The Fourier Transform

In Fourier series, we actually studied two procedures which turned out to be inverses of each other:

(f periodic)

$$\begin{tikzcd}
f(x) \quad x \in \mathbb{R} \arrow[r, bend left, "\text{fourier coeffs } \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx"] & \hat{f}(n) \quad n \in \mathbb{Z} \arrow[l, bend left, "\text{Fourier series } \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{inx}"]
\end{tikzcd}$$

The point is that this gives us two different representations of the same object ($f(x)$ and $\hat{f}(n)$ both describe the same function), and this is useful because some operations/structures are much easier/more transparent in one representation as opposed to the other (eg. $\frac{d}{dx}$ becomes multiplication).

The flowchart for how to use this is similar to what we saw in diagonalization:

$$\begin{tikzcd}
\hat{f} \arrow[r, "\text{easy}"] & \widehat{\text{solution}} \arrow[d] \\
f \arrow[u] \arrow[r, "\text{hard}"] & \text{solution}
\end{tikzcd}$$

The Fourier transform is an extension of this to non periodic functions. Here is a (very) Heuristic derivation.

The high-level idea is that it is a limit of Fourier series as the period tends to infinity.

Recall that for a $2L$-periodic function, the Fourier expansion is in terms of $e^{inx\frac{2\pi}{2L}}$ (which are also $2L$-periodic):

<u>coeffs</u>: $$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-inx\frac{\pi}{L}}\,dx$$

<u>series</u>: $$f(x) = \sum_{n\in\mathbb{Z}} C_n e^{inx\frac{\pi}{L}}.$$

Notice that the frequencies are now $\alpha_n = \frac{n\pi}{L}$ rather than $n \in \mathbb{Z}$, and that as $L \to \infty$ these are getting closer together: $\Delta\alpha = \alpha_n - \alpha_{n-1} = \frac{n\pi}{L} - \frac{(n-1)\pi}{L} = \frac{\pi}{L}$.

Define the fourier coefficient at the frequency α_n to be

$$\hat{f}(\alpha_n) = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{-inx}\,dx.$$

Then, $C_n = \Delta\alpha\,\hat{f}(\alpha_n)$ and we have

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\alpha_n x} = \sum_{n\in\mathbb{Z}} \hat{f}(\alpha_n) e^{i\alpha_n x} \Delta\alpha .$$

If we interpret the later as a Riemann sum

and let $L \to \infty$, $\Delta\alpha \longrightarrow 0$

and let the α_n "converge to a continuous variable α"

(this can be made rigorous, but we will not do it)

and assume f is nice enough that all integrals and series converge, then we get

$$\boxed{\hat{f}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \quad , \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} d\alpha}$$

The first equation is the <u>definition</u> of the fourier transform of f, which is the <u>function</u> $\hat{f}(\alpha)$.

The second is the <u>inversion theorem</u> which says that f can be recovered from $\hat{f}$ by a certain integral.

Notice that the two expressions are almost the same, except for the 2π and minus sign.

To increase this symmetry, some people use the alternate normalization:

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x}\, dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x}\, d\alpha$$

which also works. <u>We will use this one from now on.</u>

Note that the fourier transform is a <u>function of functions</u> : it takes $f : \mathbb{R} \to \mathbb{C}$ and outputs

$$\hat{f} : \mathbb{R} \to \mathbb{C} .$$

We will also use the notation

$$\mathcal{F} f = \hat{f}$$

to indicate this relationship, i.e. $\mathcal{F}$ is an operator which maps functions to their fourier transforms.

$$\begin{tikzcd} f \arrow[r, bend left, "\mathcal{F}"] & \hat{f} \arrow[l, bend left, "\mathcal{F}^{-1}"] \end{tikzcd}$$

The fourier inversion theorem says that (when f is sufficiently nice) $\boxed{\mathcal{F}^{-1} \mathcal{F} f = f}$.

Two Examples:

① $$f(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise.} \end{cases}$$

The Fourier transform is easily seen to be

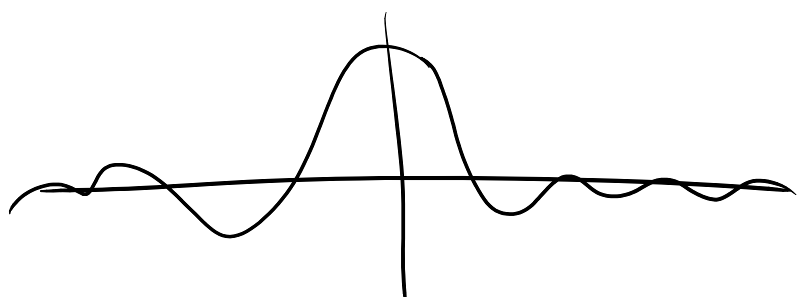
$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ix\alpha}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\alpha} - e^{i\alpha}}{(-i\alpha)}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(\alpha)}{\alpha}.$$

The latter function comes up often and has a name:

$$\operatorname{sinc}(\alpha) = \frac{\sin(\alpha)}{\alpha}.$$

It looks like this:



with zeros at the integers.

