Homework 9 Partial Solutions

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1. Determine which of the following functions on the real line are periodic. If they are, determine the fundamental period; if not, explain why.

❶ $\sin(\pi x)$

❷ e^x

❸ $e^{ix} + \sin x$

❹ $\sin(3x) + \cos(5x)$

❺ $\sin(3x) + \sin(\sqrt{5}x)$

Solution:

❶ periodic, with fundamental period 2.

❷ not periodic: $e^x = e^{x+p} \iff 1 = e^p \iff p = 0$.

❸ periodic, with fundamental period 2π.

❹ periodic, with fundamental period 2π.

❺ not periodic: no (integral) multiple of $\frac{1}{3}$ is also an integral multiple of $\frac{1}{\sqrt{5}}$, except 0.

2. If $f\colon \mathbb{R} \to \mathbb{C}$ is periodic with period $2L$, what are the periods of

❶ $g(x) = f(cx)$, ❷ $h(x) = f(x-t)$, and ❸ $k(x) = f(-x)$?

Calculate $\hat{h}(n)$ and $\hat{k}(n)$ in terms of $\hat{f}(n)$. Solution:

❶ Assuming $c \neq 0$, the period of $g(x)$ is $2L/c$.

❷ The period of $h(x)$ is $2L$.
Since

$$h(x) = f(x-t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in(x-t)} = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{-int}\, e^{inx},$$

we calculate

$$\hat{h}(n) = e^{-int}\hat{f}(n) \qquad (n \in \mathbb{Z}).$$

❸ The period of $k(x)$ is $2L$.
Since

$$\begin{aligned} k(x) \;=\; f(-x) &= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in(-x)} \\ &= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{-inx} \\ &= \sum_{-n\in\mathbb{Z}} \hat{f}(n)e^{-inx} \\ &= \sum_{-n\in\mathbb{Z}} \hat{f}(-(-n))e^{i(-n)x} \\ &= \sum_{m\in\mathbb{Z}} \hat{f}(-m)e^{imx}, \end{aligned}$$

we calculate

$$\hat{k}(n) = \hat{f}(-n) \qquad (n \in \mathbb{Z}).$$

3. Calculate, for all integers m and n, the integrals:

❶ $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$

❷ $\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx$.

Then calculate

❸ $\int_{-1}^{1} \sin(nx)\sin(mx)\,dx$

❹ $\int_{-1}^{1} e^{inx}e^{-imx}\,dx$.

Solution:

❶ First:

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx &= 2\int_0^{\pi} \sin(nx)\sin(mx)\,dx \\ &= 2\int_0^{\pi} \tfrac{e^{inx}-e^{-inx}}{2i}\tfrac{e^{imx}-e^{-imx}}{2i}\,dx \\ &= \int_0^{\pi} \tfrac{e^{i(n-m)x}+e^{-i(n-m)x}}{2} - \tfrac{e^{i(n+m)x}+e^{-i(n+m)x}}{2}\,dx \\ &= \int_0^{\pi} \left(\cos[(n-m)x] - \cos[(n+m)x]\right)\,dx.\end{aligned}$$

But

$$\int_0^{\pi} \cos[(n-m)x]\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m, \end{cases}$$

and

$$\int_0^{\pi} \cos[(n+m)x]\,dx = \begin{cases} 0 & \text{if } n+m \neq 0 \\ \pi & \text{if } n+m = 0. \end{cases}$$

Therefore

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } n = m = 0 \ \text{ or } \ |n| \neq |m| \\ \pi & \text{if } n = m \neq 0 \\ -\pi & \text{if } n = -m \neq 0. \end{cases}$$

❷

$$\begin{aligned}\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx &= \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx \\ &= \int_{-\pi}^{\pi} \left(\cos\left[(n-m)x\right] + i\sin\left[(n-m)x\right]\right)\,dx \\ &= 2\int_{0}^{\pi} \cos\left[(n-m)x\right]dx \\ &= \begin{cases} 0 & \text{if } n-m \neq 0 \\ 2\pi & \text{if } n-m = 0. \end{cases}\end{aligned}$$

