Homework 12 Solutions

1.

❶ Chapter 14, Section 7, Problem 56.
Find the inverse Laplace transform of the function $F(p) = \frac{1}{p^4-1}$ by using (7.16).

❷ Chapter 14, Section 7, Problem 62.
Find the inverse Laplace transform of the function $F(p) = \frac{(p-1)^2}{p(p+1)^2}$ by using (7.16).

Solution:

❶ We first find the poles of $F(z)e^{zt}$ and factor the denominator to get

$$F(z)e^{zt} = \frac{e^{zt}}{(z-1)(z+1)(z-i)(z+1)}.$$

Evaluating the residues at the four simple poles, we find

residue at $z = 1$	is	$\frac{e^t}{4}$
residue at $z = -1$	is	$-\frac{e^{-t}}{4}$
residue at $z = i$	is	$-\frac{e^{it}}{4i}$
residue at $z = -i$	is	$\frac{e^{-it}}{4i}$

Then by (7.16) we have

$$f(t) = \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{e^{it}}{4i} + \frac{e^{-it}}{4i} = \frac{\sinh t - \sin t}{2}.$$

❷ We first find the poles of $F(z)e^{zt}$ and factor the denominator to get

$$F(z)e^{zt} = \frac{(z-1)^2 e^{zt}}{z(z+1)^2}.$$

Evaluating the residues at the two poles, we find

residue at $z = 0$	is	1
residue at $z = -1$	is	$-4te^{-t}$

Then by (7.16) we have

$$f(t) = 1 - 4te^{-t}.$$

2.

❶ Chapter 8, Section 11, Problem 5.
Solve the differential equation $y'' + \omega^2 y = f(t)$, $y_0 = y_0' = 0$, with $f(t)$ given by the functions in Figure 11.4, by the following methods.

(a) Use the convolution integral, being careful to consider separately the three intervals 0 to t_0, t_0 to $t_0 + \frac{1}{n}$, and $t_0 + \frac{1}{n}$ to ∞.

(b) Write $f_n(t)$ as a difference of unit step functions as in $L25$, and use $L25$ to find $L(f_n)$. Expand $\frac{1}{p(p^2+\omega^2)}$ by partial fractions and use $L28$ to find $y_n(t)$. Your result should agree with (a).

(c) Let $n \to \infty$ and show that your solution in (a) and (b) tends to the same solution (11.5) obtained using the functions of Figure 11.3; that is, either set of functions gives, in the limit, the same solution (11.11) obtained by using the δ function. Note that, when you let $n \to \infty$, you do not need to consider the interval t_0 to $t_0 + \frac{1}{n}$ since, if $t > t_0$, then for sufficiently large n, $t > t_0 + \frac{1}{n}$.

❷ Chapter 8, Section 11, Problem 7.
Using the δ function method, find the response of $y'' + 2y' + y = \delta(t - t_0)$ to a unit impulse.

❸ Chapter 8, Section 11, Problem 11.
Using the δ function method, find the response of $\frac{d^4y}{dt^4} - y = \delta(t - t_0)$ to a unit impulse.

Solution:

❶ (a) Taking the Laplace transform of each term, substituting the initial conditions, and solving for Y, we get

$$p^2Y + \omega^2 Y = L(f_n(t))$$
$$Y = \frac{1}{p^2+\omega^2} L(f_n(t))$$

According to $L3$, the inverse transform of $\frac{1}{p^2+\omega^2}$ is $\frac{\sin(\omega t)}{\omega}$. Therefore,

$$Y = L\left(\frac{\sin(\omega t)}{\omega}\right) L(f_n(t)).$$

According to $L34$,

$$y(t) = \int_0^t \frac{\sin[\omega(t-\tau)]}{\omega} f_n(\tau)\, d\tau$$

$$= \begin{cases} 0 & \text{if} \quad t < t_0 \\ \frac{n}{\omega^2}\left(1 - \cos[\omega(t-t_0)]\right) & \text{if} \quad t_0 < t < t_0 + \frac{1}{n} \\ \frac{n}{\omega^2}\left(\cos[\omega(t - t_0 - \frac{1}{n})] - \cos[\omega(t-t_0)]\right) & \text{if} \quad t_0 + \frac{1}{n} < t \end{cases}$$

