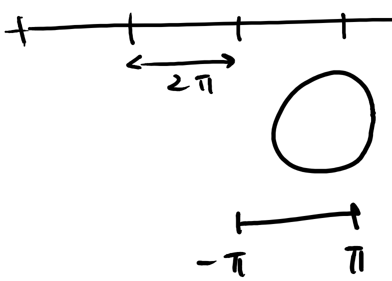
4/1/15

Fourier Series, 2

Last time, we ended with Fourier's original application of using the exponential Fourier series to solve the heat equation on a circle.

The important points were:

(1) There are three equivalent ways to think of a periodic function with period 2π:



- as a periodic function on $\mathbb{R}$
- as an arbitrary function on the unit circle, denoted S^1.
- as a function on any interval of length 2π, with the endpoints identified eg. $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$, $f'(-\pi) = f'(\pi)$, etc.

These are called periodic boundary conditions.

The length of the interval / circumference of the circle are important, since we crucially use that $\sin(nx), \cos(nx), e^{inx}$ are periodic with respect to this length, ie $e^{in(\pi)} = e^{in(-\pi)}$, etc.

Everything can be made to work for intervals of another length $2L$, such as $[-L, L]$, if we replace $\sin(nx)$ by $\sin\left(\frac{2\pi n x}{L}\right)$, e^{inx} by $e^{\frac{2\pi i n x}{L}}$, etc.

(2) Perodicity allows us to use integration by parts to "move" the $\frac{d^2}{dx^2}$ operator from one term in a product to the other:

$$\int_{-\pi}^{\pi} \frac{d^2 v}{dx^2} e^{-inx}\, dx = \cancelto{0}{e^{-inx} \frac{dv}{dx} \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} \frac{dv}{dx} \frac{de^{-inx}}{dx}\, dx$$

$$= -\left[\cancelto{0}{\frac{de^{-inx}}{dx} \cdot v \Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} v \cdot \frac{d^2}{dx^2} e^{-inx}\, dx \right]$$

$$= \int_{-\pi}^{\pi} v \cdot \frac{d^2}{dx^2} e^{-inx}\, dx.$$

<u>Note</u>: This argument would have worked if we replaced e^{inx} by <u>any</u> periodic function with the same period:

$$\int_{-\pi}^{\pi} \frac{d^2}{dx^2} f \cdot g\, dx = \int_{-\pi}^{\pi} f \cdot \frac{d^2}{dx^2} g\, dx$$

for any f, g periodic with period 2π or satisfying periodic boundary conditions on $[-\pi, \pi]$.

Thus, it is more a property of the $\frac{d^2}{dx^2}$ operator + boundary conditions than the particular functions in this problem.

(3) $\frac{d^2}{dx^2} e^{inx} = (in)^2 e^{inx} = -n^2 e^{inx}$.

i.e., the exponential functions are eigenvectors of the $\frac{d^2}{dx^2}$ operator. This is a very special property which $\sin(nx)$ and $\cos(nx)$ also share, but which no other functions have.

(4) The functions e^{inx} are "orthogonal" in that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}.$$

This allows us to compute the Fourier coefficients in $f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}$ just by computing integrals

Notation: We will denote the n^{th} fourier coefficient c_n of f by $\hat{f}(n)$. This is handy when there are multiple functions involved.

Thus, we will write $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$.

The most coherent way to understand what is going on is in terms of <u>inner product spaces</u>.

The relevant space here is $L^2[-\pi,\pi]$ "ell two"

$$= \left\{ f : [-\pi,\pi] \to \mathbb{C} \ : \ \int_{-\pi}^{\pi} |f(x)|^2 < \infty \right\}$$

with the <u>inner product</u>

$$\langle f | g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx.$$

You can check that this is a <u>vector space</u> and that $\langle f|g\rangle$ satisfies all the axioms of an inner product (see week 3 notes). The inner product allows us to define a <u>norm</u>

$$\|f\| = \sqrt{\langle f|f\rangle}$$

and together they behave just like the Euclidean dot product/norm in $\mathbb{R}^n$. In particular, they satisfy:

1. The triangle Inequality $\|f+g\| \leq \|f\| + \|g\|$
2. Pythagoras' theorem : $\|f+g\|^2 = \|f\|^2 + \|g\|^2$ when $\langle f|g\rangle = 0$

The point is that this allows us to use our geometric intuition and concepts like projection, length, orthogonality when thinking about L^2.

