Lecture notes for Green's Functions

Motivation: Suppose I am interested in solving systems of linear equations

$$Ax = b_1, \quad Ax = b_2, \quad \dots \quad Ax = b_m$$

in some $n \times n$ matrix A and many different vectors $b_1 \dots b_m$.

Instead of solving them separately, I could compute the inverse A^{-1} once. With this inverse in hand, the solutions are simply given by matrix multiplication

$$y_1 = A^{-1} b_1, \quad y_2 = A^{-1} b_2, \quad \dots \quad y_m = A^{-1} b_m .$$

One way to conceptualize what the inverse means is the following:

① Solve the system for the "basic" right hand sides

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \dots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and call the solutions $g_1 \dots g_n$ i.e.

$$A g_1 = e_1, \quad \dots \quad A g_n = e_n. \qquad \text{———} \quad (A)$$

② Note that every right hand side b can be written as a linear combination of the e_i:

$$b = \beta_1 e_1 + \beta_2 e_2 \dots + \beta_n e_n$$

for some coefficients $\beta_1 \cdots \beta_n$

But now, summing the equations ($\ast$) with the same weights we find

$$\beta_1 A g_1 + \beta_2 A g_2 \cdots + \beta_n A g_n = \beta_1 e_1 + \cdots \beta_n e_n$$

$$\Downarrow$$

$$A(\beta_1 g_1 + \beta_2 g_2 \cdots \beta_n g_n) = b$$

i.e. the solution is given by the same linear combination of the basic solutions:

$$\beta_1 g_1 + \cdots + \beta_n g_n = \begin{bmatrix} | & | & | & & | \\ g_1 & g_2 & g_3 & \cdots & g_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$=: G b$$

Thus we have $G = A^{-1}$, and the columns of A^{-1} are simply the basic solutions $g_1, \ldots, g_n$

The point is that once you know the basic solutions, you can solve the system for any b simply by taking a linear combination of them, which is the same thing as matrix multiplication.

<u>Green's Functions</u> are a way of doing the same thing for <u>linear</u> differential equations, except that:

- the "basic" e_i are replaced by delta functions
- linear combinations are replaced by integrals.

Specifically, suppose I want to solve second order systems:

$$A y''(x) + B y'(x) + C y(x) = f(x)$$

for $y(x)$ defined on some domain with some boundary conditions.

for many different forcing functions $f(x)$.

Let $L = A\frac{d^2}{dx^2} + B\frac{d}{dx} + C$ denote the differential operator corresponding to the LHS, so the equation is just:

$$Ly = f.$$

The strategy is to solve this equation for the (infinitely many) "basic" forcing functions $\{\delta(x-x')\}_{x' \in \mathbb{R}}$

where we recall that $\delta(x-x')$ is defined by

$$\int_{-\infty}^{\infty} \delta(x-x') f(x)\, dx = f(x') \quad \text{——} \; (\cancel{AA})$$

for every test function $f(x)$ and looks like an "infinitely peaked spike" at x'.

The Green's function $G(x,x')$ is defined as follows: for every x', the univariate function $G(x,x')$ (in x) is the solution of the DE

(3) $\boxed{Ly(x) = \delta(x-x')}$ (subject to boundary cond.).

If you think of functions as being infinite vectors, then G is an "infinite matrix" whose columns are the solutions to (3) indexed by x'. (Thus $G(x,x')$ is analogous to the inverse matrix G from the linear equation setting).

The upshot is that the solution to $Ly=f$ is now given by

$$\int_{-\infty}^{\infty} G(x,x')\, f(x')\, dx'$$

since

$$L\int_{-\infty}^{\infty} G(x,x')\, f(x')\, dx'$$

$$= \int_{-\infty}^{\infty} L\, G(x,x')\, f(x')\, dx' \quad \text{since } L \text{ is a differential operator in } \underline{\underline{x}}$$

(4) $$= \int_{-\infty}^{\infty} \delta(x-x')\, f(x')\, dx' \quad \text{by definition}$$

$$= \underline{\underline{f(x)}}.$$

The integral transform $\int G(x,x')f(x')dx'$ is the analogue of the matrix vector product, and (4) is a continuous way of writing that $f(x)$ is a "linear combination" of $\delta(x-x')$ with coefficients $f(x')$

How to compute Green's Functions

You can find the fundamental solutions $G(x,x')$ (also called point source solutions) to $Ly = \delta(x-x')$ sub. boundary conditions

any way you like. More than anything else, the choice of method is dictated by the boundary conditions.

