Homework 4 Solutions

Update: the squares were missing in the original version that I uploaded.

Section 1

(1) $v = \dfrac{x^2}{x^2+y^2} = \dfrac{1}{1+\frac{y^2}{x^2}}$.

$$\frac{\partial v}{\partial x} = \frac{-1}{\left(1+\frac{y^2}{x^2}\right)^2} \cdot \frac{-2y^2}{x^3}$$

$$\frac{\partial v}{\partial y} = \frac{-1}{\left(1+\frac{y^2}{x^2}\right)^2} \cdot \frac{2y}{x^2}.$$

(2) $v = e^x \cos y$.

(a) $$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} e^x \cos y\right) = \frac{\partial}{\partial x}\left(-e^x \sin y\right) = -e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} e^x \cos y\right) = \frac{\partial}{\partial y} e^x \cos y = -e^x \sin y$$

(b) $$\frac{\partial^2}{\partial x^2} e^x \cos y + \frac{\partial^2}{\partial y^2} e^x \cos y$$

$$= e^x \cos y + e^x(-(\cos y)) = 0.$$

Section 4

(2) letting $y = \sqrt{x}$, we have $dy = \dfrac{dx}{2\sqrt{x}}$

So $\Delta y \approx x + \dfrac{\Delta x}{2\sqrt{x}}$.

Taking $x = n$ and $\Delta x = a$, $\sqrt{n+a} - \sqrt{n} \approx \dfrac{a}{2\sqrt{n}}$.

In the specific example, we have

$$\sqrt{10^{26}+5} - \sqrt{10^{26}} \approx \frac{5}{2\sqrt{10^{26}}} = \underline{\underline{2.5 \times 10^{-13}}}.$$

Section 5

(1) $z = xe^{-y}$, $x = \cosh t$, $y = \cos t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= e^{-y} \cdot \sinh t + (-xe^{-y}) \cdot (-\sin t)$$

$$= e^{-y}(\sinh t + x \sin t).$$

(7) $c = \sin(a-b)$, $b = ae^{2a}$.

$$\frac{dc}{da} = \frac{\partial c}{\partial a}\frac{da}{da} + \frac{\partial c}{\partial b} \cdot \frac{db}{da}$$

$$= \cos(a-b) + (-\cos(a-b)) \cdot (e^{2a} + a \cdot 2e^{2a})$$

Section 7

(2) $P = r\cos t$, $r \sin t - 2te^{r} = 0$.

There are 3 variables and 2 equations, so only one degree of freedom. Thus P is a function of t and $\frac{dP}{dt}$ makes sense.

Taking total differentials:

$$dP = \frac{\partial P}{\partial r} dr + \frac{\partial P}{\partial t} dt$$

$$= \cos t \, dr + (-r \sin t) dt$$

and

$$0 = \frac{\partial}{\partial r}(r \sin t - 2te^r) dr + \frac{\partial}{\partial t}(r \sin t - 2te^r) dt$$

$$= (\sin t - 2te^r) dr + (r \cos t - 2e^r) dt.$$

Substituting $dr = \dfrac{-(r\cos t - 2e^r)\, dt}{\sin t - 2te^r}$ into

the first equation, we get

$$dP = -\cos t \frac{(r \cos t - 2e^r)}{\sin t - 2te^r} dt - r \sin t \, dt$$

So

$$\frac{dP}{dt} = \frac{-\cos t (r \cos t - 2e^r)}{\sin t - 2te^r} - r \sin t$$

$$= \frac{-r\cos^2 t - 2e^r \cos t - r \sin^2 t + 2te^r \sin t}{\sin t - 2te^r}$$

$$= \frac{2te^r \sin t - 2e^r \cos t - r}{\sin t - 2te^r}.$$

(7) $x = r\cos\theta, \quad y = r\sin\theta.$

(a) Since y is an explicit function of r, θ,

$$\left(\frac{\partial y}{\partial \theta}\right)_r = r\cos\theta$$ is easy.

