Set Theory

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These notes cover introductory set theory. Starred sections below are optional. They discuss interesting mathematics connected to concepts covered in the course. Thanks to Cecelia Higgins, Jacob Manaker, Forte Shinko, Marlon Trifunovic, Spencer Unger, and Eric Wang, for corrections and helpful conversations about the material in the notes.

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1 Introduction

Set theory began with Cantor's proof in 1874 that the natural numbers do not have the same cardinality as the real numbers. Cantor's original motivation was to give a new proof of Liouville's theorem that there are non-algebraic real numbers[1]. However, Cantor soon began researching set theory for its own sake. Already by 1878 he had articulated the continuum problem: whether there is any cardinality between that of the natural numbers and the real numbers. Cantor's ideas had a profound influence on mathematics, and by 1900, Hilbert included the continuum problem as the first in his famous list of 23 problems for mathematics in the 20th century.

Lets recall Cantor's definition of cardinality. If X and Y are sets, say that X **has cardinality less than or equal to** Y and write $|X| \leq |Y|$ if there is an injective function from X to Y. Say that X and Y have the **same cardinality and write** $|X| = |Y|$ if there is a bijection from X to Y. These definitions agree with our usual ways of counting the number of elements of finite sets. Cantor's insight was to also use these definitions to compare the size of infinite sets.

Lets recall a few basic facts about cardinality[2]:

Exercise 1.1. *If X is a nonempty set, then $|X| \leq |Y|$ if and only if there is a surjection from Y to X.*

Say that a set X is **finite** if it has the same cardinality as a set of the form $\{0, \ldots, n-1\}$ for some natural number n. If X is not finite, say that X is **infinite**. The smallest size of infinite set is that of the natural numbers $\mathbb{N}$ (see Exercise 1.2). Finally, say a set X is **countable** if $|X| \leq |\mathbb{N}|$.

Exercise 1.2. *If X is a set, either X has the same cardinality as a finite set, or $|\mathbb{N}| \leq |X|$.*

Exercise 1.3 (A countable union of countable sets is countable.)**.** *If X_i is a countable set for every $i \in \mathbb{N}$, then $\bigcup_i X_i$ is countable.*

Exercise 1.4. *If X is an infinite set, and Y is a countable set, then $|X| = |X \cup Y|$.*

Exercise 1.5 (Cantor-Shröder-Bernstein)**.** *$|X| = |Y|$ if and only if $|X| \leq |Y|$ and $|Y| \leq |X|$.*

We write $X \subseteq Y$ if X is a **subset** of Y. That is, $\forall z(z \in X \to z \in Y)$. $\mathcal{P}(X)$ denotes the collection of all subsets of X:

$$\mathcal{P}(X) = \{Y : Y \subseteq X\}.$$

[1] Recall that a real number is called algebraic if is a root of a nonzero polynomial with rational coefficients. For example, $\sqrt{2}$ is algebraic since is a root of the equation $x^2 - 2$. Cantor showed that the cardinality of the real numbers is greater than that of the algebraic numbers. Thus, there must be non-algebraic numbers.

[2] In this section, we freely use the axiom of choice

Exercise 1.6. *Show that there is a bijection from $\mathcal{P}(\mathbb{N})$ to the real numbers $\mathbb{R}$. [Hint: $x \mapsto \frac{2}{\pi}\tan^{-1}(x)$ is a bijection from $\mathbb{R}$ to $(0,1)$. Then show there is a bijection from $(0,1)$ to $\mathcal{P}(\mathbb{N})$ using binary expansions and Exercise 1.4.]*

Recall Cantor's diagonal argument that $\mathbb{N}$ has strictly smaller cardinality than $\mathcal{P}(\mathbb{N})$ (and hence $\mathbb{R}$). Suppose $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is any function. Then f is not onto $\mathbb{P}(\mathbb{N})$ and so $|\mathcal{P}(\mathbb{N})| \not\leq |\mathbb{N}|$ by Exercise 1.1. We prove this by constructing a subset of $\mathbb{N}$ that is not in $\operatorname{ran}(f)$. Let $D = \{n \in \mathbb{N} \colon n \notin f(n)\}$. Then this set D diagonalizes against f. Since $n \in D \leftrightarrow n \notin f(n)$, D cannot equal $f(n)$ for any n. Hence, $D \notin \operatorname{ran}(f)$ and f is not onto.

