Homework 3 Soltions

① (9.17)

(a) If AB is symmetric, we must have

$$(AB)^T = B^T A^T = AB.$$

But since $A = A^T$, $B = B^T$ this means

$BA = AB$, so A and B commute.

(b) Suppose A and B are orthogonal: $A^{-1} = A^T$, $B^{-1} = B^T$.

Then $(AB)^T = B^T A^T = B^{-1} A^{-}$

So $(AB)^T AB = B^{-1} A^{-1} AB = B^{-1} B = I$,

whence $(AB)^T = (AB)^{-1}$, so AB is also orthogonal.

(c) If AB is Hermitian, must have

$$(AB)^\dagger = (AB)^{*T} = (A^* B^*)^T = B^{*T} A^{*T} = B^\dagger A^\dagger = AB.$$

Since $A = A^\dagger$ and $B = B^\dagger$, this means

$BA = AB$, or A and B commute.

(d) Suppose $A^{-1} = A^\dagger$ and $B^{-1} = B^\dagger$.

then $(AB)^\dagger = B^\dagger A^\dagger = B^{-1} A^{-1}$

so $(AB)^\dagger AB = B^{-1} A^{-1} AB = I$,

which means $(AB)^\dagger = (AB)^{-1}$ i.e.

AB is also unitary.

② (11.41)

$$H^\dagger = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}^\dagger = \begin{pmatrix} 2^* & (-i)^* \\ i^* & 2^* \end{pmatrix} = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} = H$$

So H is Hermitian.

To find its eigenvalues, we solve the characteristic equation $\det(\lambda 1 - H) = 0$, which is

$$\begin{vmatrix} \lambda-2 & -i \\ i & \lambda-2 \end{vmatrix} = (\lambda-2)^2 + i^2 = \lambda^2 - 4\lambda + 3 = 0 = (\lambda-3)(\lambda-1)$$

So the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

To find the eigenvectors, we solve the linear equations

$$\begin{pmatrix} 3-2 & -i \\ i & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1-2 & -i \\ i & 1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \qquad \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

whose solutions are the eigenvectors

$$\overline{b_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad \overline{b_2} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

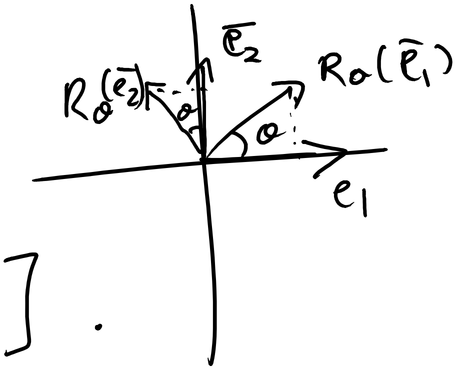
The matrix $U = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ is indeed unitary since the columns are orthogonal in the (complex) inner product.

So we have

$$\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = U^\dagger H U = U^{-1} H U = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

③ (a) Using trigonometry, we compute

$$[R_\theta(\bar{e}_1)] = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



and $$[R_\theta(\bar{e}_2)] = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

For an arbitrary $\bar{x} = x_1\bar{e}_1 + x_2\bar{e}_2$, we have by linearity:

$$R_\theta(\bar{x}) = R_\theta(x_1\bar{e}_1 + x_2\bar{e}_2) = x_1 R_\theta(\bar{e}_1) + x_2 R_\theta(e_2)$$

Which means that the coordinate vectors of both sides in the standard basis must be equal:

$$[R_\theta(\bar{x})] = x_1 [R_\theta(\bar{e}_1)] + x_2 [R_\theta(\bar{e}_2)]$$

$$= x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So, the standard matrix is

$$[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(b) If we write the new basis as columns of a matrix: $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

then the change of basis formula (Section 2 of lecture notes) tells us:

$$[R_\theta]_B = B^{-1}[R_\theta]B$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{1-(-1)} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta - \sin\theta & -\cos\theta - \sin\theta \\ \sin\theta + \cos\theta & -\sin\theta + \cos\theta \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{bmatrix} = [R_\theta]$$

Update: There was a mistake in the original hw3sol.pdf. I used $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ instead of $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Warning:

This is a degenerate case! In general, we will have $[R_\theta] \neq [R_\theta]_B$. What happened here is extremely rare and special, and the reason they come out to be equal is that B is *itself* a rotation matrix (for $\theta = \pi/4$), so $B^{-1}[R_\theta]B$

$$= [R_{-\pi/4}][R_\theta][R_{\pi/4}].$$

A very special fact about $\mathbb{R}^2$ is that all rotation matrices commute, so this becomes

$[R_\theta]$. This is not true in general!

