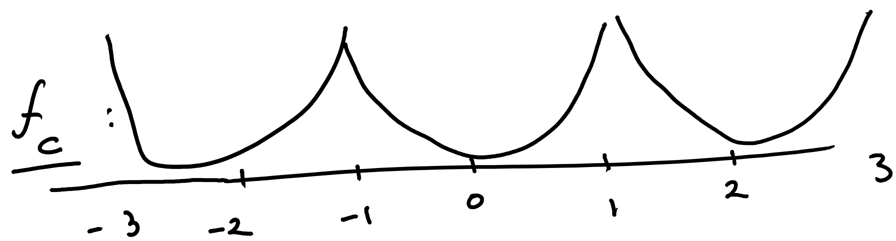
Section 9

(1) (a) $e^{inx} = \underbrace{\cos nx}_{\text{even}} + \underbrace{i\sin nx}_{\text{odd}}.$

(b) $xe^{x} = \underbrace{\dfrac{xe^{x} - xe^{-x}}{2}}_{\text{even}} + \underbrace{\dfrac{xe^{x} - xe^{-x}}{2}}_{\text{odd}}$

(20)



Period $2L$, $L = 1$.

$$a_n = \frac{2}{L}\int_0^1 x^2 \cos(\pi n x)\,dx$$

$$= 2\left[\cancel{x^2 \frac{\sin(\pi n x)}{\pi n}}^{\,\to 0}\Big|_0^1 - \int_0^1 2x \frac{\sin \pi n x}{\pi n}\,dx\right]$$

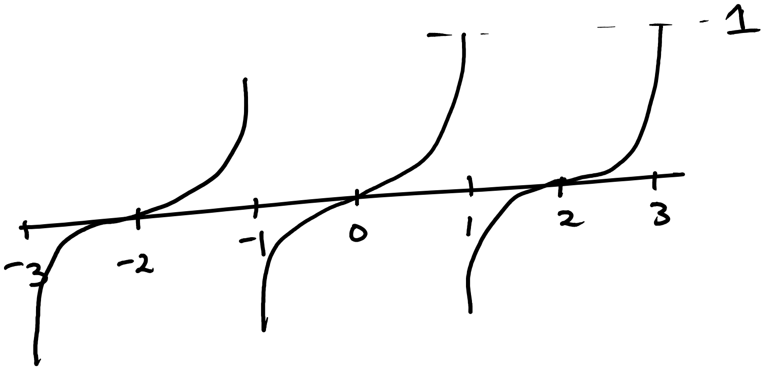
$$= -\frac{4}{\pi n}\left[x\left(-\frac{\cos \pi n x}{\pi n}\right)\Big|_0^1 - \int_0^1 -\frac{\cos \pi n x}{\pi n}\,dx\right]$$

$$= \frac{-4}{\pi n}\left[-\frac{\cos(n\pi)}{\pi n} + \cancel{\frac{1}{(\pi n)^2}\sin(\pi n x)\Big|_0^1}^{\,\to 0}\right]$$

$$= \frac{4(-1)^n}{\pi^2 n^2} \qquad a_0 = \int_0^1 x^2 = \frac{1}{3}$$

So $f_c(x) = \frac{1}{3} + \sum_{n\geq 1} \frac{(-1)^n 4}{\pi^2 n^2}\cos(n\pi x)$.

f_s:



$$f_s(x) = \begin{cases} x^2 & x \in [0,1] \\ -x^2 & x \in [-1,0] \end{cases}$$

$$\frac{b_n}{2} = \int_0^1 x^2 \sin(\pi n x)\,dx = x^2\left(-\frac{\cos \pi n x}{\pi n}\right)\Big|_0^1 - \int_0^1 2x\left(-\frac{\cos \pi n x}{\pi n}\right)dx$$

$$= \frac{-\cos \pi n}{\pi n} + \frac{2}{\pi n}\left[\underbrace{\cancel{x\,\frac{\sin \pi n x}{\pi n}\Big|_0^1}}_{0} - \int_0^1 \frac{\sin \pi n x}{\pi n}\,dx \right]$$

$$= \frac{(-1)^{n+1}}{\pi n} + \frac{2}{\pi n}\left[\frac{\cos \pi n x}{(\pi n)^2}\Big|_0^1 \right]$$

$$= \frac{(-1)^{n+1}}{\pi n} + \frac{2}{(\pi n)^3}\left(\cos(\pi n) - 1\right)$$

So $f_s(x) = 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{\pi n}\sin(n\pi x) + 2\sum_{\substack{n\geq 1 \\ \text{odd}}} \frac{4}{(\pi n)^3}\sin(\pi n x)$

