Homework 7 Solutions

Section 3

(19) e^{3z} is analytic everywhere, and in particular around C,
So Cauchy's Integral formula implies that

$$\frac{1}{2\pi i}\oint_C \frac{e^{3z}}{z-\ln 2}\,dz = e^{3(\ln 2)} = 2^3 = 8$$

Thus the value of the integral is

$$2\pi i \cdot 8 = \underline{\underline{16\pi i}}.$$

(21) Write $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw.$

Differentiating under the integral gives

$$\frac{d}{dz} f(z) = \frac{1}{2\pi i}\oint \frac{d}{dz}\frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\oint \frac{f(w)}{(w-z)^2}\,dw$$

The general formula is obtained by induction:

$$f^{(n)}(z) = \frac{d}{dz} f^{(n-1)}(z) = \frac{d}{dz}\frac{(n-1)!}{2\pi i}\oint \frac{f(w)}{(w-z)^n}\,dw = \frac{(n-1)!}{2\pi i}\oint \frac{d}{dz}\frac{f(w)}{(w-z)^n}\,dw$$

$$= \frac{n!}{2\pi i}\oint \frac{f(w)}{(w-z)^{n+1}}\,dw$$

(22)

$$\oint_C \frac{\sin 2z \, dz}{(6z-\pi)^3} = \frac{1}{6^3} \oint \frac{\sin 2z \, dz}{(z-\pi/6)^3} dz$$

$$= \frac{1}{6^3} \frac{2\pi i}{2!} \frac{d^2}{dz^2} \sin(2z) \Big|_{z=\pi/6} \quad \text{by } \#21$$

$$= \frac{\pi i}{216} \cdot 2 \cdot 2 \cdot (-\sin 2z) \Big|_{z=\pi/6}$$

$$= \frac{-4 \sin \pi/3 \, \pi i}{216} = \frac{-4(\sqrt{3}/2)\pi i}{216} = -\frac{\sqrt{3}\pi i}{108}$$

Problem 2

Parametrize C as $z = \gamma(t)$, $t \in [a,b]$, so $\gamma(a) = z_1$, $\gamma(b) = z_2$.

Then, by the chain rule, $\frac{d}{dt} F(\gamma(t))$

$$= \frac{dF}{dz}(\gamma(t)) \cdot \frac{d\gamma}{dt}(t)$$

$$= f(\gamma(t))\gamma'(t).$$

Thus $$\int_a^b \frac{d}{dt} F(\gamma(t)) \, dt = \int_a^b f(\gamma(t))\gamma'(t) \, dt$$

$$= \int_C f(z) \, dz.$$

By the fundamental thm of calculus (for functions of a real variable),

$$\int_a^b \frac{d}{dt} F(\gamma(t))dt = F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1).$$

So $$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

In the example, $f(z) = e^z + \sin z$ has an antiderivative :

$F(z) = e^z - \cos z$.

Thus, $$\int_\gamma (e^z + \sin z)dz = F(e^{i\pi}) - F(e^{i0})$$

$$= e^{-1} - \cos(-1) - (e^1 - \cos(1))$$

$$= \underline{\underline{e^{-1} - e}} \quad \text{since } \cos(-1) = \cos(1).$$

Problem 3

- $f(z) = e^z + \sin z$ is analytic on and inside $|z-2|=3$ (since it is analytic everywhere) so

$$\oint_{|z-2|=3} \frac{f(z)}{z-0}\,dz = 2\pi i f(0) = 2\pi i\,(e^0) = \underline{\underline{2\pi i}}$$

- $$\oint_{C'} \frac{z^2 e^z}{2z+i}\,dz = \frac{1}{2} \oint_{C'} \frac{z^2 e^z}{\left(z+\frac{i}{2}\right)}\,dz$$

because C' is negatively oriented

$$= -\frac{2\pi i}{2}\left(\left(-\tfrac{i}{2}\right)^2 e^{-i/2}\right)$$

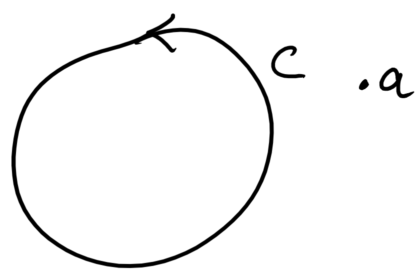
since $z^2 e^z$ is analytic on/inside C' and $-i/2 \in \text{interior}(C')$

$$= -i\pi\left(-\frac{1}{4}\left(e^{-i/2}\right)\right)$$

$$= \frac{\pi}{4} e^{-\frac{i}{2}+\pi/2} \quad \text{in polar form}$$

$$= \frac{i\pi}{4}\left(\cos(-1/2) + i\sin(-1/2)\right) \quad \text{in cartesian form.}$$

