Homework 6 Solutions

Section 14

(21) $-1 = e^{i(\pi+2\pi k)}$, $k \in \mathbb{Z}$ so

$\log(-1) = i(\pi + 2\pi k)$,

$$\begin{aligned}\cos(i \log(-1)) &= \cos(i^2(\pi+2\pi k)) \\ &= \cos(-\pi - 2\pi k) \\ &= \cos(-\pi) \quad \text{since } \cos \text{ is periodic with period } 2\pi \\ &= \underline{\underline{-1}}.\end{aligned}$$

(24) Since we have to show it can have more values, it suffices to exhibit an example. We will do the two suggested in the book:

(a)
$$\begin{aligned}(-i)^{2+i} &= e^{(2+i)\log(-i)} \\ &= e^{(2+i)(i(\frac{3\pi}{2}+2\pi k))} \quad k \in \mathbb{Z} \\ &= e^{2i(\frac{3\pi}{2}+2\pi k) - (\frac{3\pi}{2}+2\pi k)} \\ &= e^{i3\pi} \cdot e^{i4\pi k} \cdot e^{-(\frac{3\pi}{2}+2\pi k)} \\ &= -e^{-(\frac{3\pi}{2}+2\pi k)} \quad \text{since } e^{i3\pi} = -1.\end{aligned}$$

$k \in \mathbb{Z}$.

Let the value corresponding to k be z_k.

We have:

$$\begin{aligned}(z_k)^{2-i} &= e^{(2-i)\log z_k} \\ &= e^{(2-i)(-(\frac{3\pi}{2}+2\pi k) + i(\pi+2\pi h))} \quad h \in \mathbb{Z} \\ &= e^{-2(\frac{3\pi}{2}+2\pi k)} e^{2i\pi + i4\pi h} e^{i(\frac{3}{2}\pi+2\pi k)} e^{(\pi+2\pi h)} \\ &= e^{-3\pi - 4\pi k + \pi + 2\pi h} e^{i\frac{3\pi}{2}} \quad \text{since } e^{2i\pi + i4\pi h}, e^{i2\pi k} = 1\end{aligned}$$

$$= -i \; e^{-2\pi - 4\pi k + 2\pi h} \qquad k, h \in \mathbb{Z}$$

$$= -i \; e^{-2\pi + 2\pi(2k - h)}$$

On the other hand, $(-i)^5 = -i$, which corresponds to the cases $2k - h = 1$.

However, when (for instance) $k=0, h=0$ we get

$$-i \, e^{-2\pi} \neq -i.$$

For an explanation of when a^b has a single, finitely many, and infinitely many values, see my email dated March 5.

(b) $i^i = e^{i \log i} = e^{i(i(\pi/2 + 2\pi k))} = e^{-(\pi/2 + 2\pi k)}$ $\quad k \in \mathbb{Z}$

$$(i^i)^i = e^{i \log(i^i)} = e^{i(-(\pi/2 + 2\pi k) + i 2\pi h)} \qquad h \in \mathbb{Z}$$

$$= e^{-i\pi/2} e^{i 2\pi k} e^{-2\pi h}$$

$$= -i \, e^{-2\pi h}, \quad h \in \mathbb{Z}$$

On the other hand, $i^{-1} = -i$, corresponding to $h = 0$.

Section 17

(16) Writing $z = x+iy$, $e^{i\pi/2} = i$, this is the set

$$\mathrm{Re}(i(x+iy)) = \mathrm{Re}(ix-y) = -y > 2.$$

So it is the set of points strictly below the line $\mathrm{Re}(z) = 2$.

(17) If $w = \arcsin z$ then

$$\sin w = \frac{e^{iw} - e^{-iw}}{2i} = z$$

letting $v = e^{iw}$, this becomes

$$v - \frac{1}{v} = 2iz \iff v^2 - 1 - 2ivz = 0.$$

The solutions are:

$$v = \frac{2iz \pm \sqrt{4i^2z^2+4}}{2}$$

$$= iz \pm \sqrt{1-z^2}.$$

But $iw = \log v$, so

$$w = \frac{1}{i}\log\left(iz \pm \sqrt{1-z^2}\right)$$

$$= -i\log\left(iz \pm \sqrt{1-z^2}\right)$$

as desired.