(b) We write $f_n(t)$ as a difference of unit step functions:

$$f_n(t) = n\left(u(t-t_0) - u(t - t_0 - \tfrac{1}{n})\right)$$

According to $L25$,

$$L(f_n(t)) = n\frac{e^{-t_0 p} - e^{-(t_0 + \frac{1}{n})p}}{p}$$

We expand $\frac{1}{p(p^2+\omega^2)}$ by partial fractions to get

$$\frac{1}{p(p^2+\omega^2)} = \frac{1}{p} - \frac{p}{p^2+\omega^2}$$

Putting it all together, we have

$$\begin{aligned} Y &= \frac{1}{p^2+\omega^2} L(f_n(t)) \\ &= \frac{1}{p^2+\omega^2} n \frac{e^{-t_0 p} - e^{-(t_0+\frac{1}{n})p}}{p} \\ &= \frac{n}{p(p^2+\omega^2)}\left(e^{-t_0 p} - e^{-\left(t_0+\frac{1}{n}\right)p}\right) \\ &= n\left(\frac{1}{p} - \frac{p}{p^2+\omega^2}\right)\left(e^{-t_0 p} - e^{-\left(t_0+\frac{1}{n}\right)p}\right) \\ &= n\left(\frac{1}{p}e^{-t_0 p} - \frac{1}{p}e^{-\left(t_0+\frac{1}{n}\right)p} - \frac{p}{p^2+\omega^2}e^{-t_0 p} + \frac{p}{p^2+\omega^2}e^{-\left(t_0+\frac{1}{n}\right)p}\right) \end{aligned}$$

According to $L28$, the inverse Laplace transform of

$$\begin{array}{rcr} \frac{1}{p}e^{-t_0 p} & \text{is} & u\,(t-t_0) \\ \frac{1}{p}e^{-\left(t_0+\frac{1}{n}\right)p} & \text{is} & \cos\left(\omega\,(t-t_0)\right)u\,(t-t_0) \\ \frac{p}{p^2+\omega^2}e^{-t_0 p} & \text{is} & u\left(t - t_0 - \frac{1}{n}\right) \\ \frac{p}{p^2+\omega^2}e^{-\left(t_0+\frac{1}{n}\right)p} & \text{is} & \cos\left(\omega\left(t-t_0-\frac{1}{n}\right)\right)u\left(t-t_0-\frac{1}{n}\right) \end{array}$$

Therefore,

$$y_n(t) = \begin{cases} 0 & \text{if} \quad t < t_0 \\ \frac{n}{\omega^2}\left(1 - \cos[\omega(t-t_0)]\right) & \text{if} \quad t_0 < t < t_0 + \frac{1}{n} \\ \frac{n}{\omega^2}\left(\cos[\omega(t - t_0 - \frac{1}{n})] - \cos[\omega(t-t_0)]\right) & \text{if} \quad t_0 + \frac{1}{n} < t \end{cases}$$

(c) Letting $n \to \infty$, we see that

$$\lim_{n\to\infty} y_n(t) = \begin{cases} 0 & \text{if } \ t < t_0 \\ \frac{\sin(\omega(t-t_0))}{\omega} & \text{if } \ t > t_0 \end{cases}$$

❷ From Problem 6c, we get the initial conditions $y_0 = y_0' = 0$. So, taking Laplace transforms, we get

$$(p+1)^2 Y = L\left(\delta(t-t_0)\right).$$

According to $L6$, the Laplace transform of te^{-t} is $\frac{1}{(p+1)^2}$. Therefore,

$$Y = L\left(te^{-t}\right) L\left(\delta(t-t_0)\right).$$

According to $L28$,

$$y(t) = (t-t_0)e^{-(t-t_0)}u(t-t_0).$$

❸ According to 35, taking Laplace transforms yields

$$(p^4-1)Y = L\left(\delta(t-t_0)\right).$$

According to Problem 1 of HW12, the Laplace transform of $\frac{\sinh t - \sin t}{2}$ is $\frac{1}{p^4-1}$. Therefore,

$$Y = L\left(\tfrac{\sinh t - \sin t}{2}\right) L\left(\delta(t-t_0)\right).$$

According to $L28$,

$$y(t) = \tfrac{\sinh(t-t_0) - \sin(t-t_0)}{2} u(t-t_0).$$

3.

❶ Chapter 8, Section 12, Problem 2.
Use (12.6) to solve (12.1) when $f(t) = \sin\omega t$.

❷ Chapter 8, Section 12, Problem 6.
For Problem 10.17, show that $G(t,t') = \begin{cases} 0 & 0 < t < t' \\ \frac{\sinh a(t-t')}{a} & t > t' \end{cases}.$
Thus write the solution to 10.17 as an integral similar to (12.6) and evaluate it.

❸ Chapter 8, Section 12, Problem 8.
Solve the differential equation $y'' + 2y' + y = f(t)$, $y_0 = y_0' = 0$, where

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a. \end{cases}$$

As in Problems 6 and 7, find the Green function for the problem and use it in

$$y(t) = \int_0^\infty G(t,t') f(t')\, dt'. \tag{12.4}$$

Consider the cases $t < a$ and $t > a$ separately.