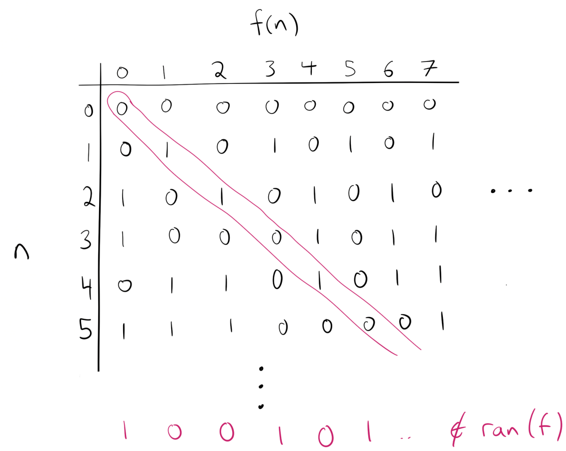

Figure 1: Cantor's diagonal argument. In this figure we're identifying subsets of $\mathbb{N}$ with infinite binary sequences via their characteristic functions. That is, letting the nth bit of the infinite binary sequence be 1 if n is an element of the set, and 0 otherwise.

This exact same argument generalizes to show that given any set X, its powerset $\mathcal{P}(X)$ has larger cardinality.

Exercise 1.7. *Show that for every set X, there is no surjection $f\colon X \to \mathcal{P}(X)$, and hence $|\mathcal{P}(X)| \not\leq |X|$. [Hint: define $D = \{x \in X \colon x \notin f(x)\}$. Then show $D \notin \operatorname{ran}(f)$.]*

Cantor had realized that as a consequence of this theorem, there can be no universal set: a set containing all other sets. Every set would inject (via the identity function) into a universal set. But Exercise 1.7 shows that the powerset of the universal set could not inject into the universal set. Bertrand Russell traced through this argument (letting f be the identity function and X be a supposed universal set), and isolated the resulting contradiction into what is now known as Russell's paradox. If

$$D = \{x \colon x \notin x\},$$

then is $D \in D$? If $D \notin D$, then $D \in D$ by definition, contradiction. But if $D \in D$, then $D \notin D$ by definition, contradiction.

Russell's writings about this paradox caused a brief crisis in the foundations of set theory. Allowing ourselves to construct a set containing all mathematical objects satisfying some given property leads to contradictions. What sets, then, should we be allowed to construct? Is the whole enterprise of set theory inconsistent?

The resolution to Russell's paradox that set theorists have adopted is the so called iterative conception of set theory[3]. All sets are arranged into a cumulative hierarchy. We begin with a simple collection of sets, and then apply some basic operations to iteratively create more sets. This produces the hierarchy V of all sets. The precise set existence axioms we will use will be discussed in the next section. They are known as Zermelo-Frankel set theory or ZF. We use ZFC to denote $\mathsf{ZF}+$ the axiom of choice. The first part of this class will be discussing these axioms of ZFC and axiomatic set theory.

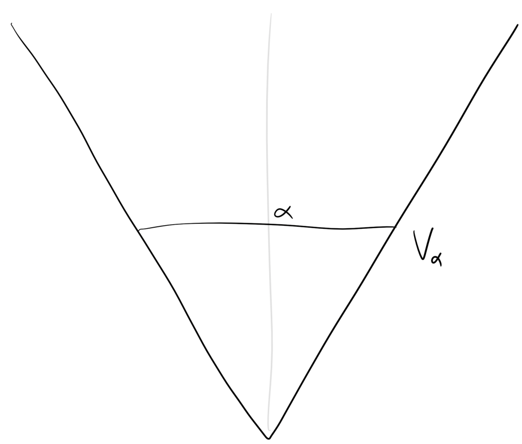

Figure 2: A picture of the set theoretic universe, known as V. At step α, we construct all sets of "rank" α. V_α denotes all sets of rank less than α.

