Math 121A Spring 2015, Homework 3

Due February 13 at 10am

1. Chapter 3 Section 9: 17.

2. Chapter 3 Section 11: 41.

3. Let $R_\theta : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation which rotates its input by an angle of θ counterclockwise.

   (a) Find the matrix $[R_\theta]$ of R_θ in the standard basis.

   (b) Find the matrix $[R_\theta]_B$ of R_θ in the basis $B = \{(1,1)^T, (-1,1)^T\}$.

   (c) Show that $\|[R_\theta]x\| = \|x\|$ for every $x \in \mathbf{R}^2$. Is this also true for $[R_\theta]_B$?

4. Suppose U is an $n \times n$ unitary matrix. Show that $U^\dagger$ is also unitary, and that $\|Ux\| = \|U^\dagger x\| = \|x\|$ for all $x \in \mathbf{C}^n$. *Hint: remember that* $\|x\|^2 = \sum_i |x_i|^2 = \sum_i x_i^* \cdot x_i$ *for complex vectors.*

5. Let P_2 be the vector space of polynomials in t of degree at most 2, with complex coefficients. Consider the linear operators $S : P_2 \to P_2$ and $T : P_2 \to P_2$ defined by:

$$S(p) = \frac{\partial}{\partial t} p \qquad \text{and} \qquad T(p) = t \frac{\partial}{\partial t} p.$$

   Write the matrices of S and T with respect to the standard basis of monomials $\{1, t, t^2\}$. Find the eigenvalues and eigenvectors of S and T. Is either one diagonalizable?

6. We say that A is *similar* to B if there is an invertible matrix C such that $A = CBC^{-1}$. Show that if A is similar to B then B is similar to A. Show that similar matrices have the same eigenvalues. What is the relationship between the eigenvectors of A and those of B?

7. The *trace* of a matrix is defined as $\mathrm{tr}(A) = \sum_i A_{ii}$, the sum of the diagonal entries. Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Use this to conclude that if A is diagonalizable, then $\mathrm{tr}(A)$ is equal to the sum of the eigenvalues of A.

   Bonus: Show that $\mathrm{tr}(A)$ *is always equal to the sum of the eigenvalues, even if* A *is not diagonalizable.*

8. Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Use diagonalization to compute $A^5 - A^3 + A$, as well as $\mathrm{tr}(A^{10})$.

9. The Fibonacci sequence $1, 1, 2, 3, 5, \ldots$ is defined by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, with initial conditions $f_0 = 1$ and $f_1 = 1$. In matrix notation, this may be written as:

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}.$$

   (a) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

   (b) Use this to compute an expression for

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   (c) Use this to derive a formula for f_n. What is $\lim_{n\to\infty} \frac{f_n}{2^n}$?

10. Consider the arrangement of masses and springs shown in figure 12.1 (page 165), and also discussed in class. In class, we assumed that the masses were both equal to 1 and that the spring constants were all equal to k. Now consider the situation where the spring constants are k, h and k, i.e., the middle spring is different.

Letting $x_1(t)$ and $x_2(t)$ denote the positions of the masses at time t, this system is described by the second-order differential equations:

$$\frac{\partial^2 x_1(t)}{\partial t^2} = -kx_1(t) - h(x_1(t) - x_2(t)),$$

$$\frac{\partial^2 x_2(t)}{\partial t^2} = -kx_2(t) - h(x_2(t) - x_1(t)),$$

Use diagonalization to decouple and solve these equations, for initial conditions $x_1(0) = 1$, $x_2(0) = 0$, and $\dot{x_1}(0) = \dot{x_2}(0) = 0$. What are the normal modes? What happens as $h \to 0$? As $h \to \infty$?