❸ Since $2\sin(nx)\sin(mx) = \cos((n-m)x) - \cos((n+m)x)$, we must have

$$\int_{-1}^{1} \sin(nx)\sin(mx)\,dx = \int_{0}^{1} \cos\left[(n-m)x\right]dx - \int_{0}^{1} \cos\left[(n+m)x\right]dx.$$

But

$$\int_0^1 \cos\left[(n-m)x\right]dx = \begin{cases} \frac{\sin(n-m)}{n-m} & \text{if } n-m \neq 0 \\ 1 & \text{if } n-m = 0 \end{cases}$$

and

$$\int_0^1 \cos\left[(n+m)x\right]dx = \begin{cases} \frac{\sin(n+m)}{n+m} & \text{if } n+m \neq 0 \\ 1 & \text{if } n+m = 0, \end{cases}$$

so

$$\int_{-1}^{1} \sin(nx)\sin(mx)\,dx = \begin{cases} \frac{\sin(n-m)}{n-m} - \frac{\sin(n+m)}{n+m} & \text{if } |n| \neq |m| \\ 0 & \text{if } n = m = 0 \\ -\frac{\sin(2m)}{2m} & \text{if } n = m \neq 0 \\ \frac{\sin(2m)}{2m} & \text{if } n = -m \neq 0. \end{cases}$$

❹ To calculate $\int_{-1}^{1} e^{i(n-m)x}\,dx$, we first note that if $n-m=0$, then the integral is 0.

Otherwise, let $\blacksmiley = n - m$. Then

$$\int_{-1}^{1} e^{i(n-m)x}\,dx \;=\; \int_{-1}^{1} e^{i\blacksmiley x}\,dx = \int_{-1}^{1} (\cos(\blacksmiley x) + i\sin(\blacksmiley x))\,dx$$

$$= \left.\frac{\sin(\blacksmiley x) - i\cos(\blacksmiley x)}{\blacksmiley}\right|_{x=-1}^{1}$$

$$= \frac{2\sin\blacksmiley}{\blacksmiley} \;=\; \frac{2\sin(n-m)}{n-m}.$$

5.

(a) Evaluate the (exponential) Fourier coefficients $\hat{f}(n)$ of the sawtooth function:

$$f(x) = x \qquad (-\pi \le x < \pi).$$

(b) Use Parsevals theorem to conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) Use a similar method applied to the function

$$f(x) = x^2 \qquad (-\pi \le x < \pi)$$

to find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution:

(a) First: $\hat{f}(0) \;=\; \langle 1 \mid x \rangle \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx \;=\; 0.$
For $n \neq 0$,

$$\begin{aligned} \hat{f}(n) \;=\; \left\langle e^{inx} \mid x \right\rangle \;&=\; \tfrac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx}\, dx \\ &= \tfrac{1}{2\pi} \int_{-\pi}^{\pi} x(\cos(nx) - i \sin(nx))\, dx \\ &= \tfrac{1}{2\pi n^2} \left(\begin{array}{c} \cos(nx) + nx \sin(nx) \\ +i\, nx \cos(nx) - i \sin(nx) \end{array} \right) \Bigg|_{x=-\pi}^{\pi} \\ &= \tfrac{\cos(n\pi)}{n} \end{aligned}$$

So $\hat{f}(n) = \frac{(-1)^n}{n}$ when $n \neq 0$.