Note that we will never define what a set *is* in these notes. We're taking an axiomatic viewpoint. ZFC includes *some* true principles about sets, but not all of them. We caution that it is false to say "a set is an element of a model of set theory". First, this would be circular; a model is defined in model theory using sets. Second, there are strange models of set theory which we do not want to use to define what sets are. It would be similarly wrong to say that a natural number is an element of a model of PA; there are nonstandard models of PA with infinite elements greater than any natural number[4].

[3]Other alternatives to ZFC have been also explored such as Russell's type theory, or Quine's new foundations. They are rarely considered in modern set theory.

[4]Indeed, Gödel's incompleteness theorem says that it's hopeless to try and axiomatize all

The point in examining models of set theory for us will not be to build the "correct" model. Rather, our goal in examining models of set theory will be to understand what the axioms of set theory can prove.

1.1 Independence in modern set theory*

In the second part of our class, we'll begin to discuss some topics around independence in set theory.

In reaction to Russell's paradox, many mathematicians hoped to find a foundation for set theory that could be proved to be free of paradoxes. Gödels work in 1931 shattered this hope; we can never prove that the ZFC axioms of set theory are consistent using simple means. Gödel showed that any computable set of axioms which can interpret and prove basic theorems about the natural numbers cannot prove its own consistency. From a modern viewpoint, mathematical theories are arranged along a hierarchy of consistency strength, where $T_1 \leq_{\mathrm{CON}} T_2$ if $\mathrm{Con}(T_2) \to \mathrm{Con}(T_1)$. That is, the consistency of T_2 implies the consistency of T_1.

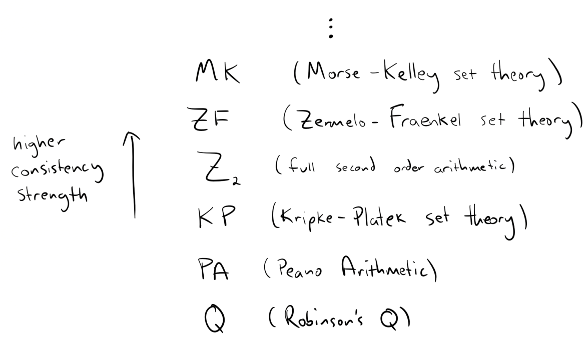

Figure 3: A picture of some common theories arranged by their consistency strengths.

An important class of set theoretic assumptions with strong consistency strength are large cardinal assumptions. These are assumptions that there exist "very large" cardinal numbers. For example, an **inaccessible cardinal** is an uncountable cardinal number κ so that κ is regular ($\mathrm{cf}(\kappa) = \kappa$) and κ is a **strong limit** (i.e. $\lambda < \kappa$ implies $2^\lambda < \kappa$). Informally, this means κ cannot be reached from below by adding smaller cardinals or applying the powerset operation to smaller cardinals. If κ is an inaccessible cardinal, then if we stop building the set-theoretic universe at stage κ (i.e. if we take V_κ), then we obtain a model of ZFC. Since ZFC + "there exists an inaccessible cardinal" proves there is a model of

true sentences about the natural numbers. It is similarly hopeless to try and axiomatize all true principles about sets.

ZFC, by Gödel's completeness theorem, $\mathsf{ZFC}+$"there exists an inaccessible cardinal" implies Con(ZFC), and therefore ZFC cannot prove there is an inaccessible cardinal. This is a typical phenomenon. If κ is a large cardinal, then the universe restricted to height κ satisfies ZFC, and more generally will contain many "smaller" large cardinals.

Many other interesting set theoretical statements end up being equivalent in consistency strength to large cardinal assumptions. For example, ZF + "all sets of real numbers are Lebesgue measurable", is equiconsistent with an inaccessible cardinal, and ZFC + "there is a saturated ideal on ω_1" is equiconsistent with a Woodin cardinal. One of the most important open problems in modern set theory is proving that the proper forcing axiom PFA is equiconsistent with a supercompact cardinal.