(b) For $\left(\frac{\partial y}{\partial \theta}\right)_x$, we take differentials:

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta \quad \text{——— (1)}$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta \quad \text{——— (2)}.$$

Since x is fixed, we set $dx = 0$, which yields (for fixed x):

$$\cos\theta\, dr = r\sin\theta\, d\theta$$

Substituting the expression for dr into (2),

$$dy_x = \sin\theta\left(\frac{r\sin\theta}{\cos\theta}\, d\theta\right) + r\cos\theta\, d\theta$$

subscript means keeping x fixed

$$= \frac{r\sin^2\theta\, d\theta + r\cos^2\theta\, d\theta}{\cos\theta}$$

$$= \frac{r\, d\theta}{\cos\theta}.$$

Thus, $\left(\frac{\partial y}{\partial \theta}\right)_x = \frac{r}{\cos\theta}.$

(c) Dividing our original equations gives

$$\frac{x}{y} = \frac{\cos\theta}{\sin\theta} \implies \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

So $\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$

On the other hand, eqn (2) $-$ $\tan\theta \times$ eqn (1) is:

$$dy - \tan\theta\, dx = \left(r\cos\theta + \frac{r\sin^2\theta}{\cos\theta} \right) d\theta$$

So $$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{r\cos\theta + \frac{r\sin^2\theta}{\cos\theta}} = \frac{\cos\theta}{r}.$$

This is the same as what we got from the first method because

$$\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2}$$
$$= \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}.$$

$\frac{\partial \theta}{\partial y}$ and $\frac{\partial y}{\partial \theta}$ are reciprocals only when the variable being held constant is the same.

(14) $v = x^2+y^2+xyz$, $x^4+y^4+z^4 = 2x^2y^2z^2+10$.

Taking differentials:

$$dv = (2x+yz)\,dx + (2y+xz)\,dy + xy\,dz$$

$$4x^3dx + 4y^3\,dy + 4z^3dz = 4xy^2z^2dx + 4yx^2z^2dy + 4zx^2y^2dz.$$

Since z is held constant, we are interested in the case when $dz=0$, and the equations simplify to:

$$dv = (2x+yz)dx + (2y+xz)dy$$

$$4x^3dx + 4y^3dy = 4xy^2z^2dx + 4yx^2z^2dy$$

At the point $(x,y,z) = (2,1,1)$, this becomes.

$$dv = 5dx + 4dy$$

$$24dx + 4dy = 8dx + 16dy. \iff 16dx = 12dy$$

So $dv = 5dx + 4\left(\frac{16}{12}\right)dx = \frac{31}{3}dx$

So $\left(\frac{\partial v}{\partial x}\right)_z (2,1,1) = \frac{31}{3}$.

(18) $m = a+b$, $n = a^2+b^2$. Want: $\left(\frac{\partial b}{\partial m}\right)_n$, $\left(\frac{\partial m}{\partial b}\right)_a$.

Take differentials:

$$dm = da + db$$

$$dn = 2ada + 2bdb.$$

When n is held constant:

$$dm = da + db$$

$$0 = 2ada + 2bdb$$

$$\Rightarrow da = -\frac{b}{a}db.$$

Substituting: $dm = \left(1 - \frac{b}{a}\right)db$

So $\left(\frac{\partial b}{\partial m}\right)_n = \frac{1}{1-\frac{b}{a}}$ //

When a is held constant:

$$dm = db$$

$$dn = 2bdb$$

So $\left(\frac{\partial m}{\partial b}\right)_a = 1.$ //

(22) $w = f(ax+by)$. Let $z = ax+by$.

We have $dw = \frac{df}{dz} \cdot dz$

$$= \frac{df}{dz}\left(a\,dx + b\,dy\right)$$

$$= a\frac{df}{dz}dx + b\frac{df}{dz}dy.$$

Thus, $\left(\frac{\partial w}{\partial x}\right)_y = a\frac{df}{dz}$, $\left(\frac{\partial w}{\partial y}\right)_x = b\frac{df}{dz}$

So $b\frac{\partial w}{\partial x} - a\frac{\partial w}{\partial y}$

$$= ba\frac{df}{dz} - ab\frac{df}{dz} = 0.$$

(23) $v = f(x-ct) + g(x+ct)$. We are interested in:

$$\frac{\partial^2 v}{\partial x} = \frac{\partial^2}{\partial x} f(x-ct) + \frac{\partial^2}{\partial x} g(x+ct).$$

Applying the conclusion of the previous question to f and g, we get

$$\boxed{-c\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}} \text{ and } \boxed{c\frac{\partial g}{\partial x} = \frac{\partial g}{\partial t}} \quad \text{——} (*)$$

Applying $\frac{\partial}{\partial x}$ again gives:

