Homework 2 Solutions

① (a) Diverges: by the preliminary test, because $\lim_{n\to\infty} \sin(\log n) \neq 0$ (actually does not exist).

Proving this is kind of tricky because we only care about $\sin(\log n)$ for integer n, and thus may have different limiting behavior from $\sin(\log x)$ for real x as $x \to \infty$ (for example, $\sin(2\pi x)$ oscillates as $x \to \infty$ but $\sin(2\pi n) = 0$ for all integers n).

Proof: Observe that $\log(n+1) - \log(n)$

$$= \log\left(\frac{1+n}{n}\right) = \log\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$$

$\leq \frac{1}{n}$ since the series alternates and terms are decreasing and tending to zero.

Thus, for $n \geq 100$ (say), the sequence $b_n = \log n$ is increasing by increments of at most $\frac{1}{100}$, i.e. $b_{n+1} - b_n \leq \frac{1}{100}$.

Since $b_n \to \infty$, this means that for every real $x \geq 100$ there is some $|b_n - x| \leq \frac{1}{100}$, which implies (in particular) that b_n is in the interval $\left[(2k+1)\frac{\pi}{2} - \frac{1}{100}, (2k+1)\frac{\pi}{2} + \frac{1}{100}\right]$ for infinitely many integers k, so $\lim_{n\to\infty} \sin(b_n) \neq 0$.

① (b) $\displaystyle\sum_{n=1}^{\infty} \log\left(n \sin\left(\tfrac{1}{n}\right)\right) = \sum_{n=1}^{\infty} \log\left(n\left(\tfrac{1}{n} - \tfrac{1}{3!\,n^3} + O\left(\tfrac{1}{n^5}\right)\right)\right)$

$$= \sum_{n=1}^{\infty} \log\left(1 - \frac{1}{6n^2} + O\left(\frac{1}{n^4}\right)\right)$$

$$= \sum_{n=1}^{\infty} -\left(\frac{1}{6n^2} - O\left(\frac{1}{n^4}\right)\right) - \frac{1}{2}\left(\frac{1}{6n^2} - O\left(\frac{1}{n^4}\right)\right)^2 - \frac{1}{3}\left(\frac{1}{6n^2} - O\left(\frac{1}{n^4}\right)\right)^3 \cdots$$

$$= \sum_{n=1}^{\infty} -\frac{1}{6n^2} + O\left(\frac{1}{n^4}\right)$$ since all further terms are of degree at least 4

which converges since $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n^4}$ converge.

People have been confused about how to use $O(\,)$ notation. One simple way to think of it is this:
Since $f = O(g)$ means $\lim_{n\to\infty} \left|\frac{f}{g}\right|$ is finite,
$f = O(g)$ just means there is a constant C such that $|f| \le C|g|$ for sufficiently large n. So the $O(\,)$ just "hides" this constant, and ignores lower order terms, eg:

$$\frac{1}{6n^2} = O\left(\frac{1}{n^2}\right), \qquad \frac{1}{6n^2} - \frac{1}{5!\,n^4} = \frac{1}{6n^2} - O\left(\frac{1}{n^4}\right).$$

Note that this notation does not care if f is negative or positive, so we can even write

$-\frac{1}{6n^2} = O\left(\frac{1}{n^2}\right)$, which is useful when we don't care about the sign.

① (c) Notice that for $\log n \geq 3$, we have

$$\frac{1}{(\log n)^{\log n}} \leq \frac{1}{3^{\log n}} = \frac{1}{e^{\log n \log 3}}$$
$$= \frac{1}{n^{\log 3}}.$$

Since $\log 3 > 1$, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\log 3}}$ converges.

Let N be the first integer greater than e^3, so that $\log N \geq 3$.

Writing $$\sum_{n=2}^{\infty} \frac{1}{\log n^{\log n}} = \sum_{n=2}^{N} \frac{1}{\log n^{\log n}} + \sum_{n=N+1}^{\infty} \frac{1}{\log n^{\log n}}$$

and comparing the latter part with $\sum_{N+1}^{\infty} \frac{1}{n^{\log 3}}$, we conclude that the series converges.

② Section 10

(4) $$\rho = \lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{x^{2n}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{x^2}{2} \left(\frac{n}{n+1}\right)^2 \right| = \frac{x^2}{2}.$$

So the series converges when $|x| < \sqrt{2}$.

at $x = \sqrt{2}$: $\sum_{n=1}^{\infty} \frac{(\sqrt{2})^{2n}}{2^n n^2} = \sum \frac{1}{n^2}$ converges.

$x = -\sqrt{2}$: $\sum_{n=1}^{\infty} \frac{(-\sqrt{2})^{2n}}{2^n n^2} = \sum \frac{1}{n^2}$ converges.