Because of Gödel's incompleteness theorem, none of these large cardinals can be proved to exist from ZFC (and we cannot prove they are consistent without assuming the consistency of even "larger" cardinals). However, they are a vital part of the study of modern set theory, and they are viewed as the "natural" way to increase the consistency strength of the theory of ZFC. The consistency strength of all "natural" theories has been empirically found to be linearly ordered and indeed wellordered. This is important evidence that these theories are mathematically important[5]. Large cardinals also create beautiful and intricate structure in the set theoretic universe which has important and concrete mathematical consequences (for example, in our understanding of the real numbers). They are freely used and investigated in modern set theory. We cannot prove they are consistent, but we deeply believe they are because of the beautiful and important mathematical structures they create.

This, then, is one source of independence in set theory. Any statement that implies Con(ZFC) must be either false or independent of ZFC.

However, there is a completely different method for proving independence from the axioms of ZFC: forcing and inner models. In 1938, Gödel proved that inside any model of ZF set theory, there is an inner model L consisting of what are called the **constructible sets**. This is in a sense the smallest possible universe of set theory. It contains only the sets one must have by virtue of these sets being explicitly definable. Gödel showed that this inner model known as L always satisfies both the axiom of choice and the continuum hypothesis. This was reassuring to mathematicians who were worried about the validity and acceptability of the axiom of choice. By Gödel's theorem if ZF is consistent and has a model, then ZF + the axiom of choice is also consistent. So using the axiom of choice cannot add new inconsistencies to set theory.

A huge importance of inner models such Gödel's L is that they have an extremely detailed and canonical structure. Indeed, there is a whole study of "fine structure theory", which analyzes these canonical models in great detail. Unfortunately, Gödel's L is deficient in that sufficiently large cardinals (e.g. measurable cardinals) cannot exist inside L. One aim of modern inner model

[5]The naturalness assumption is very important here; neither linearity or wellfoundedness are true if we consider *all* theories extending ZFC.

theory is to construct inner models that are compatible with having all large cardinals, and understanding their structure.

Complementing Gödel's constructible universe was Cohen's 1963 invention of the method of forcing. Given a countable model of ZFC, Cohen showed how one can add sets to the model to create a larger "outer model" of ZFC. Cohen used this technique to show that given any countable model of ZFC, one can add many real numbers to it in order to find an outer model where the continuum hypothesis is false.

These two results combine to show that if there is a model of ZFC, then there is a model of both $\mathsf{ZFC} + \mathsf{CH}$ and another model of $\mathsf{ZFC} + \neg\mathsf{CH}$. Thus, ZFC cannot prove that CH is either true or false, and CH is independent from ZFC. Philosophers of set theory still fiercely debate questions such as whether there could be new intuitively justified axioms for set theory that resolve the continuum hypothesis, or whether CH even has a definite truth value[6].

This, then, is the second source of independence in set theory: we can prove a statement φ is independent from ZFC by constructing outer or inner models that satisfy both φ and its negation. An imperfect analogy is that starting with any field, we can study its subfields and field extensions. If we find two different fields, one of which has property ψ and the other does not, then we know the field axioms do not imply ψ.

The invention of forcing led to a renaissance of independence results in set theory, many of which had stood open for many decades. For example, Suslin's problem[7], which had been open since 1920 was shown to be independent of ZFC by Solovay and Tennenbaum in 1971. Forcing is also intimately tied to inner model theory. The canonical structure given by inner models is often a necessary starting point for a good understanding of the outer models we force to create. Forcing and inner models also found applications in many different fields of mathematics. For example, Kaplansky's conjecture in functional analysis, and Whitehead's problem in group theory are independent of ZFC.

There is a deep contrast between the type of independence that comes from having consistency strength, and the type that comes from forcing/inner models. While very simple statements (e.g. Π^0_1 statements in arithmetic) can be independent of ZFC by virtue of having consistency strength, statements which are shown to be independent of ZFC by forcing and inner models must be very complicated by so-called absoluteness results. For example, we cannot use forcing to show any Σ^1_2 sentence is independent of ZFC by Shoenfield absoluteness. Indeed, assuming certain large cardinals exist, CH is in some sense the "simplest" statement that can be proved independent from ZFC by forcing.[8]

Set theory remains a vibrant and active field of research, and many open

[6]Classical large cardinals axioms cannot help resolve this question by a theorem of Levy and Solovay.