The inversion theorem tells us that

$$f(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin(\alpha)}{\alpha} e^{i\alpha x}\, d\alpha .$$

To appreciate the depth of this identity, plug in $x = 0$:

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha)}{\alpha}\,d\alpha \iff \int_{-\infty}^{\infty} \frac{\sin\alpha}{\alpha}\,d\alpha = \pi,$$

which we proved using contour integration a few weeks ago

This may be seen as a hint as to why the Fourier transform is able to represent a discontinuous function like $f(x)$ by an integral of a continuous function: contour integrals also exhibit discontinuous behavior depending on which poles are inside the contour.

Example 2: This is a very important example for physics, statistics, and many other fields.

The Gaussian distribution with mean zero and variance σ^2 has the probability density:

"Bell Curve"

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{x^2}{2\sigma^2}}$$

Then:

$$\hat{f}(\alpha) = \frac{1}{2\pi}\frac{1}{\sigma}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\, e^{-ix\alpha}\,dx$$

$$= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\left(\frac{x}{\sigma}+i\sigma\alpha\right)^2-(i\sigma\alpha)^2\right)}\,dx \qquad \text{completing the square}$$

$$= \frac{e^{-\frac{\sigma^2\alpha^2}{2}}}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,\sigma\,dy \qquad \text{taking } y = \frac{x}{\sigma} + i\sigma\alpha, \quad dy = \frac{dx}{\sigma}$$

$$= e^{-\frac{\sigma^2}{2}\alpha^2}\left(\int^{\infty} \frac{1}{\ } e^{-\frac{y^2}{2}}\,du\right)$$

$$= \frac{e^{-\frac{\sigma^2 \alpha^2}{2}}}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right) \qquad dy = \frac{dx}{\sigma}$$

$$= \frac{e^{-\frac{\sigma^2}{2}\alpha^2}}{\sqrt{2\pi}}$$

since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy = 1$

because the Gaussian is a probability density.

In particular, ignoring the $\sqrt{2\pi}$'s we have

$$\ast \qquad \mathcal{F}\left(\frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}}\right) = e^{-\frac{\sigma^2\alpha^2}{2}}.$$

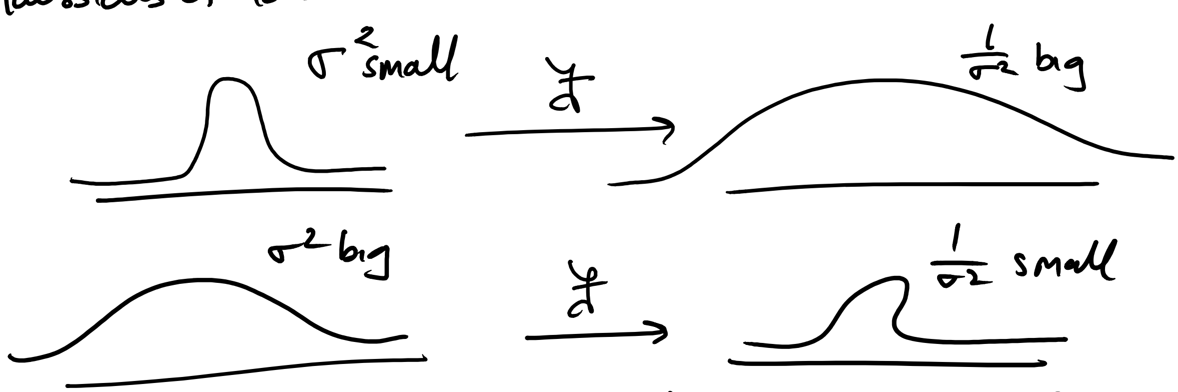
So the Fourier transform of a Gaussian of variance σ^2 is a (scaling of) a Gaussian of variance $\frac{1}{\sigma^2}$.

On the board in class, I wrote it as $\mathcal{F}\left(\frac{1}{\sqrt{2\pi}\sqrt{\sigma}} e^{-\frac{x^2}{2\sigma^2}}\right)$

$$= \sqrt{\frac{\sigma}{2\pi}}\, e^{-\frac{\sigma^2\alpha^2}{2}}$$

which is also correct.
But neither are strictly speaking Gaussian densities — they are <u>scalings</u> of Gaussian densities by $\frac{1}{\sqrt{\sigma}}$ and $\sqrt{\sigma}$ respectively).

So the Fourier transform maps Gaussians of high variance to Gaussians of low variance and vice versa:



It turns out that it is impossible for both a function f and its Fourier transform $\hat{f}$ to have low variance — this is the uncertainty principle.

It turns out that it is impossible for both a function f and its Fourier transform $\hat{f}$ to have low variance — this is the <u>uncertainty principle</u>.

Also note that when $\sigma = 1$ we have

$$\mathcal{F}\left(e^{-\frac{x^2}{2}} \right) = e^{-\frac{q^2}{2}}$$

i.e. the Gaussian with unit variance is its own Fourier transform. This is closely related to the <u>central limit theorem</u> in statistics.