$f_p(x) =$

Exponential series:

$$c_0 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$c_n = \int_0^1 x^2 e^{-2\pi i n x}\,dx$$

$$= x^2 \frac{e^{-2\pi i n x}}{-2\pi i n}\Big|_0^1 - \int_0^1 2x \frac{e^{-2\pi i n x}}{-2\pi i n}\,dx$$

$$= \frac{1}{-2\pi i n} + \frac{2}{2\pi i n}\left[x \frac{e^{-2\pi i n x}}{-2\pi i n}\Big|_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n}\,dx \right]$$

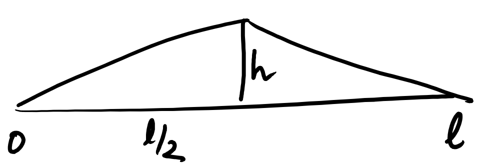
$$= \frac{1}{-2\pi i n} + \frac{1}{\pi i n}\left[\frac{1}{-2\pi i n} + \cancel{\frac{1}{2\pi i n} \times \frac{e^{-2\pi i n x}}{-2\pi i n}\Big|_0^1}^{\,0} \right]$$

$$= \frac{1}{-2\pi i n} + \frac{1}{4\pi^2 n^2}$$

So $$f_p(x) = \frac{1}{3} + \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} \left(\frac{1}{-2\pi i n} + \frac{1}{4\pi^2 n^2} \right) e^{2\pi i n x}.$$

(23) The function is
$$f(x) = \begin{cases} \frac{2x}{\ell} \cdot h & x \in [0, \ell/2] \\ h - h\left(\frac{2x}{\ell} - 1\right) & x \in [\ell/2, \ell] \end{cases}$$
$$= 2h - \frac{2hx}{\ell}$$

We will expand this in $\sin\left(\frac{n\pi x}{\ell}\right)$.



$$b_n = \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \underbrace{\frac{2h}{\ell}\int_0^{\ell/2} x \sin\left(\frac{n\pi x}{\ell}\right)dx}_{I_1} + \underbrace{2h\int_{\ell/2}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)dx}_{I_2} - \underbrace{\frac{2h}{\ell}\int_{\ell/2}^{\ell} x\sin\left(\frac{n\pi x}{\ell}\right)dx}_{I_3}$$

$$I_1 = \frac{2h}{\ell}\left[\left. \frac{x\left(-\cos\left(\frac{n\pi x}{\ell}\right)\right)}{n\pi/\ell}\right|_0^{\ell/2} - \int_0^{\ell/2} \frac{\left(-\cos\frac{n\pi x}{\ell}\right)}{n\pi/\ell}dx \right]$$

$$= \frac{2h}{\ell}\left[\frac{\ell^2}{2\pi\ell}\left(-\cos\left(\frac{n\pi}{2}\right)\right) + \frac{1}{n\pi/\ell} \left.\frac{\sin\left(\frac{n\pi x}{\ell}\right)}{n\pi/\ell}\right|_0^{\ell/2} \right]$$

$$= \frac{2h}{\ell}$$

13.8

(a) x^3 and $\sinh 2x$ are odd so their averages are $\underline{\underline{\text{zero}}}$.

$\cos 3\pi x$ has period $\frac{2}{3}$ and the interval $(-5,5)$ consists of exactly 15 periods $\left(15 \times \frac{2}{3} = 10\right)$.
Since the average on every period is zero, the average on $(-5,5)$ is $\underline{\underline{\text{zero}}}$.

$\sin^2 \pi x$ has period 2, with average $\frac{1}{2}$ on every period. So the average on $(-5,5)$, which consists of 5 periods, is also $\underline{\underline{2}}$.
Thus, the average of the whole function is $\underline{\underline{2}}$.

(b) The last three terms are odd so they contribute nothing.

$\sin^2 3x$ has period $\frac{2\pi}{3}$ and average $\frac{1}{2}$ on every period.
So the average of $2\sin^2 3x$ on $(-\pi,\pi) = 3$ periods is $2 \times \frac{1}{2} = \underline{\underline{1}}$.