(25) (a) $$\overline{\cos z} = \overline{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \cdots}$$

$$= \overline{1} - \frac{\overline{z^2}}{2!} + \frac{\overline{z^4}}{4!} - \frac{\overline{z^6}}{6!}$$

Since $\overline{S} = \overline{\lim_{N\to\infty} S_N} = \lim_{N\to\infty} \overline{S_N}$ for a series

and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$$= 1 - \frac{(\overline{z})^2}{2!} + \frac{(\overline{z})^4}{4!} - \frac{(\overline{z})^6}{6!} \cdots$$

Since $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

$$= \cos(\overline{z}).$$

(b) Yes, for the same reason.

(c) $\overline{f(z)} = \overline{1+iz} = 1 - i\overline{z} \neq f(\overline{z})$.

Chapter 14

Section 1

(4) $|z| = |x+iy| = \sqrt{x^2+y^2} + 0i$ so

$u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$.

⑧ $\sin(x+iy) = \dfrac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$

$$= \frac{-i}{2}\left(e^{-y}e^{ix} - e^{y}e^{-ix}\right) \quad \text{since } \frac{1}{i} = -i$$

$$= \frac{-i}{2}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)$$

$$= \frac{1}{2}\left(e^{y}(i\cos x + \sin x) - e^{-y}(i\cos x - \sin x)\right)$$

$$= \frac{1}{2}(e^{y} + e^{-y})\sin x \;+\; i\,\frac{1}{2}(e^{y} - e^{-y})\cos x$$

$$= \sin x \cosh y \;+\; i\,\cos x \sinh y$$

So $v(x,y) = \sin x \cosh y$

$V(x,y) = \cos x \sinh y.$

⑨ $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2} = \dfrac{x-iy}{x^2+y^2}$

So $v(x,y) = \dfrac{x}{x^2+y^2}$ $\quad V(x,y) = \dfrac{-y}{x^2+y^2}$.

Section 2

(22) $u = y$ so $u_x = 0$ $v_x = 1$

$v = x$ $u_y = 1$ $v_y = 0$

$u_y \neq v_x$ so not analytic.

(28) By the definition,

$$g'(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}$$

So $g(z+\Delta z) = g(z) + g'(z)\Delta z + \epsilon_1 \Delta z$

where $\epsilon_1 \to 0$ as $\Delta z \to 0$

In other words, $g(z+\Delta z) = g(z) + g'(z)\Delta z + o(\Delta z)$.

This is the linear approximation of g at z.

Similarly, at the point $y = g(z)$, we have

$$f(y+\Delta y) = f(y) + f'(y)\Delta y + o(\Delta y).$$

for any perturbation Δy.

We now apply both of these facts to our case of interest:

$$f(g(z+\Delta z)) = f(\underbrace{g(z)}_{y} + \underbrace{g'(z)\Delta z + o(\Delta z)}_{\Delta y})$$

$$= f(g(z)) + f'(g(z))\left(g'(z)\Delta z + o(\Delta z)\right) + o(\Delta y)$$

$$= f(g(z)) + f'(g(z))g'(z)\Delta z + o(\Delta z)$$

Since $o(\Delta y) = o(\Delta z)$

and $f'(g(z))\, o(\Delta z) = o(\Delta z)$

But now $\frac{d}{dz} f(g(z)) =$

$$\lim_{\Delta z \to 0} \frac{f(g(z+\Delta z)) - f(g(z))}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f'(g(z))g'(z)\Delta z + o(\Delta z)}{\Delta z}$$

$$= f'(g(z))g'(z) + \lim_{\Delta z \to 0} \frac{o(\Delta z)}{\Delta z}$$

$$= f'(g(z))g'(z).$$

(46) Write

$$f(z) = u(r,\theta) + iv(r,\theta)$$

with $z = re^{i\theta}$.

Applying the Chain Rule, we have

$$\frac{\partial f}{\partial r} = \frac{df}{dz} \cdot \frac{\partial z}{\partial r} = \frac{df}{dz} \cdot e^{i\theta} = \frac{df}{dz} \cdot \frac{z}{r}$$

$$\frac{\partial f}{\partial \theta} = \frac{df}{dz} \cdot \frac{\partial z}{\partial \theta} = \frac{df}{dz} \cdot ire^{i\theta} = \frac{df}{dz} \cdot iz$$

On the other hand, by linearity:

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

Setting these equal, we find:

$$\frac{z}{r}\frac{df}{dz} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \qquad iz\frac{df}{dz} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

Equating $z\frac{df}{dz}$:

$$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{i}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}.$$

Equating real & imaginary parts, we have:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

which are the Cauchy Riemann equations in polar coordinates.

