Homework 11 Solutions

Section 8

(10)
$$\frac{2p-1}{p^2-2p+10} = \frac{2p-2}{(p-1)^2+9} + \frac{1}{(p-1)^2+9}$$

$$= 2\frac{(p-1)}{(p-1)^2+9} + \frac{1}{3}\frac{3}{(p-1)^2+9}$$

Now $\mathcal{L}^{-1}\left[\frac{p-1}{(p-1)^2+9}\right] = e^t \mathcal{L}^{-1}\left[\frac{p}{p^2+9}\right]$

since $\mathcal{L}\left[e^{-at}f(t)\right] = F(p+a)$

$= e^t \cos 3t$ by (L4)

Similarly: $\mathcal{L}^{-1}\left[\frac{3}{(p-1)^2+9}\right] = e^t \mathcal{L}^{-1}\left[\frac{3}{p^2+9}\right] = e^t \sin(3t)$ by (L3)

So $\mathcal{L}^{-1}\left[\frac{2p-1}{p^2-2p+10}\right] = 2e^t\cos(3t) + \frac{1}{3}e^t\sin(3t)$

(11) $$\frac{3p+2}{3p^2+5p-2} = \frac{3p+2}{3(p+1)(p-\frac{1}{3})}$$

$$p = \frac{-5 \pm \sqrt{25+24}}{6} = \frac{-5\pm 7}{6} = -1 \text{ or } \frac{1}{3}$$

$$= \frac{\cancel{3p+3}}{\cancel{3(p+1)}(p-\frac{1}{3})} - \frac{1}{3(p+1)(p-\frac{1}{3})}$$

So $$\mathcal{L}^{-1} = \mathcal{L}^{-1}\left[\frac{1}{p-\frac{1}{3}}\right] - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{(p+1)(p-\frac{1}{3})}\right]$$

$$= e^{t/3} - \frac{1}{3}\,\frac{e^{-t}-e^{t/3}}{(-\frac{1}{3}-1)} \text{ by (L7)}$$

$$= e^{t/3} + \frac{e^{-t}}{4} - \frac{e^{t/3}}{4}$$

$$= \frac{3}{4}e^{t/3} + \frac{e^{-t}}{4}$$

(14) There are many ways to do this but here is one.

Assume $C \neq 0$ since otherwise $\frac{Ap+B}{Ep+F} \not\to 0$ as $|p| \to \infty$ so the Bromwich inversion integral does not converge and the inverse Laplace transform does not exist.

(the one useful exception to this case is $\mathcal{L}(\delta(t-a)) = e^{-pa}$, whose inverse Laplace transform is a δ function, which is not even really a function)

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Then we can factor $Cp^2+Ep+F = C(p-\lambda_1)(p-\lambda_2)$

for $\lambda_{1,2} = \dfrac{-E \pm \sqrt{E^2-4CF}}{2C}$, possibly complex

and $\mathcal{L}^{-1}\left[\dfrac{Ap+B}{Cp^2+Ep+F}\right]$

$$= \frac{A}{C}\mathcal{L}^{-1}\left[\frac{p}{(p-\lambda_1)(p-\lambda_2)}\right] + \frac{B}{C}\mathcal{L}^{-1}\left[\frac{1}{(p-\lambda_1)(p-\lambda_2)}\right]$$

$$= \frac{A}{C}\underbrace{\frac{-\lambda_1 e^{\lambda_1 t}+\lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2}}_{\text{by } L8} + \frac{B}{C}\underbrace{\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}}_{\text{by } L7}.$$

(26) By (L28),

$$\mathcal{L}[\,u(t-a)f(t-a)\,] = e^{-pa}\mathcal{L}[f(t)]$$

where $u(t-a) = \begin{cases} 1 & \text{if } t>a \\ 0 & \text{else}\end{cases}$

is the step function.