Solution:

❶ We have from (12.6) that

$$\begin{aligned} y(t) &= \int_0^t \tfrac{1}{\omega} \sin\omega(t-\tau) f(\tau)\, d\tau \\ &= \tfrac{1}{\omega} \int_0^t \sin\omega(t-\tau) \sin\omega\tau\, d\tau \\ &= \tfrac{1}{\omega} \int_0^t (\sin\omega t \cos\omega\tau - \sin\omega\tau \cos\omega t) \sin\omega\tau\, d\tau \\ &= \tfrac{1}{\omega} \int_0^t (\sin\omega t \cos\omega\tau \sin\omega\tau - \sin\omega\tau \cos\omega t \sin\omega\tau)\, d\tau \\ &= \tfrac{1}{\omega} \left(\tfrac{1-\cos 2\omega t}{4\omega} \sin\omega t - \cos\omega t [\tfrac{t}{2} - \tfrac{\sin 2\omega t}{4\omega}] \right). \end{aligned}$$

❷ The Green function satisfies

$$\frac{d^2}{dt^2}G(t,t') - a^2 G(t,t') = \delta(t-t')$$

with G and $\frac{d}{dt}G$ equal to 0 on $t=0$. To solve for G, we take the Laplace transform of the differential equation to get

$$(p^2 - a^2)L[G] = L[\delta(t-t')].$$

According to $L9$, $L[\frac{\sinh at}{a}] = \frac{1}{p^2-a^2}$. Therefore

$$L[G] \;=\; \frac{e^{-t'p}}{p^2-a^2} \;=\; L[\frac{\sinh at}{a}]L[\delta(t-t')].$$

By $L28$,

$$G(t,t') = \frac{\sinh a(t-t')}{a} u(t-t').$$

Now we solve for $y = y(t)$:

$$\begin{aligned} y(t) \;&=\; \int_0^\infty G(t,t') f(t')\,dt' \;=\; \int_0^\infty G(t,t')\,dt' \\ &=\; \int_0^t \frac{\sinh a(t-t')}{a}\,dt' \\ &=\; -\frac{\cosh a(t-t')}{a^2}\bigg|_{t'=0}^{t} \\ &=\; \frac{\cosh at - 1}{a^2}. \end{aligned}$$

❸ The Green function satisfies

$$\frac{d^2}{dt^2}G(t,t') + 2\frac{d}{dt}G(t,t') + G(t,t') = \delta(t-t')$$

with G and $\frac{d}{dt}G$ equal to 0 on $t=0$. To solve for G, we take the Laplace transform of the differential equation to get

$$(p^2+2p+1)L[G] = L[\delta(t-t')].$$

Since $\frac{1}{p^2+2p+1} = \frac{1}{(p+1)^2}$, $L[te^{-t}] = \frac{1}{(p+1)^2}$. Therefore

$$L[G] \;=\; \frac{e^{-t'p}}{(p+1)^2} \;=\; L[te^{-t}]L[\delta(t-t')].$$

By $L28$,

$$G(t,t') = (t-t')e^{-(t-t')}u(t-t').$$

Now we solve for $y=y(t)$:

$$\begin{aligned} y(t) \;&=\; \int_0^\infty G(t,t')f(t')\,dt' \;=\; \int_0^a G(t,t')\,dt' \\ &= \begin{cases} \int_0^t (t-t')e^{-(t-t')}\,dt' & \text{if } 0<t<a \\ \int_0^a (t-t')e^{-(t-t')}\,dt' & \text{if } 0<a<t. \end{cases} \end{aligned}$$

Since

$$\int (t-t')e^{-(t-t')}\,dt' = (t-t'+1)e^{-(t-t')},$$

if $0<t<a$, then

$$y(t) = (t-t'+1)e^{-(t-t')}\Big|_{t'=0}^{t} \;=\; 1-(t+1)e^{-t};$$

if $t>a$, then

$$y(t) = (t-t'+1)e^{-(t-t')}\Big|_{t'=0}^{a} \;=\; (t-a+1)e^{-(t-a)}-(t+1)e^{-t}.$$

4.

Use

$$y(x) = -\cos x \int_0^x (\sin \bar{x}) f(\bar{x})\, d\bar{x} - \sin x \int_0^{\pi/2} (\cos \bar{x}) f(\bar{x})\, d\bar{x} \tag{12.17}$$

to find the solution of

$$y'' + y = f(x) \tag{12.7}$$

with $y(0) = y(\pi/2) = 0$ when the forcing function is given $f(x)$.