Section 3

3 For the first integral, we have the curves

$$\gamma_1(t) = (1-t)(-1) + t(1) = -1+2t, \quad \gamma_1'(t) = 2 \quad t\in[0,1]$$
$$\gamma_2(t) = (1-t)(1) + t(1+i) = 1+ti, \quad \gamma_2'(t) = i \quad t\in[0,1]$$
$$\gamma_3(t) = (1-t)(1+i) + t(-1+i) = 1+i-2t, \quad \gamma_3'(t) = -2 \quad t\in[0,1]$$
$$\gamma_4(t) = (1-t)(-1) + t(-1+i) = -1+i-ti, \quad \gamma_4'(t) = -i \quad t\in[0,1]$$

So: $\oint_C z^2 dz = \int_0^1 (-1+2t)^2 \cdot 2\, dt + \int_0^1 (1+ti)^2 \cdot i\, dt$

$$+ \int_0^1 (1+i-2t)^2 (-2)\, dt + \int_0^1 (-1+i-ti)^2 (-i)\, dt$$

$$= \int_0^1 (\cancel{2} - \cancel{8t} + \cancel{8t^2}) + (i - \cancel{2t} - \cancel{it^2}) + (-\cancel{2} + \cancel{2} - \cancel{8t^2} - 4i + \cancel{8t} + 8ti) + (-\cancel{i} + \cancel{i} + \cancel{t^2 i} - \cancel{2} + \cancel{2t} - 2ti)\, dt$$

$$= \int_0^1 -3i + 6ti\, dt = -3it \Big|_0^1 + \frac{6t^2 i}{2}\Big|_0^1$$

$$= -3i + 3i = \underline{\underline{0}}$$

phew.

The second integral may be parameterized as $\gamma_1 + \gamma_2$ with

$\gamma_1(t) = t, \quad t \in [-1, 1]$

$\gamma_2(t) = e^{it} \quad t \in [0, \pi], \; \gamma_2'(t) = ie^{it}$

So $\displaystyle\int_{\gamma_1} z^2 dz + \int_{\gamma_2} z^2 dz$

$$= \int_{-1}^{1} t^2 dt + \int_0^{\pi} e^{2it} \cdot ie^{it} dt$$

$$= \frac{t^3}{3}\Big|_{-1}^{1} + i \int_0^{\pi} e^{3it} dt$$

$$= \frac{2}{3} + i\int_0^{\pi} \cos 3t + i\sin 3t \; dt$$

$$= \frac{2}{3} + i\left(\frac{\sin 3t}{3}\Big|_0^{\pi} - i\frac{\cos 3t}{3}\Big|_0^{\pi} \right)$$

$$= \frac{2}{3} + i\left(-i\left(\frac{-1-1}{3}\right)\right)$$

$$= \frac{2}{3} - \frac{2}{3} = \underline{\underline{0}}.$$

(6) <u>first path:</u>

$$\int_0^{\pi/2} e^{it}(ie^{it})\,dt$$

$$= i\int_0^{\pi/2} e^{2it}\,dt$$

$$= i\;\frac{e^{2it}}{2i}\Big|_0^{\pi/2}$$ Using the antiderivative

$$= \frac{e^{i\pi} - e^{0}}{2} = \underline{\underline{-1}}$$

<u>Second path:</u>

$$\int_0^{1} (1+ti)(i)\,dt + \int_1^{0} (t+i)\,dt$$

$$= \int_0^{1} (i - t - t - i)\,dt = -t^2\Big|_0^1 = \underline{\underline{-1}}.$$

(12)

Path 1: $\gamma_1(t) = t(1+2i)$, $\gamma_1'(t) = 1+2i$, $t\in[0,1]$

$$\int_{\gamma_1} |z|^2 dz = \int_0^1 (t^2 + 4t^2)(1+2i)\,dt$$

$$= (1+2i) \left.\frac{5t^3}{3}\right|_0^1 = \frac{(1+2i)5}{3}.$$

Path 2: $\gamma_2(t) = 2it$, $t\in[0,1]$, $\gamma_2'(t) = 2i$

$\gamma_3(t) = 2i+t$, $t\in[0,1]$, $\gamma_3'(t) = 1$

$$\int_{\gamma_2} |z|^2 dz + \int_{\gamma_3} |z|^2 dz = \int_0^1 4t^2 \cdot 2i\,dt + \int_0^1 (4+t^2)\,dt$$

$$= 2i \left.\frac{4t^3}{3}\right|_0^1 + \left. 4t + \frac{t^3}{3}\right|_0^1$$

$$= \frac{8i}{3} + 4 + \frac{1}{3} = \frac{13+8i}{3}$$

which is different.

This is not surprising because $|z|^2$ is not analytic.