[7]Suslin's problem asks the following: suppose $(L, <_L)$ is a dense complete linear order without endpoints in which every collection of pairwise disjoint open intervals is countable. Then must L be order-isomorphic to the real numbers?

[8]It is a theorem of Woodin that assuming the existence of large cardinals, all Σ^2_1 statements are absolute for set forcing, assuming CH (which is itself a Σ^2_1 statement).

problems remain. Indeed, even Cantor's original goal of understanding basic cardinal arithmetic is still an unfinished puzzle; very simple-seeming questions about the possible behavior of cardinality in ZFC remain open. For example, there are deep open questions about the possible behaviors of the exponential function $\kappa \mapsto 2^\kappa$ in models of set theory.

Open Problem 1.8. *If $\aleph_\omega$ is a strong limit, is $2^{\aleph_\omega} < \aleph_{\omega_1}$?*

A weaker theorem along these lines follows from work of Shelah in pcf theory (see [J] for an introduction).

Theorem 1.9 (Shelah)**.** *If $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.*

2 The axioms of ZFC

In this section, we will introduce the axioms of ZFC. The axioms of ZFC are in the **language of set theory** $\mathcal{L}_\in$ which consists of a single binary relation $\in$ of set membership. Throughout this section, we will introduce notation for certain sets, functions and relations which are defined in terms of the $\in$ relation. For example, $x \subseteq y$ will abbreviate $\forall z(z \in x \to z \in y)$. We will also use bounded quantifiers freely: $(\exists y \in x)\phi$ is defined to mean $\exists y(y \in x \wedge \phi)$, and $(\forall y \in x)\phi$ is defined to mean $\forall y(y \in x \to \phi)$. The exists unique quantifier: $\exists! y\varphi(y)$ abbreviates $\exists y(\varphi(y) \wedge (\forall y' \varphi(y') \to y' = y))$.

The axiom of Extensionality: Every set is determined by its members.

$$\forall x \forall y[x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)]$$

This axiom essentially defines what it means to be a set. A set x is determined precisely by what elements its contains. (A set has no order or other data).

The axiom of Foundation: Every nonempty set contains a $\in$-minimal element.

$$\forall x[x \neq \emptyset \to \exists y \in x \forall z \in x(z \notin y)]$$

Here $x \neq \emptyset$ abbreviates $\exists y(y \in x)$. The axiom of foundation says that the relation $\in$ on every set has a minimal element: some $y \in x$ with no predecessors under $\in$ in x.

The axiom of foundation also defines what it means to be a set, but in a more technical sense. We will prove shortly that the axiom of foundation is equivalent to the statement that every set is an element of the von Neumann universe V of sets; those that can be obtained from $\emptyset$ by iteratively applying the set existence axioms.[9]

These first two axioms define what it means to be a set. All the other axioms of ZFC are set existence axioms which state that certain sets exist.

The axiom of Pairing: Given two sets x and y, there is a set containing exactly these two sets.

$$\forall x \forall y \exists w[x \in w \wedge y \in w \wedge \forall z(z \in w \to z = x \vee z = y)]$$

We let $\{x, y\}$ denote this set w whose only two elements are x and y. Similarly, we'll use $\{x\}$ to denote the set whose only element is x. The existence of the set $\{x\}$ is by the pairing axiom when $x = y$, so $\{x\} = \{x, x\}$.

Proposition 2.1. *The pairing axiom and the axiom of foundation imply that there is no set x such that $x \in x$.*

Proof. Assume for a contradiction there is such a set x, and consider $\{x\}$. Then the only element of $\{x\}$ is x. However, since $x \in x$, x is not ϵ-minimal. $\square$

[9] Precisely, we'll show that ZFC $-$ Foundation proves that Foundation $\leftrightarrow \forall x(x \in V)$. Here V is defined as the class of sets that are in V_α for some ordinal α, where $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$.

Since $\{x, y\} = \{y, x\}$ we will also define an ordered pair, where the order of the two elements matters.

