There are two types of improper integrals which the book refers to as type 1 and type 2; type 1 integrals are infinite while type 2 have discontinuous integrands. Today we will discuss type 1 integrals and related theorems.

Infinite integrals are integrals in which one or both of the endpoints are $\pm\infty$. We define them as follows:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \tag{1}$$

$$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx \tag{2}$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_a^\infty f(x)\,dx + \int_{-\infty}^a f(x)\,dx \tag{3}$$

The integrals are **convergent** if the limit exists and is finite (not infinity). Otherwise we say the integral is **divergent**.

There are multiple ways in which we can determine if an integral converges or diverges. The most direct way is by integration.

EXAMPLE 1. Integrate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ to determine if the integral converges or diverges.

EXAMPLE 2. Integrate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ to determine if the integral converges or diverges.

EXAMPLE 3. Integrate $\int_1^\infty \ln(x)\, dx$ to determine if the integral converges or diverges.

From these examples we can see that for the limit to exist the integrand must go to zero in order for the integral to exist and more so the integrand must go to zero *at a fast enough rate* (i.e. the difference between example 2 and 1). Let's introduce a theorem that address this pattern we see to some extent,

Theorem (p-test for integrals), The integral

$$\int_1^\infty \frac{1}{x^p}\, dx \tag{4}$$

is convergent if $p > 1$ and diverges otherwise.

Proof.

This leads us to the second way we can determine if an integral exists: by way of theorems. If the integral is exactly a power of p than we can simply use the p-test, else we can use the following theorem to simplify the integral to a power of p or a simpler integrand that we integrate.

Theorem (comparison test for integrals). Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

The proof of the comparison theorem follows directly from the fact that integrals preserve inequalities, how the proof of this fact is beyond the scope of this course. It is worth considering the following pictorial representation however:

Notice that the comparison test for integrals says nothing about the other two cases and is only for non-negative functions. There is an extension of the comparison theorem for functions that take on positive and negative values, however we will not mention it (nor need to use it).

EXAMPLE 4. Show that $\int_0^\infty e^{-x^2}\,dx$ converges.

EXAMPLE 5. Does the integral $\int_0^\infty \frac{x}{x^3+1}\,dx$ converge or diverge.

EXAMPLE 6. Does the integral $\int_2^{\infty} \frac{\sin^2 x}{x^2 - x}\, dx$ converge or diverge.