambda + 3/2 = 0$$

$$\text{So } \lambda = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{12}{2}}}{2}$$
$$= \frac{\frac{5}{2} \pm \frac{1}{2}}{2} = \frac{3}{2} \text{ or } 1$$

The corresponding eigenvectors are:

$$\begin{bmatrix} -1/2 & 1 \\ -1/2 & 1 \end{bmatrix} \bar{v}_1 = 0 \quad \text{and} \quad \begin{bmatrix} -1 & 1 \\ -1/2 & 1/2 \end{bmatrix} \bar{v}_2 = 0$$

$$\Rightarrow \bar{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \Rightarrow \bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Letting $C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3/2 & \\ & 1 \end{bmatrix}$ we have $A = CDC^{-1}$, so

$$\begin{bmatrix} B(n) \\ S(n) \end{bmatrix} = A^n \begin{bmatrix} 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (3/2)^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} 50 \\ 0 \end{bmatrix}, \text{ since } \begin{bmatrix} 100 \\ 50 \end{bmatrix} = 50\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (3/2)^n 50 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (3/2)^n 100 \\ (3/2)^n 50 \end{bmatrix}.$$

Thus, the ratio of boarders to skies remains the same:

$$\frac{B(n)}{S(n)} = \frac{(3/2)^n 100}{(3/2)^n 50} = 2$$

and the total population $B(n) + S(n) = (3/2)^n 150$ tends to infinity.

(It is not surprising that the ratio doesn't change, because the initial vector $\begin{bmatrix} 100 \\ 50 \end{bmatrix}$ was a multiple of the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and therefore itself an eigenvector with eigenvalue $3/2$).

(4) $x^2+y^3 = \sin s + \cos t.$

$xy = s-t$

Taking total differentials:

$$2x\,dx + 3y^2 dy = \cos s \cdot ds + -\sin t \cdot dt$$

$$y\,dx + x\,dy = ds - dt$$

For $\left(\frac{\partial x}{\partial y}\right)_s$: put $ds=0$. Then,

$$2x\,dx + 3y^2 dy = -\sin t\, dt = \sin t\,(y\,dx + x\,dy)$$

$$y\,dx + x\,dy = -dt$$ substitute

So $(2x - y\sin t)\,dx = (x - 3y^2)\,dy.$

$$\Rightarrow \quad dx = \frac{x-3y^2}{2x-y\sin t}\,dy$$

and $$\left(\frac{\partial x}{\partial y}\right)_s = \frac{x-3y^2}{2x-y\sin t}$$

⑤ We want:

Minimize $\quad x^2 + (y-3)^2 + (z-4)^2 = f(x,y,z)$

Subject to $\quad x^2 + y^2 + z^2 = \phi(x,y,z) = 1.$

At the optimum, we have $\nabla f = \lambda \nabla \phi$ so

(1) — $\dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial \phi}{\partial x} \implies 2x = \lambda 2x \implies \underbrace{\lambda = 1}_{\text{impossible by (2)}}$ or $x = 0$ so $\boxed{x=0}$.

(2) — $\dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial \phi}{\partial y} \implies 2(y-2) = 2\lambda y \implies y(1-\lambda) = 3$ $\quad \boxed{y = \frac{3}{1-\lambda}}$

(3) — $\dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial \phi}{\partial y} \implies 2(z-3) = 2\lambda z \implies z(1-\lambda) = 4$ $\quad \boxed{z = \frac{4}{1-\lambda}}$

Substituting these into $\phi = x^2 + y^2 + z^2 = 1$:

$$\frac{3^2}{(1-\lambda)^2} + \frac{4^2}{(1-\lambda)^2} = \frac{25}{(1-\lambda)^2} = 1.$$

So $(1-\lambda)^2 = 5 \implies \lambda = \underline{\underline{-4 \text{ or } 6}}.$

$\underline{\lambda = -4}$

$x = 0,\ y = \frac{3}{5},\ z = \frac{4}{5}$

$f(0, \frac{3}{5}, \frac{4}{5})$

$= (3 - \frac{3}{5})^2 + (4 - \frac{4}{5})^2$

$= \dfrac{12^2 + 16^2}{5^2} = \dfrac{400}{25} = \underline{\underline{16}}$

So this is the closest point

(geometrically, if $p = (0,3,4)$, this is $\dfrac{p}{\|p\|}$)

$\underline{\lambda = 6}$

$x = 0,\ y = -\frac{3}{5},\ z = -\frac{4}{5}.$

$f(0, -3/5, -4/5)$

$= (3 + 3/5)^2 + (4 + 4/5)^2$

$= \dfrac{18^2 + 24^2}{5^2} = \underline{\underline{36}}$

This is the farthest point

(Geometrically, this is $\dfrac{-p}{\|p\|}$)