Here are some prescriptions for which methods work when:

Domain	Boundary Conditions	Method
$\mathbb{R}$	$y(x) \to 0$ as $x \to \pm\infty$	Fourier Transform
$[0,\infty)$	$y(0), y'(0)$ given (initial conditions)	Laplace Transform
$[0,2L]$	$y(0)=y(2L)$, $y'(0)=y'(2L)$ (periodic BC)	Fourier Series
$[0,L]$	$y(0)=y(L)$ (Dirichlet BC)	Fourier Sine Series
$[0,L]$	$y'(0)=y'(L)$ (Neumann BC)	Fourier Cosine Series

(This is not an exhaustive list, and there are many more methods for solving differential equations.)

(Note that all these methods are suited to the $\delta(x-x')$ forcing function, which only makes sense inside an integral.)

There is also a "bare hands" way to compute the fundamental solutions without appealing to any of the above.

This method has the following steps (also described on pages 462-463 of the book), for the equation

$$Ly = \delta(x-x')$$

defined on $[a,b]$ with boundary conditions

① Find the general solutions $y_<(x)$ and $y_>(x)$ to the <u>homogeneous</u> ODE

$$Ly_<(x) = 0 \quad \text{on } [a, x')$$
$$Ly_>(x) = 0 \quad \text{on } (x', b].$$

These will have <u>four</u> undetermined coefficients for a second order L.

② Eliminate <u>two</u> of these coefficients using the boundary conditions at a and b

③ Eliminate the remaining two coefficients by matching the derivatives of $y_<(x)$ and $y_>(x)$ at x':

1) $\lim_{\epsilon\to 0^+} y_<(x-\epsilon) = \lim_{\epsilon\to 0^+} y_>(x+\epsilon)$

2) $\lim_{\epsilon\to 0} y'_<(x-\epsilon) = \lim_{\epsilon\to 0} y'_>(x+\epsilon)$

Here is an example showing how this is done and justifying the matching of derivatives.

Eg: $y''(x) - y(x) = f(x)$ on $\mathbb{R}$, subject to $y(x) \underset{x \to \pm\infty}{\longrightarrow} 0$.

The operator in this case is $L = \frac{d^2}{dx^2} - 1$.

The Green's function $G(x, x')$ is given by the solutions to

$$y''(x) - y(x) = \delta(x - x').$$

Step 1: Solve homogeneous DE to the left and right of x':

$x < x'$	$x > x'$
$y_<'' - y_< = 0$	$y_>'' - y_> = 0$
$\Rightarrow y_<(x) = c_1 e^x + c_2 e^{-x}$	$\Rightarrow y_>(x) = c_3 e^x + c_4 e^{-x}$

Step 2: Eliminate 2 of the unknown coeffs using BC:

$y_<(x) \to 0$ as $x \to -\infty$	$y_>(x) \to 0$ as $x \to \infty$
so $c_2 = 0$	so $c_3 = 0$
$y_<(x) = c_1 e^x$	$y_>(x) = c_4 e^{-x}$

Step 3 : Match solutions at x' to obtain y.

We are looking to define $y(x) = \begin{cases} y_<(x) & x<x' \\ y_>(x) = & x>x' \end{cases}$

What happens at x' is crucial, and this is where the $\delta(x-x')$ comes in

We will determine this by integrating the DE on a small interval centered at x' :

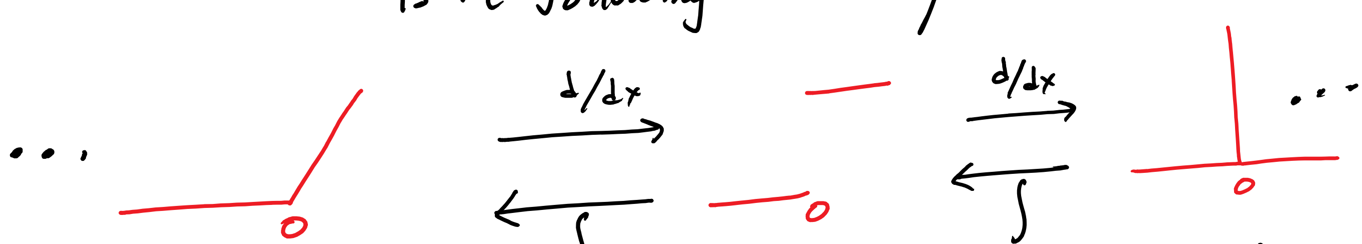
$$(5) \qquad \int_{x'-\epsilon}^{x'+\epsilon} y''(x)dx - \int_{x'-\epsilon}^{x'+\epsilon} y(x)dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')dx$$

[Recall that we are only supposed to access the δ function inside an integral.]

We now observe that

(a) The RHS is 1 by the definition of $\delta(x-x')$.

