Homework 5 Solutions

Section 5

Problem 1. $\frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i$

Problem 2. $\frac{1}{i-1} = -\frac{1}{2} - \frac{1}{2}i$

Problem 7. $\frac{3+i}{2+i} = \frac{7}{5} - \frac{1}{5}i$

Problem 28. $\left|\frac{z}{\bar{z}}\right| = 1$

Problem 34. $\left|\left(\frac{1+i}{1-i}\right)^5\right| = \left(\left|\frac{1+i}{1-i}\right|\right)^5 = (1)^5 = 1$

Problem 46.

$$\begin{aligned} \frac{x+iy}{x-iy} = -i \quad &\Longleftrightarrow \quad x + iy = (-i)(x - iy) \\ &\Longleftrightarrow \quad x + iy = -y - ix \\ &\Longleftrightarrow \quad x = -y \end{aligned}$$

Problem 50.

$$\begin{aligned} |x + iy| = y - ix \quad &\Longleftrightarrow \quad \sqrt{x^2 + y^2} = y - ix \\ &\Longleftrightarrow \quad \sqrt{x^2 + y^2} = y \text{ and } 0 = -x \\ &\Longleftrightarrow \quad y \geq 0 \text{ and } x = 0 \end{aligned}$$

Problem 60. Since

$$\begin{aligned} |z - 1 + i| = 2 \;&\Longleftrightarrow\; |z - (1 - i)| = 2 \\ &\Longleftrightarrow\; \text{the distance between } z \text{ and } 1 - i \text{ is } 2, \end{aligned}$$

$|z - 1 + i| = 2$ is the circle with center $1 - i$ and radius 2.

Section 6

Problem 1.
Hypothesis: $\sum z_n$ is absolutely convergent. Each $z_n = x_n + iy_n$, where x_n and y_n are real numbers.
Claim: $\sum z_n$ is convergent.
Proof: We need to show that $\sum x_n$ and $\sum y_n$ are convergent. According to the definition of an absolutely convergent series,

$$\sum z_n \text{ is absolutely convergent } \Longrightarrow \sum \sqrt{x_n^2 + y_n^2} \text{ is convergent.}$$

By the Comparison Test,

$$\sum \sqrt{x_n^2 + y_n^2} \text{ is convergent } \Longrightarrow \sum x_n \text{ is absolutely convergent.}$$

According to Problem 7.9 of Chapter 1,

$$\sum x_n \text{ is absolutely convergent } \Longrightarrow \sum x_n \text{ is convergent.}$$

The same argument shows that $\sum y_n$ is convergent.

Problem 2. This is a geometric series with ratio $z = 1 + i$; it converges if and only if $|z| < 1$ (see Chapter 2, Section 6, Example 3). Since $|1 + i| > 1$, the geometric series $\sum (1 + i)^n$ is divergent.

Problem 4. This is a geometric series with ratio $z = \frac{1-i}{1+i}$; it converges if and only if $|z| < 1$ (see Chapter 2, Section 6, Example 3). Since $|\frac{1-i}{1+i}| = 1$, the geometric series $\sum (\frac{1-i}{1+i})^n$ is divergent.

Problem 13. This is a geometric series with ratio $z = \frac{1+i}{2-i}$; it converges if and only if $|z| < 1$ (see Chapter 2, Section 6, Example 3). Since $|\frac{1+i}{2-i}| < 1$, the geometric series $\sum \left(\frac{1+i}{2-i}\right)^n$ is convergent.

Section 7

Problem 1. From the ratio test,

$$\rho = \lim_{n\to\infty} \left| \tfrac{z^{n+1}}{(n+1)!} \div \tfrac{z^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = 0.$$

This series converges for all values of z.

Problem 4. From the ratio test,

$$\rho = \lim_{n\to\infty} \left| z^{n+1} \div z^n \right| = \lim_{n\to\infty} |z| = |z|.$$

This series converges for $|z| < 1$, the disk centered at 0 with radius 1.

Problem 7. From the ratio test,

$$\rho = \lim_{n\to\infty} \left| \tfrac{(-1)^{n+1} z^{2(n+1)}}{[2(n+1)]!} \div \tfrac{(-1)^n z^{2n}}{(2n)!} \right| = \lim_{n\to\infty} \left| \frac{-z^2}{4n^2 + 6n + 2} \right| = 0.$$

This series converges for all values of z.

Problem 13. From the ratio test,

$$\rho = \lim_{n\to\infty} \left| \tfrac{(z-i)^{n+1}}{n+1} \div \tfrac{(z-i)^n}{n} \right| = |z-i| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |z-i|.$$

This series converges on $|z-i| < 1$, the disk centered at i with radius 1.

Section 8 Along with (8.1), we use the formulas

$$\sum_{n=0}^{\infty} a_n \cdot \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n \qquad \text{where} \qquad c_n := \sum_{k=0}^{n} a_k b_{n-k}$$

and

$$(a+b)^n = \sum_{k=0}^{n} \tfrac{n!}{k!(n-k)!} a^k b^{n-k}.$$

Problem 1.