Problem 4



$\frac{1}{z-a}$ is analytic everywhere except $z=a$, in particular on and inside C, and f is analytic on and inside C, so their product

$\frac{f(z)}{z-a}$ is also analytic on/inside C.

Thus $$\oint \frac{f(z)}{z-a} = 0$$ by Cauchy-Goursat.

Problem 5

By linearity, we have $\oint f(z)dz = \oint g(z)dz + a_1\oint \frac{1}{z}dz$

$+ \cdots a_n \oint \frac{1}{z^n} dz.$

The first term is zero by Cauchy-Goursat.

For the other terms, we parameterize $z = e^{it}$, $t \in [0, 2\pi]$

and write $\oint_{|z|=1} \frac{1}{z^n} dz = \int_0^{2\pi} \frac{1}{e^{int}} ie^{it} dt$

$$= i\int_0^{2\pi} e^{i(1-n)t}\, dt = i\int_0^{2\pi} \cos((n-1)t) + i\sin((n-1)t)\, dt$$

$$= \begin{cases} i\int_0^{2\pi} \cos(0) + i\sin(0)\, dt = 2\pi i & \text{when } n=1 \\ i\left(\frac{\sin((n-1)t)}{n-1} - i\frac{\cos((n-1)t)}{n-1} \right)\Big|_0^{2\pi} = 0 & \text{when } n \neq 1 \end{cases}$$

Thus, the only term that survives is the $n=1$ term, and

$$\oint_C f(z)dz = 2\pi i a_1$$

Problem 6 :

Parametrize $C = \{|z-a| = r\}$ as $z = a + re^{it}$ $t \in [0, 2\pi]$.
f is analytic on and inside C, so
Cauchy's Integral formula implies:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{it})}{re^{it}} \left(rie^{it} \right) dt$$

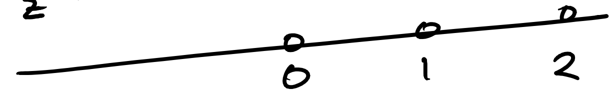
$$= \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{it}\right) dt$$

as desired.

Section 4

(3) $f(z) = \dfrac{1}{z(z-1)(z-2)}$ has singularities at $z = 0, 1, 2$.

Since we are interested in Laurent series
about $z = 0$, we leave the $\frac{1}{z}$ term
as it is and decompose



$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} =$$

for $0<|z|<1$:

update: I forgot the sign in the initial version

$$f(z) = \frac{1}{z}\left(\left(-\frac{1}{2}\right)\frac{1}{1-z/2} + \frac{1}{1-z}\right)$$

$$= \underbrace{\frac{1}{z}}_{\text{converges for } z\neq 0}\left(-\frac{1}{2}\right)\underbrace{\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)}_{\text{converges for } |z|<2} + \underbrace{\left(1+z+z^2\cdots\right)}_{\text{converges for } |z|<1}$$

$$= \frac{1}{z}\left(-\frac{1}{2} + \frac{7}{4}z + \frac{9}{8}z^2 + \cdots\right)$$

$$= \frac{1}{2z} + \frac{z}{4} + \frac{9}{8}z \cdots\cdots$$ which converges when $0<|z|<1$.