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{c}\frac{\partial^2 f}{\partial x \partial t}, \quad \frac{\partial^2 g}{\partial x^2} = \frac{1}{c}\frac{\partial^2 g}{\partial x \partial t}$$

So $$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c}\frac{\partial^2 g}{\partial x \partial t} - \frac{1}{c}\frac{\partial^2 f}{\partial x \partial t}. \quad \text{——} (1)$$

On the other hand, differentiating ~~f~~ with respect to t gives:

$$-c\frac{\partial^2 f}{\partial t\partial x} = \frac{\partial^2 f}{\partial t^2} \quad \text{and} \quad c\frac{\partial^2 g}{\partial t\partial x} = \frac{\partial^2 g}{\partial t^2}$$

Adding these, we find that

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2}$$

$$= c\frac{\partial^2 g}{\partial t\partial x} - c\frac{\partial^2 f}{\partial t\partial x} \quad \text{———} \quad (2)$$

Assuming continuity of second derivatives, we have

$$\frac{\partial^2 f}{\partial t\partial x} = \frac{\partial^2 f}{\partial x\partial t} \quad \text{and} \quad \frac{\partial^2 g}{\partial t\partial x} = \frac{\partial^2 g}{\partial x\partial t}.$$

Thus, combining (1) and (2) gives

$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2}, \quad \text{as desired.}$$

(26) $f(x,y,z) = 0$, $g(x,y,z) = 0$.

Taking total differentials: means $\left(\frac{\partial f}{\partial y}\right)_{xz}$

$$0 = df = f_x\,dx + f_y\,dy + f_z\,dz \qquad \text{—— (1)}$$

$$0 = dg = g_x\,dx + g_y\,dy + g_z\,dz. \qquad \text{—— (2)}$$

We eliminate dz by subtracting $\frac{f_z}{g_z} \times (2)$ from (1):

$$0 = \left(f_x - \frac{f_z}{g_z} g_x\right) dx + \left(f_y - \frac{f_z}{g_z} g_y\right) dy$$

Which yields the formula:

$$\frac{dy}{dx} = -\frac{f_x - \frac{f_z}{g_z} g_x}{f_y - \frac{f_z}{g_z} g_y}. \quad //$$

Section 8

(6) $f(x,y) = x^3 - y^3 - 2xy + 2$.

The partial derivatives are $f_x = 3x^2 - 2y$

$f_y = -3y^2 - 2x$.

These are zero at the critical points, so

$$\begin{aligned} 3x^2 &= 2y \\ -3y^2 &= 2x \end{aligned} \implies 3\left(-\tfrac{3}{2}y^2\right)^2 = 2y$$

$$\Downarrow$$

$$\frac{27}{8} y^3 = 1 \implies \underline{\underline{y = \tfrac{2}{3}}}.$$

So $x = -\frac{3}{2}y^2 = -\frac{3}{2}\left(\frac{2}{3}\right)^2 = -2/3$.

At the point $(-2/3, 2/3)$ we have

$$f_{xx} = 6x = -4$$

$$f_{yy} = -6y = -4$$

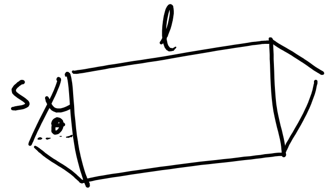
$$f_{xy} = f_{yx} = -2$$

$$\text{So} \quad f_{xx} f_{yy} = 4^2 > (f_{xy})^2$$

and thus point is a <u>maximum</u>.

<u>Section 9</u>

(1) The perimeter is $P(s, l, \theta) = 4s + 2l$



the base of each triangle is $2s\cos\theta$, and the height is $s \cdot \sin\theta$, so

the total area is

$$A(s, l, \theta) = 2sl\cos\theta + 2s^2\sin\theta\cos\theta$$
$$= 2sl\cos\theta + s^2\sin 2\theta.$$

At the optimum, we must have $\nabla A = \lambda \nabla P$ so :

$$\frac{\partial A}{\partial s} = \lambda \frac{\partial P}{\partial s} \Rightarrow \quad 2l\cos\theta + 2s\sin 2\theta = 4\lambda \quad \text{—— (1)}$$

$$\frac{\partial A}{\partial l} = \lambda \frac{\partial P}{\partial l} \Rightarrow \quad 2s\cos\theta = 2\lambda \quad \text{—— (2)}$$

$$\frac{\partial A}{\partial \theta} = \lambda \frac{\partial P}{\partial \theta} \Rightarrow \quad -2sl\sin\theta + 2s^2\cos 2\theta = 0 \quad \text{—— (3)}$$