So $\mathcal{L}^{-1}\left[\dfrac{pe^{-p\pi}}{p^2+1}\right](t) = u(t-\pi)\,\mathcal{L}^{-1}\left[\dfrac{p}{p^2+1}\right](t-\pi)$

$= u(t-\pi)\cos(t-\pi)$ by (L4)

(The conceptual point here is that multiplication of $F(p)$ by e^{-pa} corresponds to a "forward shift" by a of $f(t)$)

Section 9

(6) The Laplace transformed eqn is:

$$p^2Y - py(0) - y'(0) - 6(pY - y(0)) + 9Y$$

$$= \mathcal{L}[te^{3t}](p)$$

$$= \mathcal{L}[t](p-3)$$

$$= \frac{1}{(p-3)^2} \;//$$

plugging in $y_0 = 0, y_0' = 5$:

$$p^2Y - 5 - 6pY + 9Y = \frac{1}{(p-3)^2}$$

$$\Rightarrow \quad Y = \frac{\frac{1}{(p-3)^2} + 5}{p^2 - 6p + 9} = \frac{\frac{1}{(p-3)^2} + 5}{(p-3)^2}$$

$$= \frac{1}{(p-3)^4} + \frac{5}{(p-3)^2} = \frac{1}{3!}\frac{3!}{(p-3)^{3+1}} + 5\frac{1}{(p-3)^2}$$

$$= \underline{\underline{\frac{1}{6}e^{3t}t^3 + 5e^{3t}t}} \quad \text{by (L6)}$$

(8) $$p^2Y + 16Y = \frac{8p}{p^2+4^2}$$

$$\Rightarrow Y = \frac{8p}{(p^2+16)(p^2+16)} = \frac{8p}{(p^2+16)^2} = \frac{2\cdot 4p}{(p^2+4^2)^2}$$

So $y(t) = \underline{\underline{t\sin 4t}}$ by (L11) with $a=4$.

Alternatively, you could use the Residue formula:

$$\mathcal{L}^{-1}[Y](t) = \operatorname{Res}\left(\frac{8pe^{pt}}{(p^2+4^2)^2}, 4i\right) + \operatorname{Res}\left(\frac{8pe^{pt}}{(p^2+4^2)^2}, -4i\right)$$

which gives the same answer.

(9) $$p^2Y - py(0) - y'(0) + 16Y = \frac{8p}{p^2+4^2}$$

$$\Rightarrow p^2Y + 16Y - 8 = \frac{8p}{p^2+4^2}$$

$$\Rightarrow Y = \left(\frac{1}{p^2+4^2}\right)\left(\frac{8p}{p^2+4^2} + 8\right)$$

$$= \frac{8p}{(p^2+4^2)^2} + \frac{8}{p^2+4^2}$$

So $y(t) = \underbrace{t \sin 4t}_{\text{previous problem}} + 2\mathcal{L}^{-1}\left[\frac{4}{p^2+4^2}\right]$

$= \underline{\underline{t\sin 4t + 2\sin 4t}}$ by (L3)

(14) $p^2 Y - 1 - 4pY = \dfrac{-4}{(p-2)^2}$ by (L6)

$\Rightarrow Y = \dfrac{\dfrac{-4}{(p-2)^2} + 1}{p^2 - 4p} = \dfrac{\cancel{-4} + p^2 \cancel{+4} - 4p}{(p-2)^2(p^2-4p)}$

$= \dfrac{1}{(p-2)^2}$ // So $y(t) = \underline{\underline{te^{2t}}}$.

(30) Taking Laplace transforms of Both equations:

$$pY + 2Z = \frac{1}{p}$$

$\mathcal{L}[y] = Y(p)$

$\mathcal{L}[z] = Z(p)$

$2Y - (pZ - 1) = \dfrac{2}{p^2} \Rightarrow 2Y - pZ = \dfrac{2}{p^2} - 1$

$$\begin{pmatrix} p & 2 \\ 2 & -p \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} 1/p \\ 2/p^2 - 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} Y \\ Z \end{pmatrix} = \frac{1}{-p^2-4} \begin{pmatrix} -p & -2 \\ -2 & p \end{pmatrix} \begin{pmatrix} 1/p \\ 2/p^2-1 \end{pmatrix}$$

$$= \frac{-1}{p^2+4} \begin{pmatrix} -1 - \frac{4}{p^2} + 2 \\ -\frac{2}{p} + \frac{2}{p} - p \end{pmatrix} = \frac{1}{p^2+4} \begin{pmatrix} \frac{4}{p^2} - 1 \\ p \end{pmatrix}$$

So $$Y(p) = \frac{\frac{4}{p^2} - 1}{p^2+4} = \frac{4}{p^2(p^2+4)} - \frac{1}{p^2+4}$$

$$= \frac{1}{2} \frac{2^3}{p^2(p^2+2^2)} - \frac{1}{2} \frac{2}{p^2+2^2}$$

So $$y(t) = \underbrace{\frac{1}{2}(2t - \sin 2t)}_{\text{by (L16)}} - \underbrace{\frac{1}{2} \sin 2t}_{\text{by (L3)}} \quad //$$

and $$z(t) = \mathcal{L}^{-1}\left[\frac{p}{p^2+4}\right] = \underline{\underline{\cos 2t}}$$