❶ Chapter 8, Section 12, Problem 11.
$f(x) = \sin 2x$

❷ Chapter 8, Section 12, Problem 13.

$$f(x) = \begin{cases} x & 0 < x < \pi/4 \\ \pi/2 - x & \pi/4 < x < \pi/2 \end{cases}$$

Solution:

❶

$$\begin{aligned} y(x) &= -\cos x \int_0^x \sin \bar{x} \sin 2\bar{x}\, d\bar{x} - \sin x \int_0^{\pi/2} \cos \bar{x} \sin 2\bar{x}\, d\bar{x} \\ &= -\cos x \left(\sin \tfrac{\bar{x}}{2} - \tfrac{\sin 3\bar{x}}{6} \right) \Big|_{\bar{x}=0}^{x} - \sin x \left(\cos \tfrac{\bar{x}}{2} - \tfrac{\cos 3\bar{x}}{6} \right) \Big|_{\bar{x}=0}^{\pi/2} \\ &= -\cos x \left(\sin \tfrac{x}{2} - \tfrac{\sin 3x}{6} \right) - \tfrac{\sqrt{2}}{2} \sin x \end{aligned}$$

❷ We have two cases: $x < \pi/4$ and $x > \pi/4$.
For $x < \pi/4$,

$$\begin{aligned} y(x) &= -\cos x \int_0^x \bar{x} \sin \bar{x}\, d\bar{x} - \sin x \left(\int_0^{\pi/4} \bar{x} \cos \bar{x}\, d\bar{x} + \int_{\pi/4}^{\pi/2} (\tfrac{\pi}{2} - \bar{x}) \cos \bar{x}\, d\bar{x} \right) \\ &= -\cos x \left(\sin \bar{x} - \bar{x} \cos \bar{x} \right) \Big|_{\bar{x}=0}^{x} \\ &\quad - \sin x \left((\cos \bar{x} + \bar{x} \sin \bar{x}) \Big|_{\bar{x}=0}^{\pi/4} + \tfrac{\pi}{2} \sin \bar{x} \Big|_{\bar{x}=\pi/4}^{\pi/2} - (\cos \bar{x} + \bar{x} \sin \bar{x}) \Big|_{\bar{x}=\pi/4}^{\pi/2} \right) \\ &= -\cos x \left(\sin x - x \cos x \right) \\ &\qquad - \left((\tfrac{\sqrt{2}}{2} + \tfrac{\pi}{4}\tfrac{\sqrt{2}}{2} - 1) + \tfrac{\pi}{2}(1 - \tfrac{\sqrt{2}}{2}) - (\tfrac{\pi}{2} - \tfrac{\sqrt{2}}{2} - \tfrac{\pi}{4}\tfrac{\sqrt{2}}{2}) \right) \sin x \\ &= -\cos x \left(\sin x - x \cos x \right) - (\sqrt{2} - 1) \sin x \end{aligned}$$

For $x > \pi/4$,

$$
\begin{aligned}
y(x) &= -\cos x\left(\int_0^{\pi/4} \bar{x}\sin\bar{x}\,d\bar{x} + \int_{\pi/4}^{x} (\tfrac{\pi}{2}-\bar{x})\sin\bar{x}\,d\bar{x}\right) \\
&\qquad -\sin x\left(\int_0^{\pi/4} \bar{x}\cos\bar{x}\,d\bar{x} + \int_{\pi/4}^{\pi/2} (\tfrac{\pi}{2}-\bar{x})\cos\bar{x}\,d\bar{x}\right) \\
&= -\cos x\left((\sin\bar{x}-\bar{x}\cos\bar{x})\Big|_{\bar{x}=0}^{\pi/4} - \tfrac{\pi}{2}\cos\bar{x}\Big|_{\bar{x}=\pi/4}^{x} - (\sin\bar{x}-\bar{x}\cos\bar{x})\Big|_{\bar{x}=\pi/4}^{x}\right) \\
&\qquad -(\sqrt{2}-1)\sin x \\
&= -\cos x\left(\left(\tfrac{\sqrt{2}}{2}-\tfrac{\pi}{4}\tfrac{\sqrt{2}}{2}\right) - \tfrac{\pi}{2}(\cos x - \tfrac{\sqrt{2}}{2}) - \left(\sin x - x\cos x - \left(\tfrac{\sqrt{2}}{2}-\tfrac{\pi}{4}\tfrac{\sqrt{2}}{2}\right)\right)\right) \\
&\qquad -(\sqrt{2}-1)\sin x \\
&= -\cos x\left(\sqrt{2} - \tfrac{\pi}{2}\cos x - \sin x + x\cos x\right) - (\sqrt{2}-1)\sin x
\end{aligned}
$$