(2) and (3) simplify to:

$$\lambda = s\cos\theta$$

$$l = s\frac{\cos 2\theta}{\sin\theta}$$

Substituting these in (1), we get

$$2s\frac{\cos 2\theta\cos\theta}{\sin\theta} + 2s\sin 2\theta = 4s\cos\theta$$

Since $s \neq 0$ we cancel from both sides to get

$$\cos 2\theta\cos\theta + \sin 2\theta\sin\theta = 2\sin\theta\cos\theta$$

Or: $\cos\theta(\cos 2\theta + 2\sin^2\theta) = 2\sin\theta\cos\theta$

So either $\underbrace{\cos\theta = 0}_{} \Rightarrow \theta = \pi/2$ (impossible), or $\overset{=0}{}$

$$\cos 2\theta + 2\sin^2\theta = 2\sin\theta$$

$$\Rightarrow \quad 1 = 2\sin\theta \Rightarrow \quad \theta = \pi/6.$$

Thus, the optimal solution has $\theta = \pi/6$.

You can plug this back into the equations to solve for l and s (I will skip it here).

(4) See the midterm solutions, #5.

It turns out that in the midterm solutions I solved a different problem: I used the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}$ instead of $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{25}$.

The method is the same, even though the numbers are slightly different. Sorry about that!

(11) We want:

$$\min \quad x^2+y^2+z^2 = f(x,y,z)$$

subject to $\quad 2x+y-z = \varphi_1(x,y,z) = 1 \quad \text{——} (1)$

and $\quad x-y+z = \varphi_2(x,y,z) = 2. \quad \text{——} (2)$

The method of Lagrange Multipliers for <u>two</u> constraints tells us that at the minimum:

$$\nabla f = \lambda_1 \nabla \varphi_1 + \lambda_2 \nabla \varphi_2$$

for some λ_1, λ_2.

This implies that:

$$2x = \lambda_1(2) + \lambda_2(1) = 2\lambda_1 + \lambda_2 \quad \text{——} (3)$$

$$2y = \lambda_1(1) + \lambda_2(-1) = \lambda_1 - \lambda_2 \quad \text{——} (4)$$

$$2z = \lambda_1(-1) + \lambda_2(1) = -\lambda_1 + \lambda_2 \quad \text{——} (5)$$

Substituting these in (1), (2), we have

$$2\lambda_1 + \lambda_2 + \frac{\lambda_1 - \lambda_2}{2} - \frac{(-\lambda_1 + \lambda_2)}{2} = 1$$

and $$\frac{2\lambda_1 + \lambda_2}{2} - \frac{(\lambda_1 - \lambda_2)}{2} + \frac{(-\lambda_1 + \lambda_2)}{2} = 2$$

which after gathering terms becomes:

$$3\lambda_1 = 1 \quad , \quad \frac{3\lambda_2}{2} = 2 \implies \begin{array}{l} \lambda_1 = 1/3 \\ \lambda_2 = 4/3. \end{array}$$

Substituting back in (3),(4),(5), we get

$$x = \lambda_1 + \frac{\lambda_2}{2} = \frac{1}{3} + \frac{1}{2}\frac{4}{3} = \underline{\underline{1}}$$

$$y = \frac{\lambda_1 - \lambda_2}{2} = \frac{1/3 - 4/3}{2} = \underline{\underline{-1/2}}$$

$$z = \frac{-\lambda_1 + \lambda_2}{2} = \frac{-1/3 + 4/3}{2} = \underline{\underline{1/2}}.$$

Section 10

(1) We Want: $\min \; x^2 + y^2 \quad = f(x,y)$

subject to $x^2 - y^2 = 1. \quad = \varphi(x,y)$

Using Lagrange Multipliers, we have

$$\begin{aligned} 2x &= \lambda\, 2x \\ 2y &= \lambda(-2y) \end{aligned} \implies \begin{aligned} x &= \lambda x \\ y &= -\lambda y \end{aligned}$$

So either $x = 0, \lambda = -1$

or $y = 0, \lambda = 1$

or $x = y = \lambda = 0$.

The first and last cases are impossible since $x^2 - y^2 = 1$, so we must have $y = 0$ and $\lambda = 1$, whence $x = \pm 1$. So the two minima are $(1,0)$ and $(-1,0)$.