In this notation, the exponential functions are actually orthonormal:

$$\langle e^{inx} | e^{inx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{inx} dx = 1$$

$$\langle e^{inx} | e^{imx} \rangle = \frac{1}{2\pi} \int e^{-inx} e^{imx} dx = 0 \text{ if } m \neq n.$$

And the fourier coefficients are inner products

$$\hat{f}(n) = \langle e^{inx} | f \rangle.$$

The punch line is that the exponentials are an orthonormal basis:

Theorem: If $f \in L^2[-\pi, \pi]$ then the partial sums

$$S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} \langle e^{inx} | f \rangle e^{inx}$$

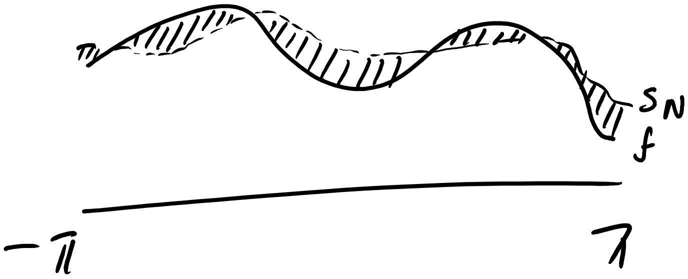
Converge in mean square to f:

$$\| f - S_N \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) - S_N(x) |^2 dx \longrightarrow 0$$

as $N \to \infty$.

This means that the vectors (in L^2) S_N get closer and closer to f in norm.

In terms of the function, it means that the "area" of the squared difference goes to zero:



This is an <u>average</u> notion of convergence which does <u>not</u> tell us that $S_N(x) \longrightarrow f(x)$ if we plug in any particular point x.

So it is <u>weaker</u> than the kind of convergence we studied in the first two weeks.

The advantages of considering it are:

- it works under extremely general and natural conditions (square-integrability is much weaker than being differentiable, continuous, etc.)
- it comes equipped with a powerful geometric intuition. (In HW9 you will show that the Nth partial sum of the Fourier Series is the least squares approximation to f using $e^{-iNx} \dots e^{iNx}$).
- It is adequate for many physical applications (such as in quantum mechanics) where we are more interested in integrals of the square of a function (which correspond to probability amplitudes or energy) than in evaluating the function at particular points. In fact, L^2 convergence is also called "convergence in energy".

One important consequence of the theorem is:

Parseval's theorem: If $f \in L^2[-\pi,\pi]$ then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \boxed{\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2} = \sum_{n\in\mathbb{Z}} |\langle e^{inx} | f\rangle|^2$$

i.e. the squared length of a vector (which is in this case a function) is equal to the sum of squares of its inner products with an orthonormal basis.

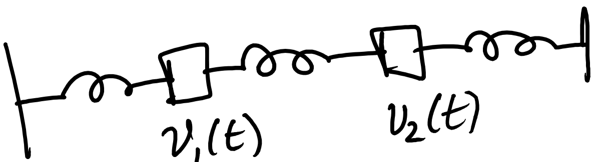
This is exactly what happens in $\mathbb{R}^n$.

It is very powerful because it allows one to evaluate integrals using series and vice versa (see HW9).

In the next lecture, we will consider a stronger notion of convergence which requires more restrictive assumptions on f.

Analogy

Coupled Oscillator Example



positions at time t: $v(t) \in \mathbb{R}^2$

$$\frac{d^2}{dt^2} v = A v$$

A is hermitian

$\Rightarrow$ eigenvectors

$A b_1 = \lambda_1 b_1$, $A b_2 = \lambda_2 b_2$

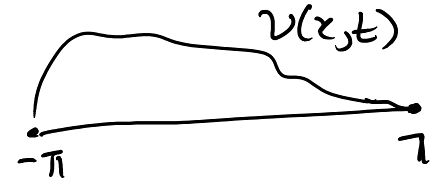
form an orthonormal basis.

Expand $v(t) = C_1(t) b_1 + C_2(t) b_1$

In eigenbasis, diff eq decouples into

$$\frac{d^2}{dt^2} c_i(t) = \lambda_i c_i(t)$$

Heat Eqn



Heat distribution at time t: function $v(t,x) \in L^2[-\pi, \pi]$

$$\frac{d}{dt} v = \alpha \frac{d^2}{dx^2} v$$

$$\left\langle f \middle| \frac{d^2}{dx^2} g \right\rangle = \left\langle \frac{d^2}{dx^2} f \middle| g \right\rangle$$

$\Rightarrow \frac{d^2}{dx^2}$ is Hermitian

$\Rightarrow$ eigenvectors $\{e^{inx}\}_{n \in \mathbb{Z}}$

form an orthonormal basis

expand

$$v(x,t) = \sum_{n \in \mathbb{Z}} c_n(t) e^{inx}$$

In eigenbasis, decouples as:

$$c_n'(t) = -n^2 c_n(t)$$

easy to solve scalar differential eqns.

$\square$