So $\mathrm{Res}(0) = \underline{\underline{1/2}}$

for $1<|z|<2$:

$$f(z) = \frac{1}{z}\left(\frac{1}{2}\frac{1}{1-z/2} - \frac{1}{z}\left(\frac{1}{1-1/z}\right)\right)$$

$$= \frac{1}{z}\left(\frac{1}{2}\underbrace{\left(1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2\cdots\right)}_{\text{converges for } |z|<2} - \frac{1}{z}\underbrace{\left(1+\frac{1}{z}+\frac{1}{z^2}\cdots\right)}_{\text{converges for } |z|>1}\right)$$

$$= \frac{1}{z}\left(\frac{1}{2}+\frac{z}{4}+\frac{z^2}{8}+\cdots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3}\cdots\right)$$

$$= \left(\frac{1}{2z} - \frac{1}{z^2} - \frac{1}{z^3}\cdots\right) + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16}\cdots$$

Converges in $1<|z|<2$

For $\underline{|z|>2}$:

$$f(z) = \frac{1}{z}\left(\frac{-1}{z-2} - \frac{1}{z-1}\right)$$

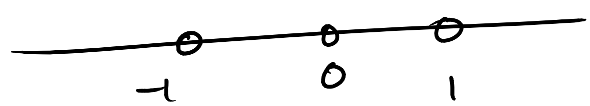
$$= \frac{1}{z}\left(-\frac{1}{z}\left(\frac{1}{1-2/z}\right) - \frac{1}{z}\left(\frac{1}{1-1/z}\right)\right)$$

$$= \frac{1}{z}\left(-\frac{1}{z}\underbrace{\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)}_{|z|>2} - \frac{1}{z}\underbrace{\left(1+\frac{1}{z}+\frac{1}{z^2}\cdots\right)}_{|z|>1}\right)$$

$$= -\frac{1}{z^2}\left(1+\frac{3}{z}+\frac{5}{z^2}+\frac{9}{z^3}\cdots\right)$$

$$= -\frac{1}{z^2} - \frac{3}{z^3} - \frac{5}{z^4}\cdots\cdots$$

(7) $f(z) = \dfrac{2-z}{1-z^2}$ has poles at $z = \pm 1$.



We compute the partial fraction decomposition:

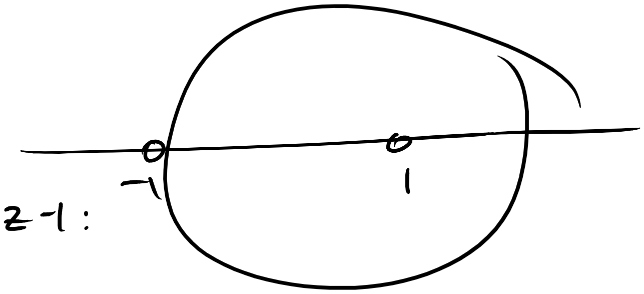
$$\frac{2-z}{1-z^2} = \frac{A}{1-z} + \frac{B}{1+z} \implies 2-z = A + Az + B - Bz$$

$$\implies \begin{matrix} A+B=2 \\ A-B=-1 \end{matrix} \implies \begin{matrix} A=1/2 \\ B=3/2. \end{matrix}$$

So $f(z) = \dfrac{1/2}{1-z} + \dfrac{3/2}{1+z}$.

There are three Annuli

1. $|z-1|<2.$



We expand about $z=1$, i.e. in powers of $z-1$:

$$f(z)=\frac{1/2}{1-z}+\frac{3/2}{1+z}$$

$$=\frac{-1/2}{z-1}+\frac{3/2}{2+(z-1)}$$

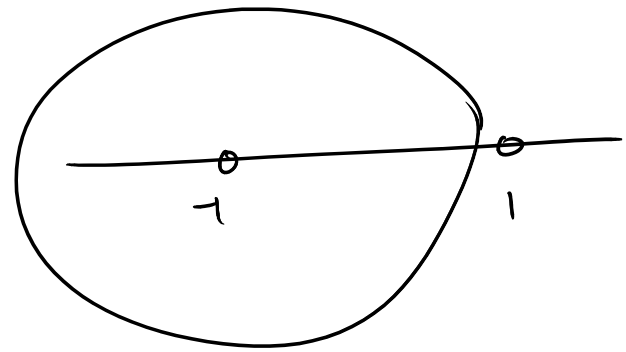
$$=\frac{-1/2}{z-1}+\frac{3/4}{1+\frac{z-1}{2}}$$

$$=\frac{-1/2}{z-1}+\underbrace{\frac{3}{4}\left(1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3\cdots\right)}_{\text{for } \left|\frac{z-1}{2}\right|<1}$$

$$=\frac{-1/2}{z-1}+\frac{3}{4}-\frac{3(z-1)}{8}+\frac{3(z-1)^2}{16}\cdots$$

So $\underline{\underline{\mathrm{Res}\ (1)=-1/2}}$

2. $|z-(-1)|<2$



Expand about $z=-1$

i.e. in powers of $z+1$.

