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Math 121A Spring 2015
Homework 1 Solutions

Section 1.1

(12) The amount of impurity removed after k stages is

$$S_k = \frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^k}$$

As the process goes on, the amount of impurity removed approaches

$$S = \lim_{k\to\infty} S_k = \frac{1}{n} + \frac{1}{n^2} + \cdots$$

$$= \frac{1/n}{1-1/n}$$ since this is a geometric series.

For $\underline{n=2}$: $S = \frac{1/2}{1-1/2} = \underline{1}$

So by taking large enough k we get as close to 1 as we want.

for $\underline{n=3}$: $S = \frac{1/3}{1-1/3} = 1/2$, so $1/2$ will always remain.

(15)

At each step, $1/4$ of the total nonblank area is removed (since $1/4$ of each triangle is removed). So the sequence of nonblank areas is

$$1, \; 3/4, \; (3/4)^2, \; \ldots \; (3/4)^n, \; \ldots$$

The corresponding blank areas removed are

$$1/4, \; 1/4\,(3/4), \; 1/4\,(3/4)^2, \; \ldots, \; 1/4\,(3/4)^n \ldots$$

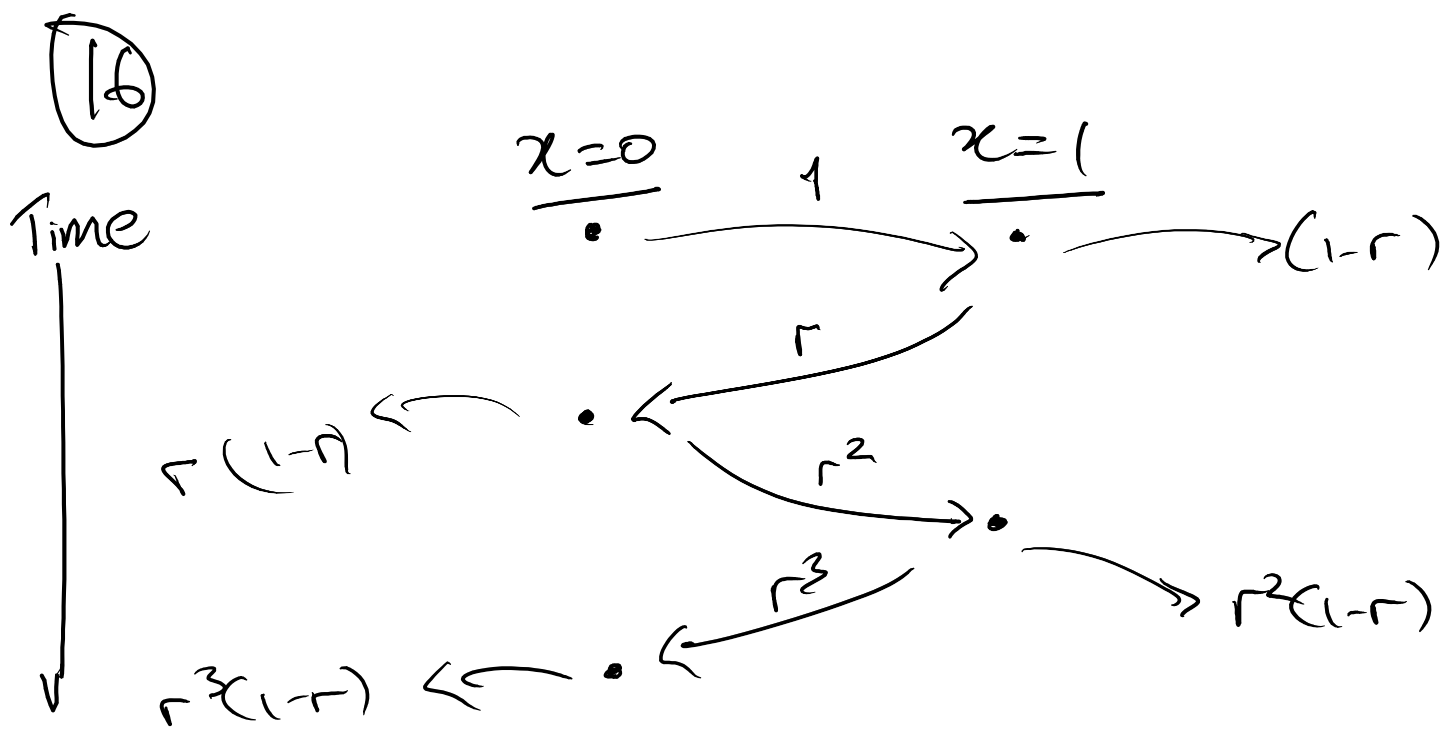
So the total blank area is

$$\frac{1}{4} \sum_{n=0}^{\infty} (3/4)^n = \frac{1}{4} \frac{1}{1-3/4} = \underline{\underline{1}}.$$