Section 12

(6) $$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} x e^{-i\alpha x}\, dx$$

$= 0$ when $\alpha = 0$

otherwise $$= \frac{1}{\sqrt{2\pi}}\left[x\frac{e^{-i\alpha x}}{-i\alpha}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{-i\alpha x}}{-i\alpha}\, dx \right]$$

$$= \frac{-1}{\sqrt{2\pi}\, i\alpha}\left[\left(e^{-i\alpha} + e^{i\alpha}\right) - \frac{e^{-i\alpha\alpha} - e^{i\alpha}}{-i\alpha} \right]$$

$$= \frac{-1}{\sqrt{2\pi}\, i\alpha}\left[2\cos(\alpha) - \frac{2}{\alpha}\sin(\alpha) \right]$$

So $$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{2}{\alpha}\sin(\alpha) - 2\cos(\alpha)}{i\alpha}\, e^{i\alpha x}\, d\alpha$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha) - \alpha\cos\alpha}{i\alpha^2}\, e^{i\alpha x}\, dx$$

12.10

TODO

12.24 (a) $$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-i\alpha x + x} dx + \int_{0}^{\infty} e^{-i\alpha x - x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{x(1-i\alpha)}}{1-i\alpha} \Big|_{-\infty}^{0} + \frac{e^{-x(1+i\alpha)}}{-(1+i\alpha)} \Big|_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\alpha^2} \right)$$

So we have $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{i\alpha x} d\alpha$$

$$= \frac{1}{\pi} \int_{-\infty}^{0} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha + \int_{0}^{\infty} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1+\beta^2} e^{-i\beta x} d\beta + \int_{0}^{\infty} \frac{1}{1+\alpha^2} e^{i\alpha x} d\alpha$$

taking $\beta = -\alpha$
$d\beta = -d\alpha$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{e^{i\alpha x} + e^{-i\alpha x}}{1+\alpha^2} d\alpha = \frac{2}{\pi} \int_{0}^{\infty} \frac{\cos \alpha x}{1+\alpha^2} d\alpha$$

(b) We have

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-|x|} \cos\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \left(\frac{e^{i\alpha x} + e^{-i\alpha x}}{2} \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x(1-i\alpha)} + e^{-x(1+i\alpha)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(e^{-x(1-i\alpha)} \Big|_0^\infty + e^{-x(1+i\alpha)} \Big|_0^\infty \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \right) = \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+\alpha^2} \right).$$

The Inverse Cosine transform gives:

$$(\star) \text{———} \quad e^{-|x|} = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{2}{1+\alpha^2} \cos\alpha x \, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{\cos\alpha x}{1+\alpha^2} d\alpha$$

which is the same.

(c) The Fourier Cosine transform of $\frac{1}{1+x^2}$ is

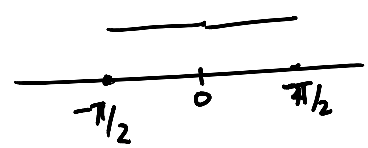
$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos\alpha x}{1+x^2} dx = \sqrt{\frac{2}{\pi}} \frac{\pi}{2} e^{-|\alpha|}$$ by $(\star)$, interchanging x and α

$$= \underline{\underline{\sqrt{\frac{\pi}{2}} e^{-|\alpha|}}}.$$

(12.27) (a) The fourier cosine integral is

$$\hat{f}_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\alpha x \, dx$$

corresponding to the fourier transform of the even extension $f_c(x) =$



$$= \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \cos(\alpha x)\, dx$$

$= \text{for } \alpha \neq 0 : \sqrt{\frac{2}{\pi}} \left. \frac{\sin(\alpha x)}{\alpha} \right|_0^{\pi/2} = \sqrt{\pi/2} \left(\frac{\sin\left(\frac{\pi\alpha}{2}\right)}{\alpha \pi/2} \right)$

$\text{for } \alpha = 0 : \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} dx = \sqrt{\frac{\pi}{2}} \left(= \lim_{\alpha \to 0} \sqrt{\frac{\pi}{2}} \frac{\sin \frac{\pi\alpha}{2}}{\left(\frac{\pi\alpha}{2}\right)} . \right)$

So $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\alpha) \cos(\alpha x)\, d\alpha$

$$= \int_0^{\infty} \frac{\sin(\pi\alpha/2)}{\pi\alpha/2} . \cos(\alpha x)\, dx$$

(b) The Fourier sine integral, corresponding to the odd extension

$f_s(x) =$

is given by $\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\pi/2} \sin(\alpha x)\, dx$

$= \text{when } \alpha \neq 0 : \sqrt{\frac{2}{\pi}} \left. \frac{-\cos(\alpha x)}{\alpha} \right|_0^{\pi/2} = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos\left(\frac{\alpha\pi}{2}\right)}{\alpha} \right)$