$$f(z) = \frac{1/2}{1-z} + \frac{3/2}{1+z}$$

$$= \frac{3/2}{z+1} + \frac{1/2}{-2+(z+1)}$$

$$= \frac{3/2}{z+1} + \frac{1/2}{-2\left(1-\frac{z+1}{2}\right)}$$

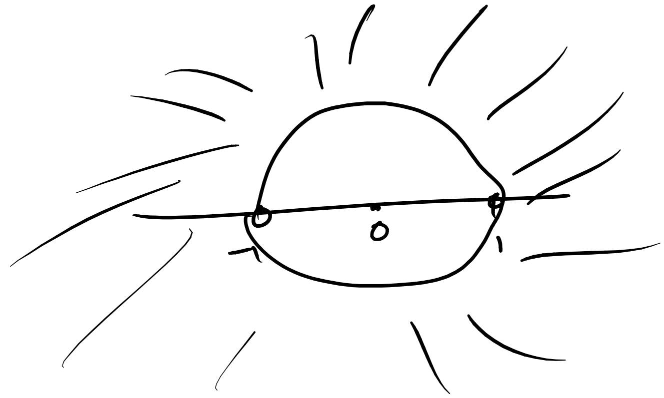
$$= \frac{3/2}{z+1} - \frac{1}{4}\left(1 + \frac{z+1}{2} + \left(\frac{z+1}{2}\right)^2 + \cdots\right)$$

$$= \frac{3/2}{z+1} - \frac{1}{4} - \frac{(z+1)}{8} - \frac{(z+1)^2}{16} \cdots$$

$$\underline{\underline{\text{Res}(-1) = 3/2}}$$

3. $|z| > 1$

expand "about ∞" i.e. in powers of $1/z$:



$$f(z) = \frac{1/2}{1-z} + \frac{3/2}{1+z}$$

$$= \frac{-1}{2z}\left(\frac{1}{1-1/z}\right) + \frac{3}{2z}\frac{1}{(1+1/z)}$$

$$= \frac{-1}{2z}\underbrace{\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)}_{\text{Convergent for } |z|>1} + \frac{3}{2z}\underbrace{\left(1-\frac{1}{z}+\frac{1}{z^2}+\cdots\right)}_{\text{conv. } |z|>1}$$

$$= \frac{1}{z} - \frac{2}{z^2} + \frac{1}{z^3} - \frac{2}{z^4} \cdots$$

No residue because 0 is not a singularity.

(10)

(c) $f(z)=\dfrac{z^3-1}{(z-1)^3} = \dfrac{(z-1)(1+z+z^2)}{(z-1)^3} = \dfrac{1+z+z^2}{(z-1)^2}$.

$= \dfrac{g(z)}{(z-1)^2}$ for $g(z) = 1+z+z^2$ analytic at 1 and $g(1) = 3 \neq 0$

So 1 is a pole of order 2.

OR Explicitly compute the Laurent Series about $z=1$:

$$f(z) = \frac{z^3-1}{(z-1)^3} = \frac{1}{(z-1)^3}\left(((z-1)+1)^3-1\right)$$

$$= \frac{1}{(z-1)^3}\left((z-1)^3+3(z-1)^2+3(z-1)+\cancel{1}\cancel{-1}\right)$$

$$= 1+\frac{3}{z-1}+\frac{3}{(z-1)^2}$$ So a pole of order 2.

(d) $$f(z) = \frac{e^z}{z-1} = e\cdot\frac{e^{z-1}}{z-1}$$

$$= \frac{e}{z-1}\left(1+(z-1)+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}\cdots\right)$$

$$= \frac{e}{z-1}+e+\frac{e(z-1)}{2!}+\cdots \Rightarrow \text{Simple pole.}$$

Problem 8

Same as problem 5! oops!