Sample Midterm 1 Solutions

① (a) Since $|\sin n| \leq 1$, we have $|a_n| = \left| \dfrac{2+\sin n}{n^2+3} \right| \leq \dfrac{3}{n^2}$.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

So by the comparison test, $\displaystyle\sum_{n=1}^{\infty} a_n$ must also converge.

(b) We apply the ratio test:

$$\rho = \lim_{n \to \infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n \to \infty} \left| \frac{2}{n+1} \right| = 0 < 1,$$

So the series converges.

② The Maclaurin Series for the numerator is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

To find the Maclaurin series for $\dfrac{1}{1+x+x^2}$, we make the substitution $y = x + x^2$. Noting that $|x+x^2| < 1$ in a neighborhood of $x=0$, we have

$$\frac{1}{1+x+x^2} = \frac{1}{1+y} = 1 - y + y^2 - y^3 \cdots$$

$$= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 \cdots$$

$$= 1 - (x+x^2) + (x^2+2x^3+x^4) - (x^3+3x^4+3x^5+x^6) \cdots$$

$$= 1 - x + x^3 \cdots\cdots$$

Multiplying these together, we have

$$\frac{\log(1+x)}{1+x+x^2} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\left(1 - x + x^3 \cdots\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - x^2 + \frac{x^3}{2} - \frac{x^4}{3} + x^4 \cdots$$

$$= x - \frac{3}{2}x^2 + \frac{5}{6}x^3 + \cdots$$

So the first four coefficients are $0, 1, -3/2, 5/6$.

(3) The characteristic equation is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -2 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - 4 = 0,$$ so the eigenvalues are

$\lambda_1 = 2$ and $\lambda_2 = -2$.

To find the eigenvectors, we solve the linear equations:

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \bar{V}_1 = 0 \quad\Downarrow\quad \bar{V}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \bar{V}_2 = 0 \quad\Downarrow\quad \bar{V}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These are the corresponding eigenvectors.

Since $\overline{v_1}$ and $\overline{v_2}$ are linearly independent, we may diagonalize A and write

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

Thus, A is diagonalizable (though this is also clear because it is symmetric)

and
$$\begin{aligned} Tr(A^{11}) &= Tr(CD^{11}C^{-1}) \\ &= Tr(C^{-1}CD^{11}) \\ &= Tr(D^{11}) \\ &= 2^{11} + (-2)^{11} \\ &= 2^{11}(1 + (-1)^{11}) = \underline{\underline{0}}. \end{aligned}$$

(4) Taking total differentials of each equation, we have:

$$\frac{\partial}{\partial x}(x^2 s + y^2 t)\,dx + \frac{\partial}{\partial y}(x^2 s + y^2 t)\,dy + \frac{\partial}{\partial s}(x^2 s + y^2 t)\,ds + \frac{\partial}{\partial t}(x^2 s + y^2 t)\,dt = 0$$

and
$$\frac{\partial}{\partial x}(x+y)\,dx + \frac{\partial}{\partial y}(x+y)\,dy = \frac{\partial}{\partial s}(st)\,ds + \frac{\partial}{\partial t}(st)\,ds$$

Computing the partial derivatives, we find that:

$$2xs\,dx + 2yt\,dy + x^2\,ds + y^2\,dt = 0$$

and
$$dx + dy = t\,ds + s\,dt.$$

To find $\left(\frac{\partial x}{\partial s}\right)_t$, we hold t constant, so $dt=0$:

$$2xs\,dx + 2ty\,dy + x^2\,ds = 0$$

$$dx + dy = t\,ds \implies dy = t\,ds - dx.$$

Substituting the second equation in the first, we get

$$2xs\,dx + 2ty\,(t\,ds - dx) + x^2\,ds = 0$$

$$\Downarrow$$

$$(2xs - 2ty)\,dx + (2t^2y + x^2)\,ds = 0$$

$$\Downarrow$$

$$\frac{dx}{ds} = \frac{-(2t^2y + x^2)}{2xs - 2ty} \quad \text{for } t \text{ fixed.}$$

Thus, $$\left(\frac{\partial x}{\partial s}\right)_t = \frac{-(2t^2y + x^2)}{2xs - 2ty}.$$

To find $\left(\frac{\partial x}{\partial t}\right)_s$, we repeat the same thing with s held constant (i.e., $ds = 0$):

$$2xs\,dx + 2yt\,dy + y^2 dt = 0$$

$$dx + dy = s\,dt \implies dy = s\,dt - dx.$$

Substituting the expression for dy:

$$2xs\,dx + 2yt\,(s\,dt - dx) + y^2 dt = 0$$

$$\Downarrow$$

$$\frac{dx}{dt} = \frac{-(2yst + y^2)}{2xs - 2yt} \quad \text{for } s \text{ fixed}$$

So $\left(\frac{\partial x}{\partial t}\right)_s = -\frac{(2yst + y^2)}{2xs - 2yt}$.

(5) Let (x,y,z) be the corner of the box which has all coordinates positive.

Then the volume is $8xyz$, and since the optimal corner lies on the ellipsoid's surface, we are interested in:

$$\max \quad V(x,y,z) = 8xyz$$

$$\text{sub} \quad \varphi(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1.$$

At the optimum, we must have $\nabla V = \lambda \nabla \varphi$ for some λ, so:

$$(1) \text{ — } \frac{\partial V}{\partial x} = \lambda \frac{\partial \varphi}{\partial x} \Rightarrow 8yz = \lambda \left(\frac{2x}{4}\right)$$

$$(2) \text{ — } \frac{\partial V}{\partial y} = \lambda \frac{\partial \varphi}{\partial y} \Rightarrow 8xz = \lambda \left(\frac{2y}{9}\right)$$

$$(3) \text{ — } \frac{\partial V}{\partial z} = \lambda \frac{\partial \varphi}{\partial z} \Rightarrow 8xy = \lambda \left(\frac{2z}{16}\right).$$

Since $x, y, z \neq 0$ we must have $\lambda \neq 0$ by (1).

Thus, dividing (2) by (1) gives:

$$\frac{x}{y} = \frac{\frac{2y}{9}}{\frac{2x}{4}} \Rightarrow \frac{x^2}{y^2} = \frac{4}{9} \Rightarrow y^2 = \frac{9}{4}x^2$$

Similarly, dividing (3) by (1) gives:

$$\frac{x}{z} = \frac{2z/16}{2x/4} \Rightarrow \frac{x^2}{z^2} = \frac{4}{16} \Rightarrow z^2 = 4x^2$$

Thus, we have

$$1 = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = \frac{x^2}{4} + \frac{1}{\cancel{9}}\frac{\cancel{9}}{4}x^2 + \frac{1}{16}4x^2$$

$$= \frac{3}{4}x^2$$

So $x = \sqrt{\frac{4}{3}} = \underline{\underline{\frac{2}{\sqrt{3}}}}$, $y = \sqrt{\frac{9}{4}\cdot\frac{4}{3}} = \underline{\underline{\sqrt{3}}}$

and $z = \sqrt{4\cdot\frac{4}{3}} = \underline{\underline{\frac{4}{\sqrt{3}}}}$

and the volume is

$$8\cdot\frac{2}{\sqrt{3}}\cdot\sqrt{3}\cdot\frac{4}{\sqrt{3}} = \underline{\underline{\frac{16}{\sqrt{3}}}}.$$