Math 121A Spring 2015, Homework 10

Due April 20 at 5pm in my office, or in class

We will denote the Fourier transform of $f : \mathbb{R} \to \mathbb{C}$ as

$$\hat{f}(\alpha) = (\mathcal{F}f)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

and the inverse Fourier transform of $\hat{f} : \mathbb{R} \to \mathbb{C}$ as:

$$(\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha.$$

Do not worry about convergence of integrals for this homework — assume all functions are nice enough so that the relevant Fourier transforms exist and the Fourier inversion theorem $\mathcal{F}^{-1}\mathcal{F}f = f$ works.

1. Boas 9.1, 9.20, 9.23, 13.8.
2. Boas 12.6, 12.10, 12.12, 12.24 (read pages 381-832), 12.27, 12.34 (read page 384).
3. (a) Show that if f is real-valued, then $\hat{f}(-\alpha) = \overline{\hat{f}(\alpha)}$.
   (b) Show that if f is even then $\hat{f}$ is even, and if f is odd then $\hat{f}$ is odd.
   What do (a) and (b) together say about the Fourier transform of a real even function? A real odd function?
   (c) Let $f^{rev}(x) = f(-x)$ be the reversal/reflection of f. Show that
   $$\mathcal{F}^{-1}f = \mathcal{F}f^{rev}.$$
   For this problem, you have to ignore the 'type' of f (i.e., whether it is a function of x or of α) and treat $\mathcal{F}$ and $\mathcal{F}^{-1}$ simply as operators which take a function and spit out another function.
4. (a) Show that $f * g = g * f$.
   (b) Use 1c and the fact that
   $$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}f \cdot \mathcal{F}g$$
   to show that
   $$\mathcal{F}(f \cdot g) = \frac{1}{\sqrt{2\pi}} \mathcal{F}f * \mathcal{F}g,$$
   i.e., that the Fourier transform turns multiplication into convolution.
5. Let
   $$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx.$$
   Express the integral
   $$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} dx dy$$
   in polar coordinates, and use this expression to conclude that $I = 1$.

6. Use the Fourier transform to solve the heat equation on an infinite line:

$$\frac{d}{dt}u(x,t) = \frac{d^2}{dx^2}u(x,t)$$

with initial conditions

$$u(x,0) = e^{-x^2/2}.$$

Describe the distribution of heat at time t in words. What happens as $t \to \infty$?

7. For a function $f : \mathbb{R} \to \mathbb{C}$ with $f(0) \neq 0$, define the *rectangular width* of f to be the width of a rectangle with height $f(0)$ and area equal to that under the graph of $f(x)$, i.e.

$$W_f = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(x)dx.$$

Show that the product of the rectangular width of a function and that of its Fourier transform is 2π, i.e.

$$W_f \cdot W_{\mathcal{F}f} = 2\pi.$$

Thus, if a function has small rectangular width, its Fourier transform must have large rectangular width.

This can be viewed as a baby version of the Uncertainty Principle, which says that the product of the variance of f and the variance of $\mathcal{F}f$ is large. (This is the same as the uncertainty principle in physics, since position and momentum are Fourier transforms of each other).