(5) $$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{x^n} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{1}{(n+1)^2} \right| = 0 \quad \text{for every } x.$$

(11) $$\rho = \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}\left(\frac{x}{5}\right)^{n+1}}{\frac{1}{n}\left(\frac{x}{5}\right)^n} \right| = \left|\frac{x}{5}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right|$$

$$= \left|\frac{x}{5}\right|.$$

So the series converges for $|x|<5$.

at $\underline{x=5}$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{5}{5}\right)^n$ diverges

$\underline{x=-5}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(-\frac{5}{5}\right)^n$ converges by the alternating test.

So the interval of convergence is $[5,5)$.

(22) Write $y = \frac{1}{x}$. Then

$$\sum_0^{\infty} \frac{n!\,(-1)^n}{x^n} = \sum_0^{\infty} n!(-1)^n y^n. \quad \text{Then}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!\, y^{n+1} (-1)^{n+1}}{n!\, y^n (-1)^n} \right| = |y| \lim_{n\to\infty} |n+1| = \infty \quad \text{for every } y$$

So the series diverges everywhere.

③ Section 13

(9)

$$\frac{1+x}{1-x} = 1+x \cdot \frac{1}{1-x} = (1+x)(1+x+x^2 - -)$$
$$= 1+2x+2x^2+2x^3 \cdots$$
$$= 1+2\sum_{n=1}^{\infty} x^n$$

(12)

$$\int_0^x \cos t^2 \, dt = \int_0^x 1 - \frac{(t^2)^2}{2!} + \frac{(t^2)^4}{4!} \cdots \, dt$$
$$= \int_0^x 1 \, dt - \int_0^x \frac{t^4}{2!} dt + \int_0^x \frac{t^8}{4!} dt \cdots$$
$$= t\Big|_0^x - \frac{t^5}{5 \cdot 2!}\Big|_0^x + \frac{t^9}{9 \cdot 4!}\Big|_0^x \cdots$$
$$= x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}$$

(23) Substitute $y = x+x^2$, and note that $|y|<1$ for x in a small interval around 0, which is what we are interested in for Maclaurin series.

Thus we may use this substitution in the geometric series:

$$\frac{1}{1+y} = 1-y+y^2 \cdots$$
$$= 1-(x+x^2)+(x+x^2)^2-(x+x^2)^3$$

$$= 1 - (x+x^2) + x^2(x+1)^2 - x^3(x+1)^3 \dots$$

$$= 1 - (x+x^2) + (x^4+2x^3+x^2) - (x^6+3x^5+3x^4+x^3) \dots$$

$$= 1 - x + x^3 - x^4 + O(x^5) \dots$$

(24) $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{1-\frac{x^2}{2!}+\frac{x^4}{4!}\dots} = 1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{24} + \dots$

long division:

$$\begin{array}{r|l} & 1 + \frac{x^2}{2!} + \frac{5x^4}{24} \dots \\ \hline 1-\frac{x^2}{2!}+\frac{x^4}{4!}\dots & 1 \\ & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ \hline & \frac{x^2}{2!} - \frac{x^4}{4!} + \dots \\ & \frac{x^2}{2!} - \frac{x^4}{4} + \frac{x^6}{2\cdot 4!} \dots \\ \hline & x^4\left(\frac{1}{4}-\frac{1}{24}\right) + \dots \end{array}$$

(28) $\sin(\log(1+x)) = \sin\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + O(x^5)\right)$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)\right) - \frac{1}{3!}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)\right)^3 + O(x^5)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{3!} - \frac{x^4}{4} - \frac{1}{3!}\left(x \cdot x \cdot \left(-\frac{x^2}{2}\right)\right) + O(x^5)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{6} + O(x^5)$$

④ Section 14

(3) The series $(1+x)^{1/2} = 1 + (\frac{1}{2})x + \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2!} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})x^3}{3!} \dots$

is alternating and the $|a_n|$ are decreasing:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n)\,x^{n+1}}{(n+1)!}}{\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-(n-1))\,x^n}{n!}} \right| = \left|\frac{(\frac{1}{2}-n)x}{n+1}\right| < 1$$

for $|x| < 1$.

So the error is bounded by the next term:

$$err(x) = \left| \sqrt{1+x} - \left(1+\frac{x}{2}\right) \right| \leq \left| \frac{(\frac{1}{2})(-\frac{1}{2})x^2}{2} \right| = \frac{1}{8}x^2.$$

This is an increasing function on $[0, \frac{1}{2}]$, so its maximum is achieved at the end point : $\frac{1}{8}(\frac{1}{2})^2 = \frac{1}{32} = 0.03125$
$< 0.032.$

Thus $|err(x)| < 0.032$ for $0 \leq x < \frac{1}{2}$.

