What to cite on Midterm 2

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Several people asked me what facts you can just cite while taking limits of contour integrals on the midterm. I expect you to at least draw the contour you are integrating over, but because the time is short, I do not expect you to parameterize every semicircular curve and go all the way down to the triangle inequality to show it vanishes. So, you can cite the following statements:

1. If P, Q are polynomials with $\deg(P) \leq \deg(Q) - 2$ then

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{P(z)}{Q(z)} dz = 0$$

where $\gamma_R(t) = Re^{it}, t \in [0, \pi]$. This follows from the triangle inequality.

2. If P, Q are polynomials with $\deg(P) \leq \deg(Q) - 1$ and $m > 0$, then

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{P(z)}{Q(z)} e^{imz} dz = 0$$

where $\gamma_R(t) = Re^{it}, t \in [0, \pi]$. This follows from Jordan's lemma, and you should cite it as 'Jordan's lemma'. If $m < 0$ then you will need a contour in the lower halfplane, i.e., with $t \in [0, -\pi]$.

3. If z_0 is a *simple* pole of f, then the limit of the integral over a semicircle of shrinking radius around z_0, parameterized as $\gamma_\rho(t) = z_0 + e^{it}, t \in [\pi, 0]$, is

$$\lim_{\rho\to 0} \int_{\gamma_\rho} f(z) dz = -\pi i Res(z_0).$$

The more general statement is: if γ_ρ is an arc with $\gamma_\rho(t) = z_0 + e^{it}, t \in [\theta_1, \theta_2]$, then the limit of the integral is $(\theta_2 - \theta_1) i Res(z_0)$.

The proof of this is: since z_0 is a simple pole, we have $f(z) = g(z)/(z - z_0)$ for some $g(z_0) \neq 0$ analytic at z_0. Then the limit of the integral over γ_ρ is equal to:

$$\lim_{\rho\to 0} \int_{\theta_1}^{\theta_2} \frac{g(z_0 + \rho e^{it})}{\rho e^{it}} i\rho e^{it} dt = i \int_{\theta_1}^{\theta_2} \lim_{\rho\to 0} g(z_0 + \rho e^{it}) dt = i \int_{\theta_1}^{\theta_2} g(z_0) dt = i Res(z_0)(\theta_2 - \theta_1).$$

Note that this argument does NOT work when we have a multiple pole on the contour. Essentially, the reason it works is that as $\rho \to 0$ the g part converges to a constant. [1]

Some people asked me for extra problems. Here are some problems that I think are intereseting and can help you test your knowledge, though the problems on the midterm will be closer to problems that already appeared on the homework. In particular, there won't be any proofs on the midterm.

[1] Note that we have used the fact that we can interchange the limit and the integral here. We will be doing this without justification in this course, but for those who have taken 104 etc the reason it is ok here is that as $\rho \to 0$ the functions $g(z_0 + \rho e^{it})$ converge uniformly on $[\theta_1, \theta_2]$ to $g(z_0)$, because g is analytic.

1. Boas Chapter 14: 2.34, 2.36, 11.6, 11.8, 11.10, 11.21.

2. Show that if f is entire (i.e., analytic everywhere) and $Re(f(z))$ is constant, then f must be constant. (hint: use the Cauchy-Riemann equations). Then show that if f is entire and $|f(z)|$ is constant, f must be constant.

3. Consider the rational function
$$R(z) = \frac{cz + d}{(z-a)(z-b)},$$
$a \neq b$. Show that
$$R(z) = \frac{Res(R;a)}{z-a} + \frac{Res(R;b)}{z-b}.$$
Hence show that the coefficients in a partial fraction decomposition are actually just residues (this works for polynomials of arbitrary degree, and is often the quickest way to compute partial fraction decompositions).
(Hint: use the fact that the integral on a small circle around a pole is equal to $2\pi i$ times the residue.)