This makes sense since the non-blank area tends to zero as $n \to \infty$:

$$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0.$$

(16)



The fractions of particles arriving at $x=1$ at odd time steps (starting at time $=1$) are

$1, r^2, r^4, \ldots\ldots$

So the total # particles escaping at $x=1$ is

$$S = (1-r) + (1-r)r^2 + (1-r)r^4 \ldots$$

this is a geometric series where each successive term is multiplied by r^2, so

$$S = (1-r)\sum_{n=0}^{\infty}(r^2)^n = \frac{1-r}{1-r^2} = \frac{1}{1+r}.$$

Similarly at $x=0$ we have

$$S' = (1-r)r + (1-r)r^3 + \cdots$$
$$= (1-r)r\sum_{n=0}^{\infty}(r^2)^n = \frac{(1-r)r}{1-r^2} = \frac{r}{1+r}.$$

Section 2

(4) There are many ways to do this.
Here is one:

Observe that $2^n > n^3$
for $n \geq 10$: $2^{10} = 1024$
whereas $10^3 = 1000$, and after
this the left hand side doubles
each time we increment n,
while the right hand side
increases by $\left(\frac{n+1}{n}\right)^3 < 2$.

Thus $\frac{2^n}{n^2} > n$ for $n \geq 10$

and $\lim_{n\to\infty} \frac{2^n}{n^2} \geq \lim_{n\to\infty} n = \infty$

(6) Observe that

$$\frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdots n \cdot n}{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}$$

$$= \left(\frac{n}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{n}{n-2}\right) \cdots \left(\frac{n}{2}\right)\left(\frac{n}{1}\right)$$

$$\geq 1 \cdot 1 \cdot 1 \cdots \left(\frac{n}{1}\right) = n.$$

So $\lim\limits_{n\to\infty} \frac{n^n}{n!} \geq \lim\limits_{n\to\infty} n = \infty.$

Note: All logarithms are base e, so $\log n = \ln n$.

(7) The $\frac{1}{\log n}$ in the exponent seems strange, so instead of considering $a_n = (1+n^2)^{1/\log n}$ we consider the sequence

$$b_n = \log(a_n) = \frac{1}{\log n} \log(1+n^2).$$

Since $\log$ is a continuous function, we have $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \log(a_n)$

$$= \log\left(\lim_{n\to\infty} a_n\right).$$

We now evaluate

$$\lim_{n\to\infty} \frac{\log(1+n^2)}{\log n}$$

$$= \lim_{n\to\infty} \frac{\log n^2}{\log n} + \lim_{n\to\infty} \left[\frac{\log(1+n^2) - \log n^2}{\log n}\right]$$

$$= \lim_{n\to\infty} \frac{2\log n}{\log n} + \lim_{n\to\infty} \frac{\log\left(\frac{1+n^2}{n^2}\right)}{\log n}$$

$$= 2 + \frac{\log 1}{\log n} = \underline{\underline{2}}$$

Thus

$$\log\left(\lim_{n\to\infty} a_n\right) = 2$$

So $\lim_{n\to\infty} a_n = \underline{\underline{e^2}}$.

(8) Observe that

$$\frac{(n!)^2}{(2n)!} = \frac{(n(n-1)\cdots 2\cdot 1)\,\cancel{(n!)}}{(2n)(2n-1)\cdots\cdots(n+1)\,\cancel{n!}}$$

$$= \left(\frac{n}{2n}\right)\cdot\left(\frac{n-1}{2n-1}\right)\cdot\left(\frac{n-2}{2n-2}\right)\cdots\left(\frac{2}{n+2}\right)\cdot\left(\frac{1}{n+1}\right)$$

$$\leq \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\cdots\frac{1}{2} = \frac{1}{2^n}$$

Since each term in the product is $\leq 1/2$.

Thus $\displaystyle\lim_{n\to\infty}\frac{(n!)^2}{(2n)!} \leq \lim_{n\to\infty}\frac{1}{2^n} = 0.$

Section 5

(3) The terms are $a_n = \dfrac{\sqrt{n+1}}{n}$,

beginning with $n=1$.

Since $\displaystyle\lim_{n\to\infty}\frac{\sqrt{n+1}}{n} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n^2}}$

$= 0$,

further testing is required.

(11) By the definition of partial sums, we have

$$S_n - S_{n-1} = a_n.$$

Taking limits of both sides, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1}$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

since both limits exist

$$= S - S = 0.$$

(the point here is that changing the index n to $(n-1)$ doesn't change the limit : $n\to\infty$ is the same thing as $n-1\to\infty$)

Section 6

Ignore the book's comments about not using lower limits. Instead, just use however large a lower limit you want.