(5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$ is alternating and the terms decrease in magnitude for $|x| < 1$, so

$err(x) = \left| 1 - \frac{x^2}{2} - \cos x \right| < \frac{x^4}{4!}$ which is increasing on $[0, \frac{1}{2}]$

and decreasing on $[-\frac{1}{2}, 0]$, so that for $|x| < \frac{1}{2}$ we have

$$\frac{x^4}{4!} < \frac{(\frac{1}{2})^4}{24} = \frac{1}{384} = 0.0026\ldots < 0.003$$

(8) The remainder is

$$R_3(x) = \sum_{n=4}^{\infty} \frac{x^n}{n^4}$$. Since the coefficients are decreasing in magnitude, we have

$$|R_3(x)| \leq \frac{1}{4^4} \sum_{n=4}^{\infty} x^n$$

$$= \frac{x^4}{4^4} \sum_{n=0}^{\infty} x^n = \frac{x^4}{256(1-x)}.$$

It is not immediately obvious that the maximum of this function over $[-1/2, 1/2]$ is achieved at the endpoints, because it is not obviously increasing or decreasing on $[-1/2, 0]$ (the numerator is decreasing and so is the denominator).

But we can estimate

$$\max_{x \in [-1/2, 1/2]} \left| \frac{x^4}{256(1-x)} \right| \leq \frac{1}{256} \max_{[-1/2, 1/2]} |x^4| \cdot \max_{[-1/2, 1/2]} \left| \frac{1}{1-x} \right|$$

$$= \frac{1}{256} \left(\frac{1}{2}\right)^4 \cdot \frac{1}{(1-1/2)}$$

$$= \frac{8}{256} = \frac{1}{32}$$

(alternately, we could observe that $R_3(-x) \leq R_3(x)$ for every $x \in [0, 1/2]$, so we know the max must be achieved on $[0, 1/2]$, on which $R_3(x)$ is increasing so it is sufficient to check the endpoint $x = 1/2$).

(5) Section 15

(2) $(1+x^4)^{-1/2} - \cos x^2$

$$= \cancel{1} + \cancel{(-1/2)x^4} + \frac{(-1/2)(-1/2-1)}{2!}x^8 + O(x^{12})$$

$$- \left(\cancel{1} - \cancel{\frac{x^4}{2!}} + \frac{x^8}{4!} + O(x^{12})\right)$$

$$= \frac{3x^8}{16} - \frac{x^8}{24} + O(x^{12})$$

Update (2/13): this should be $\frac{3x^8}{8}$

$$= \cancel{\frac{7x^8}{48} + O(x^{12})}. \qquad = \frac{(18-2)}{48}x^8 = \frac{x^8}{3}$$

which is $\approx \frac{7}{48}(0.012)^8 \qquad = \frac{(0.012)^8}{3}$

$$= \frac{7 \cdot 12^8}{48} \times 10^{-24} \qquad = 1.433 \times 10^{-16}$$

$$= \cancel{7.52 \times 10^{-16}}.$$

(3) First we calculate

$$\frac{1}{x^3}\log(1+x^2e^x) = \frac{1}{x^3}\log\left(1 + x^3 + \frac{x^4}{1!} + \frac{x^5}{2!} + \frac{x^6}{3!} + O(x^7)\right)$$

$$= \frac{1}{x^3}\left(\left(x^3+x^4+\frac{x^5}{2}+\frac{x^6}{6}+O(x^7)\right) - \frac{1}{2}\left(x^3+x^4+\frac{x^5}{2}+\frac{x^6}{6}+O(x^7)\right)^2 + O(x^9)\right)$$

$$= \frac{1}{x^3}\left(x^3+x^4+\frac{x^5}{2}+\frac{x^6}{6}+O(x^7) - \frac{x^6}{2} + O(x^7)\right)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} + O(x^4).$$

The other term is $e^{\sin x}$

$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)^2$$

$$+ \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)\right)^3$$

$$+ O(x^4)$$

$$= 1 + \left(x - x^3 + O(x^5)\right) + \frac{1}{2}\left(x^2 + O(x^4)\right)$$

$$+ \frac{1}{6}\left(x^3 + O(x^4)\right) + O(x^4)$$

$$= 1 + x + \frac{x^2}{2} - \frac{5}{6}x^3 + O(x^4).$$

So the difference is

$$e^{\sin x} - \left(\frac{1}{x^3}\right)\log(1 + x^3 e^x) = \left(1 + x + \frac{x^2}{2} - \frac{5}{6}x^3 + O(x^4)\right)$$

$$- \left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} + O(x^4)\right)$$

$$= -\frac{1}{2}x^3 + O(x^4).$$

Plugging in $x = 3.5 \times 10^{-4}$ gives

$$-\frac{1}{2}(3.5)^3 \times 10^{-12} = \underline{\underline{-6.125 \times 10^{-12}}}$$

(6) $\frac{\partial^4}{\partial x^4}\left(x^3 - \frac{x^6}{2} + \frac{x^9}{3} \cdots \right)$

$$= 0 - \frac{6\cdot5\cdot4\cdot3\cdot x^2}{2} + \frac{9\cdot8\cdot7\cdot6}{3}x^5 \cdots$$

$= 0$ at $x = 0$.

