4/10/15 Properties of the Fourier transform, Convolution

Last time, we defined the Fourier transform of $f(x)$

$$(\mathcal{F} f)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\alpha} dx = \hat{f}(\alpha)$$

and the inverse fourier transform of $\hat{f}(\alpha)$ as:

$$(\mathcal{F}^{-1} \hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{ix\alpha} d\alpha \overset{\text{by the Fourier Inversion thm}}{=} f(x).$$

Notice that they are almost the same except for the minus sign. $f, \hat{f}$ are sometimes called a pair of Fourier transforms because of this symmetry.

The Fourier transform has several good properties.
(We will not worry about issues of convergence; assume f and derivatives satisfy Dirichlet conditions.)

① Linearity: If f, g are functions and a, b are scalars:

$$\mathcal{F}(af + bg) = a\mathcal{F}f + b\mathcal{F}g$$

Or in the hat notation:

$$\widehat{af+bg} = a\hat{f} + b\hat{g}.$$

This follows from the linearity of integration, and is very useful.

(2) <u>Behavior under scaling</u>:

If $g(x) = f(cx)$ then

$$\hat{g}(\alpha) = \frac{1}{c}\hat{f}\left(\frac{\alpha}{c}\right).$$

<u>Proof</u>: $$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(cx)\, e^{-i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{c} f(y)\, e^{-\frac{\alpha y}{c}}\,dy$$ change variable $y = cx$, $dy = c\,dx$

$$= \frac{1}{c}\hat{f}\left(\frac{\alpha}{c}\right) //$$

So if we "stretch" the function f by taking $c > 1$ then the fourier transform gets "squashed".

(3) <u>Differentiation corresponds to multiplication</u>

(assuming all integrals converge and $f(x) \to 0$ as $x \to \pm\infty$)

$$\left(\mathcal{F}\frac{d}{dx}f\right)(\alpha) = i\alpha(\mathcal{F}f)(\alpha)$$

or equivalently

$$\widehat{f'}(\alpha) = i\alpha\hat{f}(\alpha).$$

<u>Proof</u>: $$\widehat{f'}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\, e^{-i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\cancel{e^{-i\alpha x} f(x)\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}(-i\alpha)\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} i\alpha \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx = i\alpha \hat{f}(\alpha).$$

④ Translation: If $g(x) = f(x-t)$ then

$$\hat{g}(\alpha) = e^{-i\alpha t} \hat{f}(\alpha).$$

Proof: $$\hat{g}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t) e^{-i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\alpha y} e^{-i\alpha t} dy$$

$$= e^{-i\alpha t} \hat{f}(\alpha).$$

optional, if you think there is time

This is actually very closely related to the differentiation property when f is smooth, as follows:

$$g(x) = f(x-t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{d^k f}{dx^k}(x)\, t^k$$ by Taylor Series

Taking Fourier transforms of both sides and applying ③:

$$\hat{g}(\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} (i\alpha)^k \hat{f}(\alpha) t^k$$

$$= \hat{f}(\alpha) e^{-i\alpha t}.$$

People say that $\frac{d}{dx}$ in this way "generates" translations.

$= \underline{\underline{f(\alpha)\, e^{\cdots}}}.$ "generates" translations!

There is one more important operation, called convolution. To motivate it, we will consider the Heat equation, this time on an infinite rod!

Let's assume the rod is such that the equation is just:

$$\frac{d}{dt} v(x,t) = \frac{d^2}{dx^2} v(x,t)$$

with initial condition $v(x,0) = f(x)$.

For every fixed t, $v(x,t)$ is a function of x; let its Fourier transform be $\hat{v}(\alpha,t)$.

Taking Fourier transforms of both sides and applying ③ gives:

$$\frac{d}{dt}\hat{v}(\alpha,t) = \mathcal{F}\left(\frac{d^2}{dx^2} v(x,t)\right)$$

$$= (i\alpha)^2 \hat{v}(\alpha,t)$$

$$= -\alpha^2 \hat{v}(\alpha,t).$$

This simple ODE has the solution

$$\hat{v}(\alpha,t) = e^{-\alpha^2 t}\hat{v}(\alpha,0)$$

$$= e^{-\alpha^2 t}\hat{f}(\alpha).$$

To recover $v(x,t)$, we need to take the inverse Fourier transform of this product.

The convolution is the operation that describes how the inverse Fourier transform (and the Fourier transform) of a product

inverse Fourier transform
of two functions is related to those of the individual functions.

⑤ Convolution:

The convolution of two functions f, g is defined to be

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

Theorem: $\widehat{f*g}(\alpha) = \sqrt{2\pi}\, \hat{f}(\alpha) \cdot \hat{g}(\alpha)$

i.e. the Fourier transform turns convolution into multiplication.

(In operator notation, $\mathcal{F}(f*g) = (\mathcal{F}f)\cdot(\mathcal{F}g)$)

In particular this implies that the inverse Fourier transform of $\hat{f}(\alpha)\hat{g}(\alpha)$ is just $\dfrac{(f*g)(x)}{\sqrt{2\pi}}$.

Proof:

$$\widehat{(f*g)}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \right) e^{-i\alpha x}\, dx$$

$$= \int_{-\infty}^{\infty} f(y) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-y)\, e^{-i\alpha x}\, dx \right) dy \qquad \text{changing the order of integration}$$

$$= \int_{-\infty}^{\infty} f(y)\, \hat{h}(\alpha)\, dy \qquad \text{where } h(x) = g(x-y) \text{ is a translation of } g$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-i\alpha y}\, \hat{g}(\alpha)\, dy \qquad \text{by property ④}$$

$$= \sqrt{2\pi}\,\hat{g}(\alpha)\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\alpha x}\,dx$$

$$= \sqrt{2\pi}\;\hat{g}(\alpha)\,\hat{f}(\alpha)$$

$\square$

(**Note**: Some authors define $(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$ so that the $\sqrt{2\pi}$ goes away.
This is also fine, we just have to be consistent.)

Coming back to our heat equation, this tells us that
$$v(x,t) = \frac{f * g}{\sqrt{2\pi}}$$

where $g(x)$ is the inverse Fourier transform of $\hat{g}(\alpha) = e^{-\alpha^2 t}$.

Observe that $\hat{g}(\alpha)$ is a multiple of a __Gaussian__ of variance $\sigma = \frac{1}{\sqrt{2t}}$. As we saw in the last lecture, the Fourier transform of a Gaussian is (upto scaling) another Gaussian:

$$\mathcal{F}\left(\frac{1}{\sigma}e^{-\frac{x^2}{2\sigma^2}}\right) = e^{-\frac{\sigma^2\alpha^2}{2}}$$

In this case, we have $\frac{\sigma^2}{2} = t \implies \sigma = \sqrt{2t}$, So $\quad g(x) = \dfrac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$. (Note that g depends on t, so it's actually a function of x, t)

and we have $\quad v(x,t) = \dfrac{(f * g)(x)}{\sqrt{2\pi}}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2t}} e^{-\frac{(y-x)^2}{4t^2}} \, dx .$$

The Factor $\dfrac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}$ is called the Heat kernel.

It is amazing that the Gaussian distribution shows up in both statistical and physical phenomena, and the reasons are explained by the properties of the Fourier transform.