This is ok because the convergence of a series is unaffected by any finite number of initial terms.

(7) Since $\frac{\partial}{\partial x} \log\log x = \frac{1}{\log x} \cdot \frac{1}{x}$

we have $\displaystyle\int_a^\infty \frac{1}{x\log x}\,dx$

$$= \log\log x \Big|_a^\infty$$

$= \infty$, taking any $a > 1$ (say).
(The reason we need sufficiently large a is that $\log\log 1 = \log 0 = -\infty$).

(15) When $\underline{p > 1}$: $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{(-p+1)} \frac{1}{x^{p-1}} \Big|_1^\infty$

$$= \frac{1}{p-1}$$

So convergent.

$\underline{p < 1}$ $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} \Big|_1^\infty$

$= \infty$

So divergent.

When $p = 1$

$$\int_1^\infty \frac{1}{x}\,dx = \log x \Big|_1^\infty$$

$$= \infty$$

So divergent.

(17) Since $0 \le e^{-x^2} \le e^{-x}$ $\forall x \geq 1$, we have

$$\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^\infty$$

$$= e$$

So $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

(25) The ratios of successive terms tend to

$$\rho = \lim_{n\to\infty} \left| \frac{e^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{e^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{e}{\sqrt{n+1}} \right| \to 0$$

so convergent.

(28) $$\rho = \lim_{n\to\infty} \left| \frac{(n+1)!\,(2n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n!\,(2n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4n^3 + O(n^2)}{27n^3 + O(n^2)} \right|$$

See the handout for Lecture 5 if you don't understand $O(\cdot)$ notation.

$$= \frac{4}{27}$$ so the series converges.

(30) $\rho > 1$ Case

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho < 1.$

Let σ be any number such that $\rho < \sigma < 1$, let $\epsilon = |\sigma - \rho|$.

By the definition of a limit, there is some finite N such that

$$\left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \epsilon$$

for all $n \geq N$. In particular, this implies

$$\left| \frac{a_{n+1}}{a_n} \right| < \rho + \epsilon = \sigma \text{ for all } n \geq N.$$

We now use this N to split the series into two parts:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$= S_N + R_N .$$

S_N is a finite sum so it is equal to a finite number.

R_N is a series, the "tail" of $\sum a_n$, and by our choice of N we know that its terms satisfy

$$|a_{N+2}| \leq \sigma |a_{N+1}|$$
$$|a_{N+3}| \leq \sigma |a_{N+2}| \leq \sigma^2 |a_{N+1}|$$
$$\vdots$$
$$|a_{N+k}| \leq \sigma^{k-1} |a_{N+1}|$$
$$\vdots$$

We will show that R_N converges absolutely. Since $|a_{N+k}| \leq \sigma^{k-1} |a_{N+1}|$, we apply the comparison test with the geometric series

$$G = |a_{N+1}| + |a_{N+1}|\sigma + |a_{N+1}|\sigma^2 \cdots = \sum_{k=1}^{\infty} |a_{N+1}| \sigma^{k-1}$$

Since $\sigma < 1$, we know that the geometric series converges.

Thus, $R_N = \sum_{k=1}^{\infty} a_{N+k}$ converges absolutely,

and $\sum_{n=1}^{\infty} a_n$ also converges.

$\underline{\rho > 1 \text{ case}}$: Let $\sigma \in (1, \rho)$.

By a similar argument as before, choose N so that $\left|\frac{a_{n+1}}{a_n}\right| \geq \sigma$ for all $n \geq N$.

In particular, this means

$$|a_{N+2}| \geq \sigma |a_{N+1}|$$

$$\vdots$$

$$|a_{N+k}| \geq \sigma^{k-1} |a_{N+1}|$$

for all k.

But now $\lim_{k \to \infty} |a_{N+k}| \geq \lim_{k \to \infty} |a_{N+1}| \sigma^{k}$

$= \infty$

So by the preliminary test the series diverges.

(35) We will compare this to the series $b_n = 1/n^2$.

Observe that:

$$\lim_{n\to\infty} \left| \left(\frac{(n-\log n)^2}{5n^4-3n^2+1}\right) \Big/ \left(\frac{1}{n^2}\right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2 - 2n\log n + \log^2 n}{5n^2 - 3 + \frac{1}{n^2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1 - \frac{2\log n}{n} + \frac{\log^2 n}{n^2}}{5 - \frac{3}{n^2} + \frac{1}{n^4}} \right|$$

$$= \frac{1}{5}.$$

Since $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges, the given series must also converge.