Definition 2.2 (Ordered pairs)**.** *We define* $(a, b) = \{\{a\}, \{a, b\}\}$

They key property of an ordered pair is the following:

Exercise 2.3. *Show that for all sets* a, b, c, d*, we have* $(a, b) = (c, d)$ *if and only if* $a = c$ *and* $b = d$*.*

The axiom of Union: Given any set of sets x, there is a set containing exactly all the element of these sets, denoted $\bigcup x$. Precisely, letting $y = \bigcup x$ denote $\forall z[z \in y \leftrightarrow \exists w \in x(z \in w)]$, the axiom of union states

$$\forall x \exists y[y = \bigcup x]$$

Writing $z = x \cup y$ for $\forall w(w \in z \leftrightarrow w \in x \vee w \in y)$, pairing and union prove that for all sets x and y, $x \cup y$ exists, since $x \cup y = \bigcup\{x, y\}$.

The axiom of Nullset: There is a set with no elements. We let $x = \emptyset$ abbreviate $\neg\exists y(y \in x)$. Nullset states:

$$\exists x[x = \emptyset]$$

The axiom of Infinity: There exists an inductive set.

$$\exists x[\emptyset \in x \wedge \forall y(y \in x \to y \cup \{y\} \in x)]$$

A set x is **inductive** if $\emptyset \in x$ and $y \in x$ implies $y \cup \{y\} \in x$, so the above axiom says that an inductive set exists. There are many different ways to axiomatize the existence of an infinite set, but the version of the axiom of infinity that we have given will work nicely with how we define the von Neumann ordinals. In Section 4 we'll define that if y is an ordinal, then $y \cup \{y\}$ is the ordinal successor of y. Note that the infinite set x whose existence is guaranteed by the axiom of infinity must have the following set as a subset: $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots\}$. We will eventually call this set ω: the set containing the ordinals $\{0, 1, 2, \ldots\}$.

The axiom of Powerset: For every set x, there is a set containing all the subsets of this set. We let $y = \mathcal{P}(x)$ abbreviate $\forall z(z \in y \leftrightarrow z \subseteq x)$

$$\forall x \exists y[y = \mathcal{P}(x)]$$

The axiom schema of Separation: If x is a set, then every subset of x that's definable (from parameters) exists. Formally, for every formula φ in the language of set theory, the following is an axiom

$$\forall x, w_1, \ldots, w_n \exists y \forall z[z \in y \leftrightarrow z \in x \wedge \varphi(x, z, w_1, \ldots, w_n)]$$

We will use $\{z \in x \colon \varphi(z, w_1, \ldots, w_n)\}$ to abbreviate the set whose existence is given by this axiom. More generally, we will use $\{z \colon \varphi(z, w_1, \ldots, w_n)\}$ to denote the collection of all sets z satisfying the formula $\varphi(z, w_1, \ldots, w_n)$. In general, this will not be a set (e.g. $\{z\colon z \notin z\}$). We will instead call such a collection a **class**, and by a **class** in these notes, we mean all sets z satisfying some formula $\varphi(z, w_1, \dots, w_n)$ where $w_1, \dots, w_n$ are fixed set parameters. In the case where such a collection is a set then $y = \{z\colon \varphi(z, w_1, \dots w_n)\}$ abbreviates $\forall z(z \in y \leftrightarrow \varphi(z, w_1, \dots, w_n))$.

If we let $z = x \cap y$ abbreviate $\forall w(w \in z \leftrightarrow w \in x \wedge w \in y)$, then Separation implies that for all x and y, there is some set z so that $z = x \cap y$. We can similarly use separation to show that the sets $x \setminus y$ and $\bigcap x$ exist. Letting $X \times Y = \{(x, y)\colon x \in X \wedge y \in Y\}$, and we can similarly use separation on $\mathcal{P}(\mathcal{P}(X \cup Y))$ to prove that $X \times Y$ is a set.