Section 10

(3) $\mathcal{L}^{-1}\left[\frac{p}{p^2-1}\cdot\frac{1}{p^2-1}\right]$

$= \mathcal{L}^{-1}\left[\frac{p}{p^2-1}\right] * \mathcal{L}^{-1}\left[\frac{1}{p^2-1}\right]$

$= \cosh(t) * \sinh(t)$

$$= \int_0^t \cosh(t-\tau)\sinh(\tau)\,d\tau.$$

Note that all functions produced by the inverse Laplace transform (and when working with the Laplace transform in general) are zero for $t<0$.

Thus, strictly speaking, we are actually working with

$u(t)\sinh(t)$ and $u(t)\cosh(t)$

where $u(t) = \begin{cases} 1 & t\geq 0 \\ 0 & \text{else} \end{cases}$

is the step function.

Here, the convolution integral becomes an integral over $[0,t]$:

$$\int_{-\infty}^{\infty} u(t-\tau)\cosh(t-\tau)\,u(\tau)\sinh(\tau)\,d\tau = \int_0^t \cosh(t-\tau)\sinh(\tau)\,d\tau.$$

This may be further simplified as:

$$\int_0^t \frac{e^{t-\tau}+e^{\tau-t}}{2} \cdot \frac{e^{\tau}-e^{-\tau}}{2}\, d\tau$$

$$= \frac{1}{4}\int_0^t e^{t} - e^{t-2\tau} + e^{2\tau-t} - e^{-t}\, d\tau$$

$$= \frac{1}{4}\left[te^{t} - \left(e^{t}\cdot \frac{e^{-2\tau}}{-2}\Big|_0^t\right) + e^{-t}\cdot \frac{e^{2\tau}}{2}\Big|_0^t - te^{-t}\right]$$

$$= \frac{1}{4}\left[te^{t} - te^{-t} - \left(\cancel{\frac{e^{-t}}{-2}} - \cancel{\frac{e^{t}}{-2}}\right) + \left(\cancel{\frac{e^{t}}{2}} - \cancel{\frac{e^{-t}}{2}}\right)\right]$$

$$= \frac{1}{4}\left(te^{t} - te^{-t}\right) = \underline{\underline{\frac{t}{2}\sinh(t)}}$$

(11) $$\mathcal{L}^{-1}\left[\frac{p}{(p^2+a^2)(p^2+b^2)}\right] = \mathcal{L}^{-1}\left[\frac{p}{p^2+a^2}\right] * \mathcal{L}^{-1}\left[\frac{1}{p^2+b^2}\right]$$

$$= \cos(at) * \frac{1}{b}\sin(bt)$$

$$= \frac{1}{b}\int_0^t \cos(a(t-\tau))\sin(b\tau)\, d\tau$$

$$= \frac{1}{2b}\int_0^t \sin(b\tau + at - a\tau) + \underbrace{\sin(b\tau - at + a\tau)}_{(A)}\, d\tau$$

$$= \frac{1}{2b}\left[\int_0^t \sin(at+(b-a)\tau)\,d\tau + \int_0^t \sin(-at+(b+a)\tau)\,d\tau\right]$$

$$= \frac{1}{2b}\left[\left.\frac{-\cos(at+(b-a)\tau)}{b-a}\right|_0^t + \left.\frac{-\cos(-at+(b+a)\tau)}{b+a}\right|_0^t\right]$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt)}{b-a} + \frac{\cos(at)}{b-a} + \frac{-\cos(bt)}{b+a} + \frac{\cos(at)}{b+a}\right]$$

$$= \frac{1}{2b}\left[\cos(at)\left[\frac{2b}{b^2-a^2}\right] - \cos(bt)\left[\frac{2b}{b^2-a^2}\right]\right]$$

$$= \frac{\cos(at)-\cos(bt)}{b^2-a^2} \quad // \quad \text{when } b \neq a$$

If $b=a$, $(*)$ becomes

$$\frac{1}{2b}\int_0^t \left(\sin(bt) + \sin(2b\tau - at)\right)d\tau$$

$$= \frac{1}{2b}\left[t\sin(bt) + \left[\left.\frac{-\cos(2b\tau-at)}{2b}\right|_0^t\right]\right]$$