$\text{when } \alpha = 0 : \sin(\alpha x) = 0 \text{ so } \hat{f}_s(0) = 0.$

So we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{1-\cos(\alpha\frac{\pi}{2})}{\alpha}\right) \sin(\alpha x)\, d\alpha$$

$$= \int_0^\infty \frac{1-\cos(\alpha\pi/2)}{\pi/2\,\alpha} \sin(\alpha x)\, dx.$$

12.34

Let f_1, f_2 be two functions satisfying Dirichlet conditions. Then, the conjugate of the Fourier transform of f_1 is:

$$\overline{\hat{f_1}}(\alpha) = \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_1(x) e^{-i\alpha x} dx}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f_1}(x)\, e^{i\alpha x} dx.$$

So the inner product $\langle \hat{f_1} | \hat{f_2} \rangle$ is

$$\int_{-\infty}^{\infty} \overline{\hat{f_1}}(\alpha)\, \hat{f_2}(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \overline{f_1}(x)\, e^{i\alpha x} dx \right) \hat{f_2}(\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} \overline{f_1}(x) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f_2}(\alpha)\, e^{i\alpha x} d\alpha \right) dx$$ exchanging the order of integration

$$= \int_{-\infty}^{\infty} \overline{f_1}(x)\, f_2(x)\, dx$$ by Fourier inversion

In particular taking $f_1 = f_2$ we get

$$\int_{-\infty}^{\infty} |\hat{f}(\alpha)|^2 \, d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 \, dx.$$

(3) (a) $\overline{\hat{f}(\alpha)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)\, e^{-i\alpha x}}\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$ since $\overline{f(x)} = f(x)$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i(-\alpha)x}\, dx = \underline{\underline{\hat{f}(-\alpha)}}$

(b) Assume $f(-x) = f(x)$.

Then, $\hat{f}(-\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$

substituting $y = -x$, $dy = -dx$

$= \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-y)\, e^{-i\alpha y}\, dy$

(☆) $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\alpha y}\, dy$ since $f(-y) = f(y)$

$= \hat{f}(\alpha)$ so $\hat{f}$ is even.

The argument for odd f is the same except in (☆) we get $-\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\alpha y}\, dy = -\hat{f}(\alpha)$ instead.

This means that a real even function has $\overline{\hat{f}(\alpha)} = \hat{f}(-\alpha) = \hat{f}(\alpha)$ so its fourier transform is real.

For a real odd function, $\overline{\hat{f}(\alpha)} = \hat{f}(-\alpha) = -\hat{f}(\alpha)$, so $\hat{f}$ is purely imaginary.

(c) We have

$$(\mathcal{F}^{-1} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) e^{i\alpha x} \, d\alpha$$

substituting $y = -\alpha$, $dy = -d\alpha$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-y) e^{-iyx} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{rev}(y) e^{-iyx} \, dy$$

$$= (\mathcal{F} f^{rev})(x).$$

4 (a) $$(g * f)(x) = \int_{-\infty}^{\infty} g(y) f(x-y) \, dy$$

Substituting $x - y = z \Rightarrow -dy = dz$

$$= -\int_{\infty}^{-\infty} g(x-z) f(z) \, dz = \int_{-\infty}^{\infty} f(z) g(x-z) \, dz$$

$$= f * g.$$

(b) Applying the fact to $\mathcal{F}f$ and $\mathcal{F}g$ we have:

$$\mathcal{F}(\mathcal{F}f * \mathcal{F}g) = \sqrt{2\pi}\ \mathcal{F}\mathcal{F}f \cdot \mathcal{F}\mathcal{F}g .$$

But by 3c, $\mathcal{F}\mathcal{F}f = \mathcal{F}\mathcal{F}^{-1}f^{rev} = f^{rev}$, so

this is the same as

$$\mathcal{F}(\mathcal{F}f * \mathcal{F}g) = \sqrt{2\pi}\, f^{rev} \cdot g^{rev} = \sqrt{2\pi}\,(f \cdot g)^{rev}$$

Taking inverse Fourier transforms of both sides:

$$\mathcal{F}f * \mathcal{F}g = \sqrt{2\pi}\ \mathcal{F}^{-1}(f \cdot g)^{rev} = \sqrt{2\pi}\,\mathcal{F}(f \cdot g)$$

by 3c

which is the desired conclusion.

