4/6/15

Last week, we learned that for a 2π-periodic square integrable function $f \in L^2[-\pi,\pi]$, the Fourier Coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Satisfy

$$\left\| \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx} - f(x) \right\|^2 \longrightarrow 0$$

as $N \to \infty$

i.e. the partial sums of the Fourier series converge in the mean square sense.

This is the cleanest theorem concerning convergence of Fourier Series, but there are other theorems with stronger conclusions assuming stronger conditions on how nice f is.

Here is a very concrete and easy theorem for the special case when f is twice differentiable (which is a much stronger requirement than being square integrable).

It can be proved using the methods that we learned in the begining of the course (unlike most of the more general theorems, which are much more delicate).

Theorem: If f is 2π-periodic and twice continuously differentiable, then

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx} = f(x) \quad \text{for all } x \in \mathbb{R}$$

i.e. the partial sums converge to $f(x)$ <u>for every point x</u>.

Proof: Let's compute the n^{th} Fourier coeff. of $f''(x)$, using integration by parts:

$$2\pi\, \widehat{f''}(n) = \int_{-\pi}^{\pi} f''(x)\, e^{-inx}\, dx$$

$$= \cancelto{0 \text{ by periodicity}}{f'(x)\, e^{-inx} \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} (-in) e^{-inx} f'(x)\, dx$$

$$= in \left[\cancelto{0}{e^{-inx} f(x) \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} (-in) e^{-inx} f(x)\, dx \right]$$

$$= (in)^2 \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx = \underline{\underline{-n^2\, 2\pi\, \hat{f}(n)}}.$$

Thus, $\hat{f}(n) = \dfrac{-1}{n^2}\, \widehat{f''}(n)$

But since $f''(x)$ is <u>continuous</u>, it must have a maximum value M on $[-\pi, \pi]$, so by the triangle inequality:

$$\left|\widehat{f''}(n)\right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f''(x)|\, |e^{-inx}|\, dx \le \frac{M \cdot 2\pi}{2\pi} = M \;//$$

Thus, $|\hat{f}(n)| \leq \frac{M}{n^2}$ and the series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$ converges absolutely for every x

by applying the comparison test with $\sum \frac{1}{n^2}$.

It can be shown that this implies it must converge to $f(x)$, but we will skip this part. ∎

An important general principle used in the proof is that differentiation turns into multiplication by (in) of the Fourier coeffs, which implies that a function with many continuous derivatives must have rapidly decaying Fourier coefficients.

In particular, repeating the above argument for the kth derivative yields:

$$\boxed{f^{(k)}(x) \text{ exists for all } x \text{ and is continuous} \iff |\hat{f}(n)| \leq \frac{M}{n^k}}$$

Understanding convergence for less smooth functions where the above only holds for $k=0,1$ or if the function is not continuous (but only integrable) can be very subtle and is a rich area with many different results and cases

Since part of the whole point is to be able to decompose such 'rough' functions into sin/cos/exp, we will mention one more general theorem which is quite useful and establishes pointwise convergence (i.e. for all x).

Dirichelet's Theorem: If $f : [-\pi, \pi] \to \mathbb{C}$ has finitely many discontinuities (including endpoints), maxima, and minima and $\int_{-\pi}^{\pi} |f(x)| < \infty$ then

this is called "L^1"

$$\lim_{N \to \infty} \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx} = \frac{f^+(x) + f^-(x)}{2}$$

for every x.

where $f^+(x) = \lim_{y \to x^+} f(y)$, $f^-(x) = \lim_{y \to x^-} f(y)$.

(This is equal to $f(x)$ at every point where $f(x)$ is continuous, and equal to the midpoint of every finite jump discontinuity of $f(x)$.)

In any case, L^2 convergence is the "right" notion of convergence for physics, because if $f, g \in L^2$ represent some physically relevant quantities and $\|f - g\| = 0$ then it takes an infinite amount of energy to perform a measurement which distinguishes f and g.

So, L^2 convergence is also called "convergence in energy" and the L^2 (mean square error) reveals how much energy is required to distinguish f and g.

Another advantage is that the L^2 error can be easily estimated using Parseval's theorem:

$$\| S_N - f \| = \| \sum_{|n|>N} \hat{f}(n) e^{inx} \|$$

$$= \sum_{|n|>N} |\hat{f}(n)|^2$$

which is guaranteed to converge since $f \in L^2$ and may be estimated more accurately using other properties of f.

On the other hand, it is in general difficult to get bounds on the point wise error

$$|f(x) - S_N(x)| = \Big| \sum_{|n|>N} \hat{f}(n) e^{inx} \Big|$$

Since for rough f the latter series converges conditionally but not absolutely (and is not alternating), so the methods of chapter 1 don't work.

Even/odd functions and sin/cos series:

f is even if $f(-x) = f(x)$

- Every function can be split into an even & odd part.
- For sin/cos series,

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx \quad \text{if even}$$

$$= \sum b_n \sin nx \quad \text{if odd.}$$

Half-period Series.

$$f : [0, L] \longrightarrow \mathbb{C}.$$

even extension: $f(-x) = f(x)$

odd extension: $f(-x) = -f(x)$.

Sin coeffs: $a_n = \frac{1}{2L} \int_{-L}^{L} f(x) \sin(nx)\, dx$

Cosine coeffs:

Differences between Taylor & Fourier Series:

Taylor

- for smooth (differentiable) functions
- Local approximation, converges near a point
- coefficients are computed by looking at derivatives at one point.
- Easy to establish (absolute) convergence

Fourier

- for integrable functions (not nec smooth)
- Global approximation, "converges" (in L^2, or in Dirichlet sense) everywhere
- Coeffs computed using integrals over a period
- Convergence is subtle and conditional (unless f is very smooth)
- Comes from an inner product + eigenbasis
  $\Rightarrow$ has "geometric" interpretation involving orthogonality, norms, etc

Odd and Even Functions

One important symmetry to keep in mind when working with Fourier series is $f(x) = f(-x)$.

A function with this property is called <u>even</u>, and a function with $f(-x) = -f(x)$ is called <u>odd</u>.

A product of an odd and even function is odd.

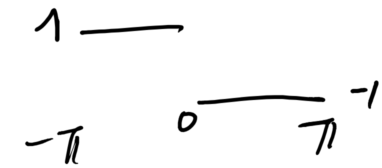
A product of two even / odd functions is even.

Thus, if f is odd, $f(x)\cos(nx)$ is odd and

$$\int_{-\pi}^{\pi} f(x)\cos(nx) = 0$$

So an odd function's sin-cos Fourier series has only sines, and similarly an even function's has only cosines.

This can be used to speed up calculations.

Eg. the square wave 

$= f(x)$

is odd, so its coefficients are just

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx \quad \text{by symmetry}$$

$$= \frac{2}{\pi}\frac{(1-\cos(n\pi))}{n}$$ which is easier than the exponential form.

Half Interval Series

A function f on $[0,\pi]$ can be extended to an odd function on $[-\pi,\pi]$

$$f_S(x) = \begin{cases} f(x) & x \in [0,\pi] \\ -f(-x) & x \in [-\pi,0] \end{cases}.$$

Since an odd function has odd cosine coefficients, its (sin-cosine) Fourier series looks like

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$ which is called a Fourier sine series.

Similarly, there is an <u>even extension</u>

$$f_C(x) = \begin{cases} f(x) & x \in [0,\pi] \\ f(-x) & x \in [-\pi,0] \end{cases}$$

with only cosine terms.

See § 7.9 for details