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= \sum_{n=0}^{\infty} \tfrac{z_1^n}{n!} \cdot \sum_{n=0}^{\infty} \tfrac{z_2^n}{n!} \\ &= \sum_{n=0}^{\infty} c_n \qquad \text{where} \qquad c_n \;=\; \sum_{k=0}^{n} \tfrac{z_1^k}{k!} \tfrac{z_2^{n-k}}{(n-k)!} \;=\; \sum_{k=0}^{n} \tfrac{z_1^k z_2^{n-k}}{k!(n-k)!} \;=\; \tfrac{(z_1+z_2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \tfrac{(z_1+z_2)^n}{n!} = e^{z_1+z_2} \end{aligned}$$

Problem 2.

$$\begin{aligned} \frac{d}{dz} e^z &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{d}{dz} \left(1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \right) = \frac{d}{dz} 1 + \frac{d}{dz} \sum_{n=1}^{\infty} \frac{z^n}{n!} \\ &= 0 + \frac{d}{dz} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \frac{d}{dz} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{d}{dz} \frac{z^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d}{dz} z^n = \sum_{n=1}^{\infty} \frac{1}{n!} n z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \;=\; e^z \end{aligned}$$

Section 9

Problem 27. For any real y,

$$|e^{iy}| \;=\; |\cos y + i \sin y| \;=\; \sqrt{\cos^2 y + \sin^2 y} \;=\; \sqrt{1} \;=\; 1.$$

Hence, for every complex $z = x + iy$,

$$|e^z| \;=\; |e^{x+iy}| \;=\; |e^x e^{iy}| \;=\; |e^x||e^{iy}| \;=\; |e^x| \;=\; e^x.$$

Section 10

Problem 1. The cube roots of 1 are 1, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Problem 11. The cube roots of -8 are -2, $1 - \sqrt{3}i$, and $1 + \sqrt{3}i$.

Problem 22. The cube roots of $-2 + 2i$ are $\sqrt{2} + i\sqrt{2}$, $\frac{-1-\sqrt{3}}{2} + i\frac{1-\sqrt{3}}{2}$, and $\frac{-1+\sqrt{3}}{2} - i\frac{1+\sqrt{3}}{2}$.

Problem 30. (Follows from Problem 31)

Problem 31. Suppose w is a complex number. Let w_1, w_2, w_3, $\ldots$, w_n be the n nth roots of w. We want to show that $\sum_{k=1}^n w_k = 0$. Since each w_k is an nth root of w, each w_k solves the equation $z^n - w = 0$. Since $z^n - w$ is a polynomial of degree n,

$$z^n - w = (z - w_1)(z - w_2)(z - w_3) \cdots (z - w_n).$$

But

$$(z-w_1)(z-w_2)(z-w_3) \cdots (z-w_n) = z^n - \left(\sum_{k=1}^n w_k \right) z^{n-1} + \cdots + (-1)^n w_1 w_2 w_3 \cdots w_n.$$

So

$$z^n - w \;=\; z^n + 0z^{n-1} + \cdots + (-w) \;=\; z^n - \left(\sum_{k=1}^n w_k \right) z^{n-1} + \cdots + (-1)^n w_1 w_2 w_3 \cdots w_n.$$

Therefore $0 = \sum_{k=1}^n w_k$.

Section 10, #31 (Alternate solution)

Suppose $z = re^{i\theta}$. The n^{th} roots of z are given by

$$r^{1/n} e^{i\left(\frac{\theta + 2\pi k}{n}\right)}, \quad k = 0, 1, \dots n-1.$$

So, the sum of the roots is

$$r^{1/n}\left(e^{i\frac{\theta}{n}} + e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)} + e^{i\left(\frac{\theta}{n} + \frac{4\pi}{n}\right)} + \dots e^{i\left(\frac{\theta}{n} + \frac{2\pi(n-1)}{n}\right)} \right)$$

$$= r^{1/n}\left(e^{i\theta/n} + e^{i\theta/n} e^{i2\pi/n} + e^{i} e^{i4\pi/n} + \dots + e^{i\theta/n} e^{i\frac{2\pi(n-1)}{n}} \right)$$

$$= r^{1/n} e^{i\theta/n}\left(1 + e^{i2\pi/n} + e^{i\frac{4\pi}{n}} + \dots + e^{i\frac{2(n-1)\pi}{n}} \right).$$

Notice that the terms in the parentheses are just the n^{th} roots of $\underline{1}$.

Letting $\omega = e^{i2\pi/n}$, their sum may be rewritten as

$$S = 1 + e^{i\frac{2\pi}{n}} + \left(e^{i\frac{2\pi}{n}}\right)^2 + \left(e^{i\frac{2\pi}{n}}\right)^3 + \dots \left(e^{i\frac{2\pi}{n}}\right)^{n-1} = 1 + \omega + \omega^2 + \omega^3 \dots \omega^{n-1}.$$

Geometrically, these numbers are vertices of a regular polygon. They are located on the unit circle, and consecutive powers are equally spaced, by an angle of $\frac{2\pi}{n}$.

The crucial geometric observation is that this polygon does not change if we rotate it by $\frac{2\pi}{n}$ counterclockwise — each vertex simply gets mapped to the next one (modulo n)

Since rotation by $\frac{2\pi}{n}$ is the same as multiplication by ω, this may be seen algebraically as:

$$\omega \cdot 1 = \omega$$
$$\omega \cdot \omega = \omega^2$$
$$\omega \cdot \omega^2 = \omega^3$$
$$\omega \cdot \omega^3 = \omega^4$$
$$\vdots$$
$$\omega \cdot \omega^{n-1} = \omega^n = 1.$$

In particular, we have

$$\omega S = \omega(1 + \omega + \omega^2 \cdots + \omega^{n-1})$$
$$= (\omega + \omega^2 + \omega^3 \cdots + 1)$$
$$= S.$$

Thus $(\omega - 1)S = 0$.

Since $\omega \neq 1$ this means $S = 0$, and we are done.

<u>Another Solution</u>:

Observe $S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}$ is a geometric series.

Thus, its sum is given by: $S = \dfrac{\omega^n - 1}{\omega - 1} = \dfrac{1-1}{\omega - 1} = \underline{\underline{0}}$.