A **binary relation** R on $X \times Y$ is a subset of $X \times Y$. We say R is a binary relation on X if it is a binary relation on $X \times X$. We sometimes write $x\ R\ y$ instead of $(x, y) \in R$. A **function** from X to Y is a subset $f \subseteq X \times Y$ so that $\forall x \in X \exists! y \in Y (x, y) \in f$. We write $f(x) = y$ for $(x, y) \in f$. We will use X^Y to denote the set of all function functions from Y to X

$$X^Y = \{f\colon f \text{ is a function from } Y \text{ to } X\}$$

which is also a set by the separation axiom. We define injections, surjections, bijections, and inverses of functions as usual. If R is a binary relation, we let $R^{-1} = \{(y, x)\colon (x, y) \in R\}$.

Note that separation is actually an axiom schema: there is a separation axiom for every formula φ of the language of set theory. We'll later prove that ZFC is not finitely axiomatizable, so this is a necessity. Recall that in PA, induction is an axiom schema, and similarly, PA is not finitely axiomatizable.

The axiom schema of Replacement: The axiom of replacement says that if F is a class function and X is a set, then $\{F(x)\colon x \in X\}$ is a set. A **class function** F is a class of ordered pairs so that there does not exist $(x, y) \in F$ and $(x, y') \in F$ so that $y \neq y'$. Formally, for each formula φ, the following is an axiom:

$$\forall v_1, \dots, v_n \forall X[(\forall x \in X \exists! y \varphi(x, y, v_1, \dots, v_n) \rightarrow \\ (\exists Y \forall y(y \in Y \leftrightarrow \exists x \in X \varphi(x, y, v_1, \dots, v_n)))]$$

Instead of the replacement axiom, sometimes ZFC is axiomatized using that axiom schema of collection. The collection axiom says that if x is a set, and φ defines a class from each element of x, then there is a set which meets all these classes.

$$\forall x, v_1, \dots, v_n \exists y \forall z \in x[\exists w \varphi(w, z, v_1, \dots, v_n) \rightarrow \exists w \in y \varphi(w, z, v_1, \dots, v_n)]$$

We will show that separation and collection are equivalent to replacement in Section 6 over the other axioms of ZF.

The axiom of Choice: Every set of pairwise disjoint nonempty sets has a "choice set".

$$\forall x[(\forall y \in x(y \neq \emptyset) \wedge (\forall y \in x)(\forall y' \in x)(y \neq y \rightarrow y \cap y' = \emptyset)) \\ \rightarrow (\exists z)(\forall y \in x)(\exists w \in y)(\forall w' \in y)(w' \in z \leftrightarrow w' = w)]$$

There are many different equivalent ways of formulating the axiom of choice. For example, assuming ZF, the axiom of choice is equivalent to Zorn's lemma and the wellordering principle. We'll prove some of these equivalences in subsequent sections.

We will let ZF denote all the above axioms except the axiom of choice. We let AC denote the axiom of choice. We let ZFC denote all the above axioms, so ZFC = ZF + AC.

There are many equivalent ways of axiomatizing ZFC, and the above is just one possibility. The axiomatization we have given above is also not minimal in the sense that some of our axioms imply others. (For example, replacement and nullset imply separation). The reason we have stated all these axioms even when there are redundancies is that we will often be interested in models of fragments of ZFC. For example, if $\alpha > \omega$ is a limit ordinal, then V_α is a model of ZFC − Replacement (and in particular it is important that V_α is a model of the separation axiom).

Exercise 2.4. *Show that the replacement axiom and the nullset axiom imply the separation axiom. (That is, if a model of set theory satisfies the replacement schema and nullset, then it satisfies the separation schema).*

Exercise 2.5. *Let n be a natural number. Show there do not exist sets $x_1, \ldots, x_n$ such that $x_1 \in x_2 \in \ldots \in x_n \in x_1$. (That is, show that for each n,* ZFC *proves the sentence*

$$\neg\exists x_1 \ldots x_n (x_1 \in x_2 \wedge x_2 \in x_3 \wedge \ldots \wedge x_n \in x_1).$$

State which axioms of ZFC *you use to prove this.*

Exercise 2.6. *Carefully prove that the following sets exist. State what axioms of* ZFC *you use:*