(11) $$\lim_{x\to0} \frac{\sin^2 2x}{x^2} = \lim_{x\to0} \frac{\left(2x - \frac{8x^3}{3!} + O(x^5)\right)^2}{x^2}$$

$$= \lim_{x\to0} \frac{4x^2 + O(x^4)}{x^2} = \underline{\underline{4}}.$$

(23a) Write $\frac{1}{x} - \frac{1}{e^x - 1} = \frac{e^x - 1 - x}{(e^x-1)x}$

$$= \frac{\frac{x^2}{2!} + O(x^3)}{\left(x + \frac{x^2}{2} + O(x^3)\right)x} = \frac{\frac{x^2}{2} + O(x^3)}{x^2 + O(x^3)}$$

So the limit is $\lim_{x\to0} \frac{\frac{x^2}{2} + O(x^3)}{x^2 + O(x^3)} = \underline{\underline{1/2}}.$

(28) $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1+\frac{1}{2}\frac{v^2}{c^2}+\cdots$

So $mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} = mc^2 + \frac{1}{2}mv^2 + \cdots$

So the second term is the kinetic energy.

(29) (a) $\frac{F}{W} = \frac{\sin\theta}{\cos\theta} = \sin\theta\sec\theta$

$$= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)\left(1+\frac{x^2}{2}+\frac{5}{24}x^4\cdots\right)$$

by Section 13 #24

$$= x+\frac{x^3}{2}-\frac{x^3}{6}+\frac{x^5}{5!}+\frac{5}{24}x^5+O(x^7)$$

$$= x+\frac{1}{3}x^3+\frac{1}{20}x^5+O(x^7)\cdots$$

(b) $\frac{x}{l} = \sin\theta$. Let $y=\frac{x}{l}$.

So $\frac{F}{W} = \frac{y}{\sqrt{1-y^2}} = y\left(1-y^2\right)^{-1/2}$

$$= y\left(1+\left(-\tfrac{1}{2}\right)\left(-y^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-y^2\right)^2-\cdots\right)$$

$$= y\left(1+\frac{y^2}{2}+\frac{3}{8}y^4\cdots\right) = \frac{x}{l}+\frac{1}{2}\left(\frac{x}{l}\right)^3+\frac{3}{8}\left(\frac{x}{l}\right)^5\cdots$$

(6)

$$\int_1^3 \frac{\sin x}{x}\,dx$$

$$= \int_1^3 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)}{x}\,dx$$

$$= \int_1^3 1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{7!} + \frac{x^8}{9!} \cdots$$

$$= x\Big|_1^3 - \frac{x^3}{3\cdot 6}\Big|_1^3 + \frac{x^5}{5\cdot 120}\Big|_1^3 - \frac{x^7}{7\cdot 7!}\Big|_1^3 + \frac{x^9}{9\cdot 9!}\Big|_1^3 \cdots$$

Notice that the series is alternating and the terms are decreasing after the two term. Thus the error of the 4-term approximation is at most

$$\frac{x^9}{9\cdot 9!}\Big|_1^3 = \frac{3^9 - 1}{9\cdot 9!}$$

$$\leq \frac{3^7}{9!} \leq \underline{\underline{0.01}}.$$

⑦ We have $\int_1^N \log x \, dx = \log x \cdot x \Big|_1^N - x\Big|_1^N$

$$= N\log N - (N-1).$$

$$\log\left((N-1)!\right) = \sum_{n=1}^{N-1} \log(n) \leq N\log N - (N-1)$$

Taking exponentials:

$$(N-1)! \leq \frac{e^{N\log N}}{e^{N-1}} = \frac{N^N}{e^N} \cdot e$$

Thus $\boxed{N! \leq eN \cdot \left(\frac{N}{e}\right)^N.}$

For the lower bound, we have

$$N\log N - (N-1) \leq \sum_{n=2}^{N} \log(n) = N!$$

Taking exponentials as before:

$$\boxed{e\left(\frac{N}{e}\right)^N \leq N!},$$ as desired.