Remember that in general, $[R_\theta] \neq [R_\theta]_B$!

(c) Observe that $[R_\theta]$ has orthonormal columns since $\cos^2\theta + \sin^2\theta = 1$ and

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = 0.$$

Thus, it is unitary, and question ④ tells us that $\|[R_\theta]x\| = \|x\|$ for all x.

Since $[R_\theta]_B = [R_\theta]$, the same is true for $[R_\theta]_B$.

(4) Since U is unitary, $U^{\dagger} = U^{-1}$

and $U^{\dagger} U = U U^{\dagger} = I$.

This means $(U^{\dagger})^{-1} = U$.

But $U = (U^{\dagger})^{\dagger}$, so

$(U^{\dagger})^{-1} = (U^{\dagger})^{\dagger}$ and $U^{\dagger}$ is

also unitary.

For the second part, observe that

$$\|x\|^2 = \sum_i x_i^* x = [\, x_1^* \cdots x_n^* \,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x^{\dagger} x.$$

Using this identity, we have for every unitary U:

$$\|Ux\|^2 = (Ux)^{\dagger}(Ux) = x^{\dagger} U^{\dagger} U x$$

$$= x^{\dagger} x = \|x\|^2$$

Since $U^{\dagger}$ is also unitary, the same is true for $\|U^{\dagger} x\|^2$.

(5) We have

$$S(1) = \frac{\partial}{\partial t} 1 = 0 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2$$

$$S(t) = \frac{\partial}{\partial t} t = 1 = 1 \cdot 1 + 0 \cdot t + 0 \cdot t^2$$

$$S(t^2) = \frac{\partial}{\partial t} t^2 = 2t = 0 \cdot 1 + 2 \cdot t + 0 \cdot t^2$$

Writing these as coordinate vectors in the basis $B = \{1, t, t^2\}$, we have

$$[S(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [S(t)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$[S(t^2)]_B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

The matrix $[S]_B$ has these as columns, so it is

$$[S]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3$$

so it has an eigenvalue of 0 with multiplicity 3. However, its rank is 2 so its nullspace has dimension 1, and there is only one eigenvector. Hence, it is <u>not</u> diagonalizable.

Just as with $S(p)$, we compute

$$T(1) = t\frac{\partial}{\partial t}(1) = 0 \quad = 0\cdot 1 + 0\cdot t + 0\cdot t^2$$

$$T(t) = t\frac{\partial}{\partial t} t = t \quad = 0\, 1 + 1\cdot t + 0\cdot t^2$$

$$T(t^2) = t\frac{\partial}{\partial t} t^2 = t\cdot 2t = 2t^2 = 0\cdot 1 + 0\cdot t + 2\cdot t^2$$

The coordinate vectors of these outputs in the B basis are:

$$[T(1)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T(t^2)]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

So $$[T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

This matrix is diagonalizable (since it is already diagonal): it has eigenvalues $0, 1, 2$ and eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, corresponding to the polynomials $1, t, t^2$.

(6) Suppose $A = CBC^{-1}$.

Multiplying on the left by C^{-1} we get

$$C^{-1}A = BC^{-1}$$

Multiplying on the right by C gives

$$C^{-1}AC = B.$$

But now $B = DAD^{-1}$ for $D = C^{-1}$ since $(D^{-1})^{-1} = C$.

So B is similar to A.

Suppose λ is an eigenvalue of A with eigenvector v, i.e. $Av = \lambda v$.

Then $CBC^{-1}v = \lambda v$, and multiplying both sides by C^{-1}, we find that this is equivalent to

$$BC^{-1}v = \lambda C^{-1}v$$

But now $Bw = \lambda w$ for $w = C^{-1}v$.

Thus every eigenvalue of A with eigenvector v is also an eigenvalue of B, with eigenvector $C^{-1}v$.

In particular, A and B have the same set of eigenvalues, and for every eigenvector v of A, $C^{-1}v$ is an eigenvector of B.

(7) Assume A and B are $n \times n$ (though this is also true for rectangular matrices).

