Laurent Series and Residue Calculus

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If f is analytic at z_0, then it may be written as a power series:

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

which converges in an open disk around z_0. In fact, this power series is simply the Taylor series of f at z_0, and its coefficients are given by

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}},$$

where the latter equality comes from Cauchy's integral formula, and the integral is over a positively oriented contour containing z_0 contained in the disk where it $f(z)$ is analytic. The existence of this power series is an extremely useful characterization of f near z_0, and from it many other useful properties may be deduced (such as the existence of infinitely many derivatives, vanishing of simple closed contour integrals around z_0 contained in the disk of convergence, and many more).

The situation is not much worse when z_0 is an isolated singularity of f, i.e., $f(z)$ is analytic in a puncured disk $0 < |z - z_0| < r$ for some r. In this case, we have:

Laurent's Theorem. If z_0 is an isolated singularity of f and $f(z)$ is analytic in the annulus $0 < |z - z_0| < r$, then

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_n}{(z - z_0)^n} + \dots, \qquad (*)$$

where the series converges absolutely in the annulus. In class I described how this can be done for any annulus, but the most useful case is a punctured disk around an isolated singularity. The terms with negative powers of $(z - z_0)$ are called the *principal part* of the Laurent series, and the singularity is classified into one of three types depending on how many terms there are in the principal part (see page 680 for a taxonomy).

The main reason we are interested in Laurent series is that given a Laurent series, it is extremely easy to calculate the value of any closed contour integral around z_0 which is contained in the annulus of convergence. In particular, if we integrate both sides of $(*)$ term by term along a small positively oriented circle centered at z_0, $\gamma(t) = z_0 + \rho e^{it}, t \in [0, 2\pi]$, we find that:

$$\oint f(z)dz = \sum_{n=0}^{\infty} a_n \oint (z - z_0)^n dz + \oint \frac{b_1}{z - z_0} dz + \sum_{n=2}^{\infty} \oint \frac{b_n}{(z - z_0)^n} dz$$
$$= \sum_{n=0}^{\infty} a_n \cdot 0 + b_1(2\pi i) + \sum_{n=2}^{\infty} b_n \cdot 0 = 2\pi i b_1.$$

since the $(z - z_0)^n$ terms vanish (they are polynomials and therefore analytic) and by a calculation we did in class,

$$\oint \frac{1}{(z - z_0)^n} dz = 0$$

whenever $n \geq 2$. By an argument similar to the proof of Cauchy's Integral formula, this may be extended to any closed contour around z_0 containing no other singular points.

Thus, the coefficient b_1 in the Laurent series is especially significant; it is called the *residue* of f at z_0, denoted $\mathrm{Res}(f, z_0)$. By a simple argument again like the one in Cauchy's Integral Formula (see page 683), the above calculation may be easily extended to any integral along a closed contour containing isolated singularities:

Residue Theorem. If f is analytic on and inside C except for a finite number of isolated singularities $z_1, \ldots, z_k$, then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j).$$

The upshot is that by virtue of the existence of a Laurent expansion, it is often easy to calculate the right hand side without doing any integration whatsoever, as long as we know all the z_j. For instance, if we actually know the Laurent series, then it is very easy to calculate the residue: we just read off the coefficient of $1/(z - z_0)$:

Example. $f(z) = ze^{1/z}$ has a singularity at $z = 0$, and is analytic everywhere else. We investigate this by computing the Laurent expansion:

$$f(z) = z\left(1 + (1/z) + \frac{(1/z)^2}{2!} + \ldots\right) = z + 1 + \frac{(1/z)}{2!} + \ldots$$

whence $\mathrm{Res}(f, 0) = 1/2$ and we know $\oint_C ze^{1/z}dz = 2\pi i \cdot 1/2$ for any closed contour containing the origin.

Even better is that we can leverage the fact that the Laurent expansion exists to compute the residue when z_0 is a pole *without necessarily even knowing the entire series.* Here is how to do it for a pole of order m at z_0. We know that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \ldots \frac{b_m}{(z - z_0)^m},$$

in some punctured disk $0 < |z - z_0| < r$. Our goal is to isolate the coefficient b_1 by performing some operations on $f(z)$ and thereby on the Laurent expansion (which is hidden from us). This can be done in three steps:

- Multiply by $(z - z_0)^m$ to turn the Laurent series into a power series:

  $$(z - z_0)^m f(z) = \sum_{n=0} a_n (z - z_0)^{n+m} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \ldots b_m.$$

  Note that this only works when we have a pole of finite order, and not when we have an essential singularity.

- Kill off the $b_2, \ldots, b_m$ terms by differentiating $m - 1$ times (this happens because they all have degree strictly less than $m - 1$ — note that the b_1 term survives)

  $$\begin{aligned} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) &= \sum_{n=0}^{\infty} a_n \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^{n+m} + (m-1) \cdot (m-2) \cdot \ldots 1 \cdot b_1 + 0 + \ldots + 0. \\ &= \sum_{n=0}^{\infty} a_n \frac{(n+m)!}{(n+1)!} (z - z_0)^{n+1} + (m-1)! b_1. \end{aligned}$$

- Kill off all the remaining terms by taking a limit as $z \to z_0$ — this happens because they all have degree strictly greater than 0, and again the b_1 term survives because it is a constant:

  $$\begin{aligned} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) &= \sum_{n=0}^{\infty} a_n \frac{(n+m)!}{(n+1)!} \lim_{z \to z_0} (z - z_0)^{n+1} + (m-1)! b_1 \\ &= (m-1)! b_1. \end{aligned}$$

  Thus, we can read off the residue by dividing the expression on the left hand side by $(m - 1)!$.