Tools for Applying Residue Calculus to Definite Integrals

Nikhil Srivastava

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1 The Triangle Inequality Argument

When applying the residue theorem to compute definite integrals, we are often faced with a situation where we have to show that the integral of some function on a semicircular contour $\gamma_R(t) = Re^{it}, t \in [0, \pi]$ in the upper half plane vanishes as $R \to \infty$.

The simplest way to do this, which works for instance when $f(z) = P(z)e^{iaz}/Q(z)$ for polynomials P, Q satisfying $\deg(Q) \geq \deg(P)+2$, is to use the triangle inequality for integrals. This gives the bound:

$$\left| \int_{\gamma_R} \frac{P(z)e^{iaz}}{Q(z)} dz \right| \leq \int_0^\pi \left| \frac{P(\gamma(t))e^{ia\gamma(t)}}{Q(\gamma(t))} \right| |\gamma'(t)| dt \leq \max_{z \in \gamma_R} \left(\left| \frac{P(z)}{Q(z)} \right| |e^{iaz}| \right) \cdot \pi R. \qquad (*)$$

When $a > 0$, we observe that every $z = x + iy$ on γ_R has $y > 0$, which gives the absolute value bound:

$$|e^{iaz}| = |e^{iax} e^{i^2 ay}| = 1 \cdot e^{-ay} \leq 1.$$

Note that this only works for $a > 0$; for $a < 0$ we would have to look at a contour in the *lower* half plane (i.e., $y < 0$).

Since Q has degree at least two larger than Q, applications of the triangle inequality for addition of complex numbers:

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|,$$

can be used to isolate the higher order terms, and conclude that

$$\left| \frac{P(z)}{Q(z)} \right| \leq O(1/R^2),$$

for large R. Plugging these estimates into $(*)$ tells us that the value of the integral is $O(1/R)$, which vanishes as $R \to \infty$.

2 Jordan's Lemma

It turns out that the above argument is wasteful; we only used that $|e^{-ay}| \leq 1$ for positive y, but most of the time, it is actually much smaller! By doing a more delicate calculation, we can make the above argument work in the more general case when $\deg(Q) \geq \deg(P)+1$, rather than 2, and this is quite useful.

The more delicate bound is called Jordan's lemma, and it says that

$$\int_0^\pi e^{-R\sin\theta} d\theta \leq \frac{\pi}{R}.$$

Before proving this, let's see why it is useful. Suppose I have some function $f(z)$, such as $f(z) = P(z)/Q(z)$ for polynomials $\deg(Q) \geq \deg(P) + 1$, whose maximum value on γ_R is M_R and $M_R \to 0$ as $R \to \infty$. Then, the triangle inequality tells us that

$$\begin{aligned} \left| \int_{\gamma_R} f(z) e^{iaz} dz \right| &\leq \int_0^\pi |f(Re^{it})||e^{iR(\cos t + i \sin t)}||iRe^{it}| dt \\ &\leq M_R \cdot R \int_0^\pi e^{-aR\sin t} dt \\ &\leq M_R \cdot R \cdot \frac{\pi}{aR} \to 0, \end{aligned}$$

as $R \to \infty$.

Now for the proof. The key observation is that on the interval $[0, \pi/2]$, the function $\sin\theta$ is at least $\frac{\theta}{\pi/2}$ — this is easy to see graphically, by plotting $y = \sin x$ and noticing that the line $y = \frac{x}{\pi/2}$ connecting the points $(0,0)$ and $(\pi/2, 1)$ lies strictly below the curve[1]. Thus, $e^{-R\sin\theta} \leq e^{-2R\theta/\pi}$ for $\theta \in [0, \pi/2]$, and we have

$$\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{e^{-2R(\pi/2)/\pi}}{-2R/\pi} - \frac{e^0}{-2R/\pi} = \frac{\pi}{2R}(1 - e^{-R}) \leq \frac{\pi}{2R}.$$

Since $\sin(\theta)$ is symmetric about $\pi/2$, we can repeat the same argument on the interval $[\pi/2, \pi]$ to get

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta} d\theta \leq \frac{\pi}{2R},$$

and adding the two gives Jordan's lemma.

All of the above arguments can be adapted to the case $a < 0$, but we must use a contour in the *lower* half plane ($y < 0$) rather than in the upper half plane.

[1] It can also be proven analytically by noticing that $\sin\theta = \frac{\theta}{\pi/2}$ at the endpoints $\theta = 0, \pi/2$, and that $(\sin\theta)'' = -\sin\theta \leq 0$ in the interval $[0, \pi/2]$ so the function is concave in this interval.

3 Cauchy Principal Values

A Cauchy Principal value is obtained by a specific way of defining an improper integral as a limit of proper integrals. Traditionally, we learn that an improper integral of the first kind is the following limit of definite integrals:

$$\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx,$$

so that we may define

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx.$$

Sometimes, neither of the integrals on the right hand side exist (for instance, take $f(x) = \sin(x)$), but there is a sense in which the integral on the left hand side does. For instance, it makes sense to assert that

$$\int_{-\infty}^\infty \sin(x) = 0$$

since $\sin(-x) = -\sin(x)$, i.e., the function is odd, and the contributions from positive and negative x cancel. This can be made formal by taking the limit *symmetrically*, and this is what we mean by the principal value:

$$\text{p.v.}\int_{-\infty}^\infty f(x)dx := \lim_{R\to\infty}\int_{-R}^R f(x)dx,$$

which does indeed give zero in the case of $f(x) = \sin(x)$. Such integrals are often useful, and are moreover easily amenable to contour integration techniques because we can close the contour with a semicircle whose center is fixed at zero, rather than having to consider semicircles whose centers vary as their radii increase.

Similarly, for improper integrals of the second kind, such as

$$\int_{-1}^1 \frac{1}{x}dx,$$

the principal value is defined by taking the limit symmetrically about the point of discontinuity:

$$\text{p.v.}\int_{-1}^1 \frac{1}{x}dx := \lim_{\epsilon\to 0^+}\left(\int_{-1}^{0-\epsilon}\frac{1}{x}dx + \int_{0+\epsilon}^1 \frac{1}{x}dx\right),$$

rather than separately on each side. When closing the contour, this corresponds to semicircles of shrinking radius $\epsilon > 0$ centered at the point of discontinuity.