Section 7

4 The terms are alternating, so we check that they are:

decreasing
$$\left|\frac{3^{n+1}}{(n+1)!}\right| < \left|\frac{3^n}{n!}\right|$$
$$\iff \frac{1}{n+1} \leq \frac{1}{3}$$
which is true for $n \geq 2$

approaching zero:

Observe that
$$\left|\frac{3^n}{n!}\right| = \left|\frac{3\cdot 3\cdot 3 \cdots 3\cdot 3\cdot 3}{n\cdot(n-1)\cdots 3\cdot 2\cdot 1}\right|$$
$$= \frac{9}{2}\left|\frac{3}{n}\right|\cdot\left|\frac{3}{n-1}\right|\cdots\left|\frac{3}{4}\right|\cdot\left|\frac{3}{3}\right|$$
$$\leq \frac{9}{2}\left|\frac{3}{n}\right| = \frac{27}{2n} \to 0$$
as $n \to \infty$.

Thus we can apply the alternating test and the series converges.

(9) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges.

Let $b_n = a_n + |a_n|$, guaranteeing that $b_n \geq 0$.

Since $b_n = a_n + |a_n| \leq 2|a_n|$,

and $\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|$ converges,

we can apply the comparison test to conclude that $\sum_{n=1}^{\infty} b_n$ converges.

But the term-wise difference of two convergent series is convergent so

$$\sum_{n=1}^{\infty} (b_n - |a_n|) = \sum_{n=1} a_n$$

must also converge.

Section 9

(9) We showed in section 2 problem 6 that $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$.

Thus, this series diverges by the

Thus, this series diverges by the preliminary test.

(16) Observe that

$$a_n = \frac{2+(-1)^n}{n^2+7} \leq \frac{3}{n^2}$$

for all $n \geq 1$.

Since $\sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$

converges, the comparison test tells us that $\sum_{n=1}^{\infty} a_n$ also converges.

(17) Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^3 + O(n^2)}{27n^3 + O(n^2)} \right|$$

$$= \frac{1}{27} < 1$$

So the series converges.

(21) Ratio test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{2n+3}\right|$$

$$= \frac{1}{2} < 1$$

So it converges.

Extra Problem (#7)

Suppose $\sum_{n=1}^{\infty} a_n$ is given.

Consider the sequence

$$b_0 = a_1$$

$$b_1 = a_2 + a_3$$

$$b_2 = a_4 + a_5 + a_6 + a_7$$

$$\vdots$$

$$b_k = a_{2^k} + \cdots + a_{2^{k+1}-1}$$

Since each partial sum of $\sum_{k=0}^{\infty} b_k$ is a partial sum of the original sequence: $\sum_{k=0}^{N} b_k = \sum_{n=1}^{2^{N+1}-1} a_n$,

and the partial sums are nondecreasing,

they must converge to the same limit, and we conclude that

$$\sum_{n=1}^{\infty} a_n \quad \text{converges iff} \quad \sum_{k=0}^{\infty} b_k$$

converges.

We now observe that

$$b_k = \underbrace{a_{2^k} + a_{2^k+1} \cdots + a_{2^{k+1}-1}}_{2^k \text{ terms}}$$

$$\leq 2^k a_{2^k}$$ since the terms are decreasing.

Thus, by the comparison test,

if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then

so must $\sum_{k=0}^{\infty} b_k$ and thereby

$\sum_{n=0}^{\infty} a_n$.

On the other hand, observe that

$$b_k = a_{2^k} + a_{2^k+1} \cdots a_{2^{k+1}-1}$$

$$\geq 2^k a_{2^{k+1}-1} \geq 2^k a_{2^{k+1}}$$

because a_n is decreasing.

Thus, by another comparison test

If $\sum_{k=0}^{\infty} 2^k a_{2^{k+1}}$

diverges, then $\sum_{k=0}^{\infty} b_k$ diverges.

But $\sum_{k=0}^{\infty} 2^k a_{2^{k+1}}$

$$= \sum_{k=1}^{\infty} 2^{k-1} a_{2^k}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} 2^k a_{2^k} .$$

$\square$

Applying this transformation to $\sum_{n=1}^{\infty} \frac{1}{n}$

we get

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

$$= \sum_{k=0}^{\infty} 2^k \frac{1}{2^k} = 1+1+1\cdots$$

which clearly diverges.

In the case of $\sum_{n=3}^{\infty} \frac{1}{n\log n}$, we

ignore the first 2 terms and get: $\sum_{k=0}^{\infty} 2^k \frac{1}{2^k \log(2^k)}$

$$= \sum_{k=0}^{\infty} \frac{1}{k \log 2}$$

which also diverges.

However, for $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^2}$ we get

$$\sum_{k=0}^{\infty} 2^k \frac{1}{2^k(\log 2^k)^2} = \sum \frac{1}{k^2 \log^2 2}$$

which converges.