$$= \frac{t\sin(bt)}{2b} + \cancel{\frac{-\cos(bt)}{2b}} + \cancel{\frac{\cos(bt)}{2b}}$$

$$= \frac{t\sin(bt)}{2b} \quad //$$

$\overline{\;\;2b\;\;}$ //

(18) Taking the Laplace transform:

$$p^2 Y + \omega^2 Y = \int_0^a e^{-pt}\,dt = \left.\frac{e^{-pt}}{-p}\right|_0^a = \frac{1-e^{at}}{p}.$$

$$\text{So } \quad Y = \frac{1}{p^2+\omega^2}\cdot\frac{1-e^{-at}}{p}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1-e^{-at}}{p(p^2+\omega^2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{p^2+\omega^2}\right] * \mathcal{L}^{-1}\left[\frac{1-e^{-at}}{p}\right]$$

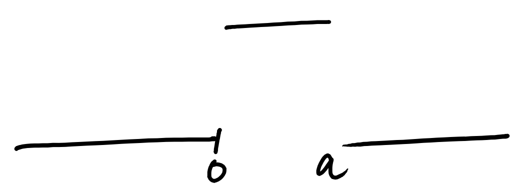
$$= \frac{1}{\omega}\sin(\omega t) * \left[v(t) - v(t-a)\right]$$

where v is the step function

$$\left(\text{Since } \mathcal{L}^{-1}\left[\frac{1}{p}\right] = v(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}\right)$$

$$= \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))\left[v(\tau) - v(\tau-a)\right]d\tau$$

$v(\tau) - v(\tau-a)$ looks like:



So the integral is :

$\underline{t<a}$

$$\frac{1}{\omega}\int_0^t \sin(\omega(t-\gamma))\,d\gamma = \frac{1}{\omega}\left.\frac{-\cos(\omega(t-\gamma))}{-\omega}\right|_0^t$$

$$= \frac{1-\cos(\omega t)}{\omega^2}$$

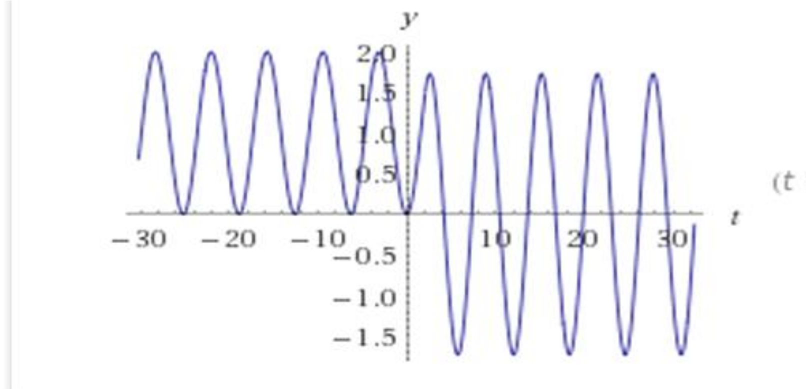
$\underline{t>a}$

$$\frac{1}{\omega}\int_0^a \sin(\omega(t-\gamma))\,d\gamma$$

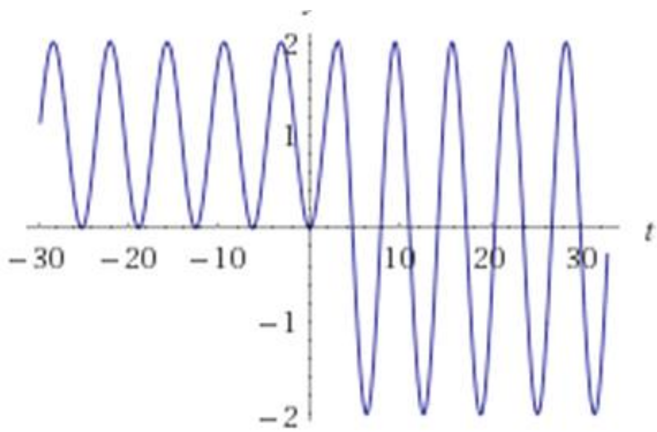
$$= \frac{1}{\omega}\left.\frac{-\cos(\omega(t-\gamma))}{-\omega}\right|_0^a$$

$$= \frac{1}{\omega^2}\Big(\cos(\omega(t-a)) - \cos(\omega t)\Big)$$

$a = \frac{\pi}{3}$:



$a = 3\pi/2$



$a = \frac{\pi}{10}$