Let A_i denote the i^{th} column of A

let A^i denote the i^{th} row of A.

Then the diagonal entries of AB are given by the dot products

$$(AB)(k,k) = A^k \cdot B_k$$

So: $$Tr(AB) = \sum_{k=1}^{n} A^k \cdot B_k$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} A_{ki} B_{ik}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ki} B_{ik}$$

$$= \sum_{i=1}^{n} A_i \cdot B^i$$

$$= \sum_{i=1}^{n} (BA)(i,i)$$

$$= Tr(BA).$$

If $A = CDC^{-1}$ for diagonal A

$$\mathrm{Tr}(A) = \mathrm{Tr}(CDC^{-1})$$

$$= \mathrm{Tr}(C(DC^{-1}))$$

$$= \mathrm{Tr}(DC^{-1}C)$$ applying $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

$$= \mathrm{Tr}(D)$$

$$= \sum_i \lambda_i$$, the sum of the eigenvalues.

<u>Bonus</u>: Observe that the characteristic polynomial does not change under a similarity transformation:

$$\det(\lambda I - A)$$

$$= \det(\lambda I - CDC^{-1})$$

$$= \det(\lambda CC^{-1} - CDC^{-1})$$

$$= \det(C(\lambda I - D)C^{-1})$$

$$= \det(C)\det(\lambda I - D)\det(C^{-1})$$

$$= \det(\lambda I - D)$$ since $\det(C^{-1}) = 1/\det(C)$.

If you expand $\det(\lambda I - A)$ as a polynomial in λ, the coefficient of λ^{n-1} is $\mathrm{Tr}(A)$.

Comparing these coefficients in $\det(\lambda I - A)$ and $\det(\lambda I - B)$, we have $Tr(A) = Tr(B)$.

(8) To find the eigenvalues we solve $\det(\lambda I - A) = 0$:

$$\begin{vmatrix} \lambda-1 & -3 \\ -3 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 9 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2)$$

So the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$.

The corresponding eigenvectors are:

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} v_1 = 0 \quad \text{and} \quad \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} v_2 = 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we may write $A = CDC^{-1}$ where $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$

Now:

$$A^5 - A^3 + A = CD^5C^{-1} - CD^3C^{-1} + CDC^{-1}$$

$$= C(D^5 - D^3 + D)C^{-1}$$

$$= C \begin{bmatrix} 4^5 - 4^3 + 4 & 0 \\ 0 & (-2)^5 - (-2)^3 - 2 \end{bmatrix} C^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1024 - 64 + 4 & 0 \\ 0 & -32 + 8 - 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 964 & 0 \\ 0 & -26 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 964 & 964 \\ -26 & 26 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 938 & 990 \\ 990 & 938 \end{bmatrix} = \begin{bmatrix} 469 & 495 \\ 495 & 469 \end{bmatrix}$$

For the trace, we have

$$\mathrm{tr}(A^{10}) = \mathrm{tr}(CD^{10}C^{-1})$$
$$= \mathrm{tr}(D^{10}) \text{ by } (\#7)$$
$$= 4^{10} + (-2)^{10} = 1049600$$

(9) (a)

$$\begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda-1) - 1$$
$$= \lambda^2 - \lambda - 1.$$

This has solutions $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$.

This gives the eigenvectors:

$$\begin{bmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} V_1 = 0 \qquad \begin{bmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} V_2 = 0$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$V_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Notice that $(V_1 \cdot V_2) = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) + 1$

$= \frac{1-5}{4} + 1 = 0$, which makes sense since A is symmetric.

(b) We have $A = CDC^{-1}$ for

$$C = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \\ & \frac{1-\sqrt{5}}{2} \end{bmatrix},$$

So $A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = CD^nC^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We calculate $C^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \frac{1}{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ -\left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix}$$

Finally, this gives

$$A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ -\left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & 0 \\ 0 & -\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}.$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right) \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix}.$$

(c) This gives the formula

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

$$\lim_{n\to\infty} \frac{f_n}{2^n} = \lim_{n\to\infty}\left(2\cdot\left(\frac{\frac{1+\sqrt{5}}{2}}{2}\right)^{n+1} - 2\cdot\left(\frac{\frac{1-\sqrt{5}}{2}}{2}\right)^{n+1}\right)$$

$$= 0$$

since $\left|\frac{\frac{1+\sqrt{5}}{2}}{2}\right| < 1$ and

$\left|\frac{\frac{1-\sqrt{5}}{2}}{2}\right| < 1.$

This formula looks slightly different from the one you might have seen in textbooks:

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}$$

The reason is that we started with $f_0 = 1$ whereas it is conventional to start with $f_0 = 0$.

