Asymptotic Notation, Applications of Series (Lecture 5)

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1 Asymptotic Notation

In the past few lectures, we have often made statements like:

$$\sin(x) = x - x^3/3! + x^5/5! + \text{“lower order terms”},$$

by which we mean that the terms we are ignoring matter less than the terms that we have written and should be regarded as "error". There is a convenient shorthand notation which makes such statements rigorous and is also very useful in calculations, called "little-oh" notation:

Notation. Given continuous functions $f(x)$ and $g(x)$, we say that $f(x) = o(g(x))$ as $x \to a$ if

$$\lim_{x\to a} \frac{f(x)}{g(x)} = 0.$$

For example, we have $x^2 = o(x)$ as $x \to 0$ since $\lim_{x\to 0} x^2/x = 0$, and similarly $x^5 = o(1)$ as $x \to 0$. Often it will be clear from the context what x is tending to, and we will not write it.

There is also an accompanying "big-oh" notation: we say that $f(x) = O(g(x))$ as $x \to a$ if

$$\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty.$$

Both of these are actually not equalities but inequalities; the first one says that f decays strictly faster than g, and the second one says that f decays at least as fast as g (up to a constant factor), as $x \to a$.

1.1 The exponential function

With this notation in hand, we can write

$$e^x = 1 + x + x^2/2! + o(x^2) \text{ as } x \to 0,$$

since the remainder term can be written as

$$R_2(x) = e^x - (1 + x + x^2/2!) = x^3/3! + x^4/4! + \ldots = x^3(1/3! + x/4! + x^2/5! + \ldots).$$

Since $|1/3! + x/4! + \ldots| \le |1 + x + x/2! + \ldots| = |e^x| \le e$ for $x < 1$, we have

$$|R_2| \le e \cdot x^3 = o(x^2) \text{ as } x \to 0.$$

Similarly, we can also write

$$e^x = 1 + x + x^2/2! + O(x^3).$$

It is important to note that what is written above is only sensible when $x \to 0$, and is not true for o and misleading for O when, for instance, $x \to 100$, in which case the term $100^3/3!$ is much *more* significant than the term $100^2/2!$ and is certainly not "lower order".

1.2 Truncating a general power series

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent for $|x| < R$. Then we can write

$$f(x) = \left(\sum_{n=0}^{N} a_n x^n\right) + a_{N+1}x^{N+1} + a_{N+2}x^{N+2} + \ldots = S_N + x^{N+1}(a_{N+1} + a_{N+2}x + \ldots + a_{N+k}x^{k-1} + \ldots).$$

Since $f(x)$ is convergent in $(-R, R)$, the series in parentheses

$$g(x) = a_{N+1} + a_{N+2}x + \ldots + a_{N+k}x^{k-1} + \ldots$$

must also be convergent in $(-R, R)$, which contains the closed interval $[-R/2, R/2]$. Thus we can take

$$M = \max_{x \in [-R/2, R/2]} |g(x)|$$

and conclude that for $|x| < R/2$:

$$|f(x) - S_N| = |x^{N+1}g(x)| \leq M|x^{N+1}| = o(x^N),$$

so that

$$f(x) = \left(\sum_{n=0}^{N} a_n x^n\right) + o(x^N).$$

1.3 Manipulation Rules

The notation is handy because of the following easy to check rules, which allow one to easily keep track of which terms are important and how quickly the error drops while adding/multiplying/dividing/composing series:

1. If $c \in \mathbb{R}$ and $f(x) = o(g(x))$ then $cf(x) = o(g(x))$.
2. If $f_1(x) = o(g_1(x))$ and $f_2(x) = o(g_2(x))$ then $f_1(x)f_2(x) = o(g_1(x)g_2(x))$.
3. If $f(x) = o(g(x))$ then $xf(x) = o(xg(x))$.
4. If $\lim_{x \to 0} g(x) = 0$ then $\frac{1}{1+g(x)} = 1 - g(x) + o(g(x))$.
5. $o(f(x)) + o(g(x)) = o(f(x) + g(x))$.
6. $o(o(f(x)) = o(f(x))$.