(b) Consequently, Parseval's theorem implies

$$\tfrac{\pi^2}{3} \;=\; \tfrac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx \;=\; \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \;=\; \sum_{n \in \mathbb{N}} \tfrac{1}{(-n)^2} + 0 + \sum_{n \in \mathbb{N}} \tfrac{1}{n^2} \;=\; 2 \sum_{n \in \mathbb{N}} \tfrac{1}{n^2}.$$

We conclude that $\sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) First: $\hat{f}(0) \;=\; \left\langle e^{0 \cdot ix} \mid x^2 \right\rangle \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx \;=\; \frac{\pi^2}{3}$. Assume $n \neq 0$. Then

Equation (9.3) implies

$$\begin{aligned}\hat{f}(n) \;=\; \tfrac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx \;&= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} x^2(\cos(nx) - i\sin(nx))\,dx \\ &= \tfrac{1}{\pi}\int_{0}^{\pi} x^2\cos(nx)\,dx \\ &= (-1)^n\tfrac{2}{n^2}.\end{aligned}$$

Moreover,

$$\tfrac{\pi^4}{5} \;=\; \tfrac{1}{2\pi}\textstyle\int_{-\pi}^{\pi} x^4\,dx \;=\; \sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2 \;=\; 8\sum_{n\in\mathbb{N}}\tfrac{1}{n^4} + \tfrac{\pi^4}{9}.$$

We conclude that $\sum_{n\in\mathbb{N}}\frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}$.

7. Suppose $f\colon [0, 2\pi] \to \mathbb{C}$ is given by $f(\theta) = F(e^{i\theta})$ where $F(z)$ is analytic on the unit circle $|z| = 1$. Show that the Laurent series for F in an annulus containing the unit circle may be used to compute the (exponential) Fourier series of f. What happens if F is analytic on and inside the unit circle? Solution:

Let A be an annulus containing the unit circle. Let the Laurent series for F in A be $\sum_{n\in\mathbb{Z}} A_n z^n$. Then

$$\begin{aligned}\hat{f}(n) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta \ &= \ \tfrac{1}{2\pi}\int_{-\pi}^{\pi} F\left(e^{i\theta}\right)e^{-in\theta}\,d\theta \\ &= \tfrac{1}{2\pi i}\int_C F(z)z^{n+1}\,dz,\end{aligned}$$

where C is the positively oriented unit circle parametrized by $\gamma(\theta)\colon\ = e^{i\theta}, \theta \in [-\pi, \pi]$. Therefore,

$$\hat{f}(n) = \tfrac{1}{2\pi i}\int_C F(z)z^{n+1}\,dz.$$

If additionally, F is analytic on and inside the unit circle, then Cauchy's theorem implies

$$\hat{f}(n) = \tfrac{1}{2\pi i}\int_C F(z)z^{n+1}\,dz = \begin{cases} 0 & n+1 \geq 0 \\ \frac{F(0)}{n!} & n+1 < 0. \end{cases}$$

8. Suppose that the partial sums of the Fourier series of $f \in L^2[-\pi, \pi]$ are

$$S_N(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

Show that for any coefficients d_n,

$$\|f - S_N\|^2 \le \left\| f - \sum_{n=-N}^{N} d_n e^{inx} \right\|^2, \tag{1}$$

i.e., the partial sums of the Fourier series minimize the mean squared error among all linear combinations of $e^{-iNx}, \ldots, e^{iNx}$. Solution:

Fix $N \in \{0, 1, 2, \ldots\}$ and suppose $\{d_n\}_{n=-N}^{N} \subset \mathbb{C}$. Let us use the following notation:

$$D_N(x) := \sum_{|n| \le N} d_n e^{inx} \qquad \text{and} \qquad R_{N+1}(x) := \sum_{|n| \ge N+1} \hat{f}(n) e^{inx}.$$

With this notation, (1) is the same as

$$\|R_{N+1}\|^2 \;\le\; \|S_N - D_N + R_{N+1}\|^2. \tag{2}$$

Since $\langle S_N - D_N \mid R_{N+1} \rangle = 0$,

$$\|S_N - D_N + R_{N+1}\|^2 \;=\; \|S_N - D_N\|^2 + \|R_{N+1}\|^2.$$

Since $\|S_N - D_N\| \ge 0$, we deduce that (2), and consequently (1), holds.