(10) Writing $\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, we have in matrix notation:

$$\frac{\partial^2 \bar{x}(t)}{\partial t^2} = \begin{bmatrix} -k-h & h \\ h & -k-h \end{bmatrix} \bar{x}(t) .$$

Call this matrix A. Observe that

$$\begin{bmatrix} -k-h & h \\ h & -k-h \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -k \\ -k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and
$$\begin{bmatrix} -k-h & h \\ h & -k-h \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -k-2h \\ k+2h \end{bmatrix} = -(k+2h) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the eigenvalues are $\lambda_1 = -k$, $\lambda_2 = -(k+2h)$ with eigenvectors $\bar{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\bar{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Changing the basis to $\bar{b}_1, \bar{b}_2$, we write

$$\bar{x}(t) = a_1(t)\,\bar{b}_1 + a_2(t)\,\bar{b}_2$$

to obtain
$$\frac{\partial^2}{\partial t^2}\left(a_1(t)\,\bar{b}_1 + a_2(t)\,\bar{b}_2 \right) = a_1(t) A\bar{b}_1 + a_2(t) A\bar{b}_2$$
$$= a_1(t)\,\lambda_1 \bar{b}_1 + a_2(t)\,\lambda_2 \bar{b}_2 .$$

Comparing the coefficients of $\bar{b}_1$ and $\bar{b}_2$ on both sides, we obtain the decoupled equations

$$\frac{\partial^2}{\partial t^2} a_1(t) = -k\, a_1(t), \qquad \frac{\partial^2}{\partial t^2} a_2(t) = -(k+2h)\, a_2(t).$$

Since $\bar{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\bar{b}_1 + \frac{1}{2}\bar{b}_2$,

and $\frac{\partial \bar{x}}{\partial t}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\,\bar{b}_1 + 0\,\bar{b}_2$, the initial conditions are $a_1(0) = a_2(0) = 1/2$

and $\dot{a}_1(0) = \dot{a}_2(0) = 0$,

Thus, must have

$$a_1(t) = a_1(0)\cos(\sqrt{-\lambda_1}\, t) = \tfrac{1}{2}\cos(\sqrt{k}\, t)$$

and $$a_2(t) = a_2(0)\cos(\sqrt{-\lambda_2}\, t) = \tfrac{1}{2}\cos(\sqrt{k+2h}\, t)$$

* See the lecture notes for a more detailed explanation.

Switching back to the standard basis, thus gives the solution:

$$\bar{x}(t) = \tfrac{1}{2}\cos(\sqrt{k}\,t)\begin{bmatrix}1\\1\end{bmatrix} + \tfrac{1}{2}\cos(\sqrt{k+2h}\,t)\begin{bmatrix}1\\-1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\cos(\sqrt{k}\,t) + \cos(\sqrt{k+2h}\,t)\\ \cos(\sqrt{k}(t)) - \cos\sqrt{k+2h}\,t\end{bmatrix}.$$

The normal modes are

$$\tfrac{1}{2}\cos\sqrt{k}t\begin{bmatrix}1\\1\end{bmatrix} \quad\text{and}\quad \tfrac{1}{2}\cos\sqrt{k+2h}\begin{bmatrix}1\\-1\end{bmatrix}$$

which correspond to the masses moving with identical and opposite velocities, respectively.

As $h \to 0$, the solution tends to $\bar{x}(t) = \begin{bmatrix}\cos\sqrt{k}\,t\\0\end{bmatrix}$,

which makes sense since the middle spring has no effect for $h=0$ and $x_2(0)=0$;

So only the first mass oscillates.

As $h \to \infty$, we have $\cos(\sqrt{k+2h}\, t)$

$$\approx \cos(\sqrt{2h}\, t),$$

So the motion is a superposition of a low frequency oscillation in the first mode and a very high frequency oscillation in the second mode.

(The reason for this perhaps unexpected behavior is that the initial condition was asymmetric: $x_1(0)=1$ whereas $x_2(0)=0$.

If instead we had had $x_1(0)=x_2(0)=1$, the middle spring would not be compressed at all, and we would only have the first mode).