(5) In polar coordinates,

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}}\, dx\, dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} e^{-\frac{(r^2\cos^2\theta + r^2\sin^2\theta)}{2}}\, r\, dr\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr\, d\theta$$

substitute $\frac{r^2}{2} = v \rightarrow r\,dr = dv$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-v}\,dv\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} -e^{-v}\Big|_0^{\infty}\,d\theta$$

$$= \frac{1}{2\pi}\cdot 2\pi = 1.$$

So $I = \pm 1$ but since $e^{-x^2/2} \geq 0$, $\underline{I = 1}$.

(6) For every t, $v(x,t)$ is a function of x, so we can take its Fourier transform with respect to x to get

$$\hat{v}(\alpha,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(x,t)e^{-ix\alpha}\,dx$$

Taking the Fourier transform (in x) of the heat equation, we have:

$$\widehat{\frac{d}{dt}v(x,t)} = \widehat{\frac{d^2}{dx^2}v(x,t)}$$

$$\Rightarrow \quad \frac{d}{dt}\hat{v}(\alpha,t) = (i\alpha)^2\hat{v}(\alpha,t) = -\alpha^2\hat{v}(\alpha,t)$$

which is an ODE in t with solution

$$\hat{v}(\alpha,t) = e^{-\alpha^2 t}\,\hat{v}(\alpha,0)$$

We recall that the Fourier transform of a Gaussian is a (scaling of) another Gaussian with reciprocal variance:

$$\mathcal{F}\left(\frac{1}{\sigma}e^{\frac{-x^2}{2\sigma^2}}\right) = e^{-\frac{\sigma^2\alpha^2}{2}}. \quad (\bigstar\bigstar)$$

In particular $v(x,0) = e^{-\frac{x^2}{2}}$ so $\sigma = 1$

and we have $\hat{v}(\alpha,0) = \mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = e^{-\frac{\alpha^2}{2}}$.

Thus,
$$\hat{v}(\alpha,t) = e^{-\alpha^2 t}\, e^{-\frac{\alpha^2}{2}} = e^{-\frac{\alpha^2}{2}(1+t)},$$

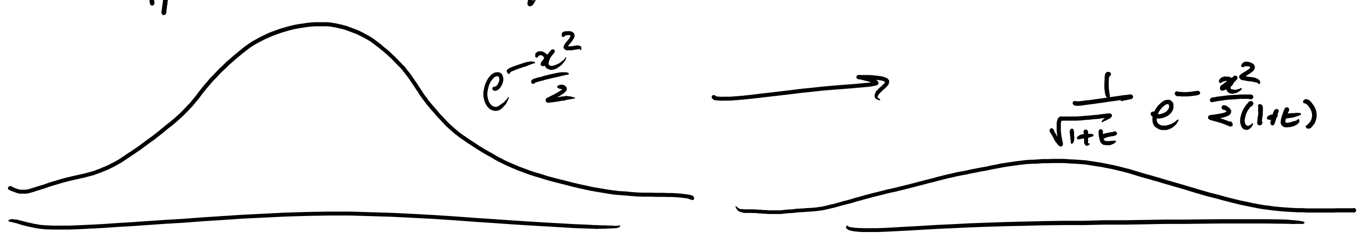
which is also a (scaling of) a Gaussian.

Applying ~~it~~ again (this time to compute the inverse Fourier transform), we find that

$$v(x,t) = \mathcal{F}^{-1}\left(e^{-\frac{\alpha^2}{2}(1+t)}\right) = \frac{1}{\sqrt{1+t}}\, e^{-\frac{x^2}{2(1+t)}}$$

Thus, the temperature at time t is given by ($\sqrt{2\pi}$ times) a Gaussian distribution with variance $\sigma^2 = 1+t$.

As $t \to \infty$ the variance goes to infinity and the temperature approaches zero everywhere.



The point of this question was to demonstrate how easy it is to deal with Fourier transforms of Gaussians — No integration required!

⑦ $$W_f = \frac{\sqrt{2\pi}}{f(0)} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \frac{\sqrt{2\pi}}{f(0)} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(0)x}\,dx$$

$$= \sqrt{2\pi}\, \frac{\hat{f}(0)}{f(0)}.$$

Similarly, $$W_{\mathcal{F}f} = \frac{\sqrt{2\pi}}{\hat{f}(0)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i(0)\alpha}\,d\alpha$$

$$= \sqrt{2\pi}\, \frac{f(0)}{\hat{f}(0)}.$$

So $$W_f \cdot W_{\mathcal{F}f} = \sqrt{2\pi}\,\frac{\hat{f}(0)}{f(0)} \cdot \sqrt{2\pi}\,\frac{f(0)}{\hat{f}(0)} = \underline{\underline{2\pi}}.$$