To see this in action, consider the example:

$$\frac{1}{\cos(x)} = \frac{1}{1 - x^2/2 + o(x^2)} = 1 + x^2/2 + o(x^2),$$

or number (15.2) from the book:

$$\begin{aligned}
\frac{1}{\sqrt{1+x^4}} - cos(x^2) &= (1+x^4)^{-1/2} - cos(x^2) \\
&= \left(1 + (-1/2)x^4 + \frac{(-1/2)(-3/2)}{2!}x^8 + o(x^8)\right) - \big(1 - (x^2)^2/2! + (x^2)^4/4! + o(x^8)\big) \\
&= \left(\frac{3}{8} - \frac{1}{4!}\right)x^8 + o(x^8) = \frac{x^8}{x} + o(x^8),
\end{aligned}$$

or something like:

$$\begin{aligned}
(1+x)^{1/x} &= \exp\left(\frac{\log(1+x)}{x}\right) \\
&= \exp(1 - x/2 + x^2/3 + o(x^2)) \\
&= e \cdot \exp(-x/2 + x^2/3 + o(x^2)) \qquad \text{noting that } -x/2 + x^2/3 + o(x^2) \to 0 \text{ as } x \to 0 \\
&= e \cdot \left(1 + (-x/2 + x^2/3 + o(x^2)) + \frac{1}{2}(-x/2 + x^2/3 + o(x^2))^2 + o(x^2)\right) \\
&= e \cdot \left(1 + (-x/2 + x^2/3 + o(x^2)) + \frac{1}{2}(-x/2)^2 + o(x^2)\right) \\
&= e \cdot \big(1 - x/2 + (11/24)x^2 + o(x^2)\big)
\end{aligned}$$

I find it much easier to do it this way than writing ... everywhere.

Similar statements are true for power series around a, and for $O()$ notation.

2 Cool Applications of Series

2.1 Maximum Overhang

This is example 16.1a in the book. (look at the picture). Identical blocks of length 2 are stacked on top of each other. The goal is to stack as many as possible without the whole thing falling down. The sufficient condition for this to happen is that for each block, the center of gravity of the blocks above it must lie above the block (i.e., not outside it). If there are N blocks, how far to the right can such a structure go?

One way to build the structure is by proceeding recursively. For $N = 1$ there is just one block, and let's say it is centered at position $x = 0$ (we will not worry about the $y-$coordinate in this problem). For $N = 2$ we shift this block by 1 and place it over the second block, so that its center of gravity is right at the edge of the lower block; we will call such a configuration *critical.* The center of gravity of this new configuration is at $x = (0 + 1)/2 = 1/2$, and it has weight 2 (i.e., 2 blocks). We then repeat, placing this critical configuration on top of the third block, and shift it as far right as possible while keeping the center of gravity above the third block. In this step, the center of gravity of the first two blocks was already at $1/2$, so the shift to the edge is just $1 - 1/2 = 1/2$. The new center of gravity is now at $(2 \cdot 1) + 1 \cdot (0))/3 = 2/3$. This configuration is also critical, and in the next step it be shifted by $1/3$.

The general pattern is that at the nth stage, there will be n blocks whose center of gravity is at $c_n < 1$. When we add the $(n+1)$st block at the bottom, we shift the first n blocks right by $1 - c_n$ so that their center of gravity is at 1. Then the new center of gravity of the $(n+1)$ blocks together is $\frac{n \cdot 1 + 1 \cdot 0}{n+1} = n/n + 1$, and the next shift is $1 - \frac{n}{n+1}$. So if we do this N times, the total shift is

$$1 + 1/2 + 1/3 + \dots$$

which is just the harmonic series! Thus if we have N blocks this strategy gets us to a distance of roughly $\log(n)$.

It was an open problem for about fifty years whether it is possible to do better. Surprisingly, the $\log(n)$ was improved to $n^{1/3}$ (which grows much, much faster) in 2008 by a group of five mathematicians, including one from this department. They also showed that you can't beat $n^{1/3}$.

See `https://math.dartmouth.edu/ pw/papers/maxover.pdf`.

2.2 Gaussian integration

The normal distribution shows up all over the sciences and in statistics, but we don't actually know how to compute the area under it. This can however be handled very easily using the series for $\exp(-x^2)$:

$$\int_{-1/2}^{1/2} 1 - x^2 + x^4/4 + o(x^4).$$

2.3 Proof that e is irrational

This proof is due to Fourier.

Suppose $e = a/b$ for some integers a, b. Then the bth order error term times $b!$:

$$x = b! \left(e - \sum_{n=0}^{b} \frac{1}{n!} \right)$$

must be an integer, since all of the denominators divide $b!$. On the other hand, we can write this as

$$x = b! \left(\sum_{n=b+1}^{\infty} \frac{1}{n!} \right) = \sum_{n=b+1}^{\infty} \frac{1}{n(n-1)\dots(b+1)}.$$

We now apply the comparison test: each term is bounded by a power of $(b+1)$:

$$\frac{1}{n(n-1)\dots(b+1)} < \frac{1}{(b+1)^{n-b}},$$

so the sum is bounded by the geometric series:

$$\sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}} = \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = \frac{(1/b+1)}{1 - 1/(b+1)} = \frac{1}{b} < 1,$$

which is a contradiction since we assumed x was an integer.