I think the book put this in this section because another way to solve it is to use the second equation to make the substitution

$$y^2 = x^2 - 1$$

which gives the unconstrained problem

$$\min \quad x^2 + x^2 - 1 = 2x^2 - 1$$

with the implicit constraint $x^2 - 1 \geq 0$ since $y^2 \geq 0$

Using calculus here gives the solution

$$4x - 1 = 0 \implies x = 1/4$$

which violates the implicit constraint.

Thus, we have to check the boundary values $x^2 = 1$ $\implies x = \pm 1$, and sure enough these are the minima we derived using Lagrange multipliers.

(8) We want: $\max \; T(x,y,z) = xyz$

sub. $x^2+y^2+z^2 = \phi(x,y,z) = 12$.

This is almost identical to #5 on the midterm as well as Section 9 #4 on this HW (with the numbers changed), and can be solved using the same Lagrange multiplier method.

The only difference is that we do not require x, y, z to be positive since they are no longer dimensions of a box, but coordinates in space.

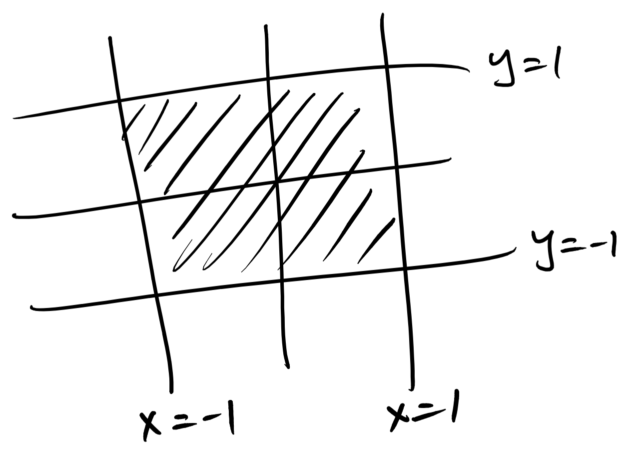
(11) The (unconstrained) max/min temperature satisfies:

$$\frac{\partial T}{\partial x} = 0 \implies 4x - 2 = 0$$

$$\frac{\partial T}{\partial y} = 0 \implies -6y = 0$$

$$\implies (x,y) = (2,0)$$

which is outside the box



So any optimum must lie on the boundary.

Let us consider the 4 restrictions:

$\underline{x=1}$

$T = 2(1)^2 - 3y^2 - 2(1) + 10$

$= -3y^2 + 10$

any interior min/max satisfies:

~~$-6y + 10 = 0$~~

~~$\Rightarrow y = \frac{10}{6} > 1$~~

~~So outside the box.~~

~~Thus, we only need to consider the corners.~~

typo!

Correction

$-6y = 10$ so $(x,y) = (1,0)$ is a critical point.

$T(1,0) = \underline{\underline{12}}$

$\underline{x=-1}$

$T = 2 - 3y^2 + 2 + 10$

$= -3y^2 + 14$

$\frac{\partial T}{\partial y} = -6y = 0$

So $(-1,0)$ is a critical point

$T(-1,0) = \underline{\underline{14}}$

$\underline{y=1}$

$T = 2x^2 - 3 - 2x + 10$

$\frac{\partial T}{\partial x} = 4x - 2$

$\Rightarrow x = 2/4$

So there is a critical point at $(1/2, 1)$

$T(1/2, 1) = \frac{1}{2} - 3 - 1 + 10$

$= \underline{\underline{\frac{13}{2}}}$

$\underline{y=-1}$

$T = 2x^2 - 3 - 2x + 10$

$\frac{\partial T}{\partial x} = 4x - 2$

$\Rightarrow x = 2/4$

So there is a critical point at $(1/2, -1)$.

$T = \frac{1}{2} - 3 - 1 + 10$

$= \underline{\underline{\frac{13}{2}}}$

The values at the corners are:

$$T(1,1) = 2-3-2+10$$
$$= \underline{\underline{7}}$$

$$T(1,-1) = -3+10 = \underline{\underline{7}}$$

$$T(-1,1) = -3+14 = \underline{\underline{11}}$$

$$T(-1,-1) = -3+14 = \underline{\underline{11}}$$

So the minimum is at $(1,-1)$ with $\underline{\underline{T=7}}$

and the maximum is at $(-1,0)$ with $\underline{\underline{T=14}}$.