(b) $y(x)$ must be continuous at x'. The reason is that integration smooths out discontinuities and differentiation amplifies them. The picture to keep in mind is the following "hierarchy:"



ramp function $r(x) = \begin{cases} 0 & x \le 0 \\ x & x>0 \end{cases}$

step function $v(x) = \begin{cases} 0 & x\le 0 \\ 1 & x>0 \end{cases}$

delta function $\int_{-\infty}^{\infty} \delta(x) f(x) = f(0)$

$\xrightarrow{d/dx}$

$\xleftarrow{\int}$

... "It's a bit of a stretch to visualize" it this way

$\delta'(x)$ = derivative of $\delta(x)$:

$$\int_{-\infty}^{\infty} \delta'(x) f(x)\, dx = -f'(0)$$

In any case, the point is that if $y(x)$ was <u>not</u> continuous, we would have $y'(x') \propto \delta(x-x')$

and $y''(x') \propto \delta'(x-x')$

since differentiation propagates/amplifies the discontinuity. But there is no $\delta'(x)$ on the right hand side, so this is impossible.

Thus, we must have

$$\lim_{\epsilon\to 0^+} y_<(x'-\epsilon) = \lim_{\epsilon\to 0^+} y_>(x'+\epsilon)$$

and

and $\displaystyle\lim_{\epsilon\to 0}\int_{x'-\epsilon}^{x'+\epsilon} y(x)\, dx = 0$. The first condition implies

$$\boxed{c_3 e^{x'} = c_4 e^{-x'}} \quad \text{———} (6).$$

The second condition tells us that (5) reduces to

$$\lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} y''(x)\, dx = 1$$

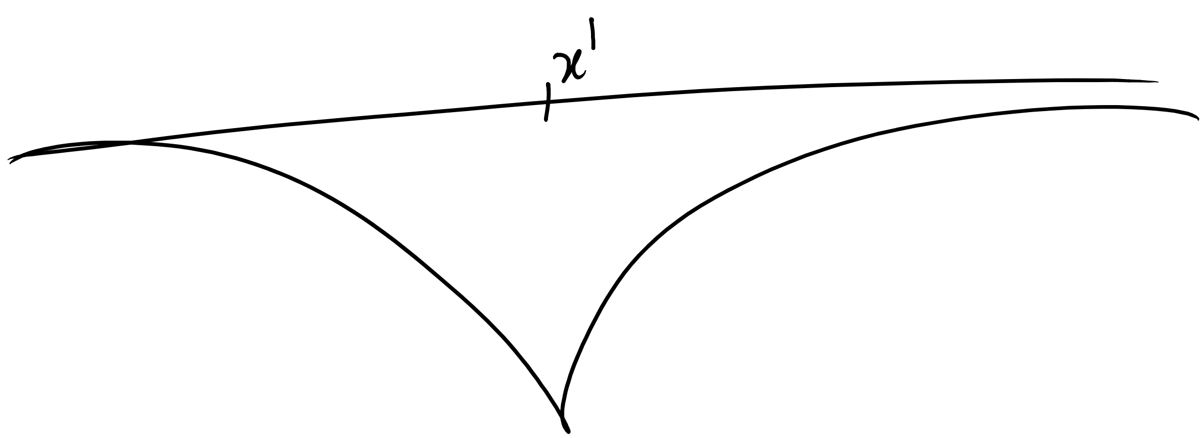
which is just $\displaystyle \lim_{\epsilon \to 0^+} y'_>(x'+\epsilon) - y'_<(x'-\epsilon) = 1$

which reveals that $\boxed{1 - C_4 e^{-x'} = C_1 e^{x} + 1}$ — (7)

Solving the linear equations (6) and (7) gives the solution $y(x) = -\frac{1}{2} e^{-|x-x'|}$

So the Green's function is

$$\underline{G(x,x') = -\frac{1}{2} e^{-|x-x'|}} .$$



Thus, the general solution of the differential eqn is given by the integral

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-x'|} f(x')\,dx'$$

which agrees with the convolution integral solution produced by using the Fourier transform.

In class, we also derived the Green's function for

$$L = \frac{d^2}{dx^2} - 1 \quad \text{on} \quad [0,\infty) \quad \text{subject to}$$

$$y(0) = 0$$
$$y(x) \to 0 \text{ as } x \to \infty.$$

Which turned out to be

$$G(x,x') = -\frac{1}{2} e^{-|x-x'|} + \frac{1}{2} e^{-|x+x'|}.$$

This can be done in the bare hands way above, or by using the "method of images" to reduce it to a problem on $\mathbb{R}$ (essentially, consider the odd extension of the problem), or by using the Fourier sine transform.