1. *For all sets x and y, there is a set z so that $\forall w(w \in z \leftrightarrow w \in x \wedge w \notin y)$.*
2. *For all nonempty sets x there is a set y so that $\forall w(w \in y \leftrightarrow \forall z \in x(w \in z))$.*
3. *For all sets a, b there is a set z so that $z = \{f : f \text{ is a function from } a \text{ to } b\}$.*

Exercise 2.7. *Show that in* ZF*, the following are equivalent.*

1. AC.
2. *For every x such that $\forall y \in x(y \neq \emptyset)$, there is a function $f : x \to \bigcup x$ such that $f(y) \in y$ for all $y \in x$.*

2.1 Classes and von Neumann-Bernays-Gödel set theory*

Classes play an important role in set theory. We've already mentioned some important classes such as the von Neumann universe V, and Gödels constructible universe L.

There are alternate ways of axiomatizing set theory where we explicitly give classes formal existence instead of just associating to each formula the class it defines. Having classes as formally defined objects is often convenient. For example, many large cardinal axioms say there is a proper class inner model M of ZFC and an class function $j\colon V \to M$ which is an elementary embedding.

One such way to axiomatize set theory and directly talk about classes is von Neumann-Bernays-Gödel set theory. In this axiomatization, all the objects of study are classes. We define a set to be a class which is an element of some other class. A **proper class** is a class which is not a set. By convention, uppercase letters in NBG denote classes, while lowercase letters denote only sets. So for example, $\exists x \varphi$ abbreviates $\exists x \exists Y(x \in Y \wedge \varphi)$. The axioms of von Neumann-Bernays-Gödel set theory, abbreviated NBG include the axiom of extensionality, and all the remaining axioms of ZF, where all quantifiers in these other axioms range just over sets[10]. There is one final axiom schema, the **class comprehension axiom scheme**: for every formula φ, the axiom:

$$\forall X_1, \ldots, X_n \exists Y[\forall x(x \in Y \leftrightarrow \varphi(x, X_1, \ldots, X_n)]$$

saying that φ defines a class.

NBG is **conservative** over ZF; it proves exactly the same formulas about sets. This is easily proved by showing every model of ZF can be extended to a model of NBG by adding all definable proper classes to our universe. Similarly, if we remove all the proper classes from a model of NBG, we obtain a model of ZF. Hence, if we want to discuss classes in this formal way, we can work with NBG without changing any of the facts we'll prove about sets. NBG also has other advantages, for example it is finitely axiomatizable, while ZF is not.

Choice in NBG is generally taken to be the axiom that there is a global choice class; a class function F so that for every nonempty set $F(x) \in x$.

[10]For example, the axiom of infinity becomes $\exists x \exists Z(x \in Z \wedge x \neq \emptyset \wedge \forall y \in x(y \cup \{y\} \in x)$.

3 Wellorderings

Wellfounded relations and wellorderings are central to the study of set theory. They are important in defining what sets are: the axiom of foundation says that the $\in$ relation on every set is wellfounded. They are also naturally lead to the definition of the ordinals, which are an essential part of set theory used to index steps in transfinite constructions and to create notions of rank. Cantor was originally lead to develop the theory of the ordinals to prove that given any two sets X and Y either $|X| \leq |Y|$ or $|Y| \leq |X|$.

A **strict partial order** is a pair $(P, <_P)$ where P is a set and $<_P$ is a binary relation on P that is irreflexive and transitive, so that $a \not<_P a$ and $a <_P b \wedge b <_P c \to a <_P c$ for all $a, b, c \in P$. We say that $(P, <_P)$ is **linear** if for all $a, b \in P$ such that $a \neq b$, either $a <_P b$ or $b <_P a$. We write $a \leq_P b$ to mean $a <_P b$ or $a = b$.

A strict partial order $(P, <_P)$ is **wellfounded** if every nonempty subset $X \subseteq P$ contains an element that is **$<_P$-minimal inside** X. That is, for every $X \subseteq P$ there is some $a \in X$ so that $b \not<_P a$ for all $b \in X$. A linear wellfounded strict partial order is called a wellordering.

Suppose $(P, <_P)$ and $(Q, <_Q)$ are strict partial orders. Then we say a function $f \colon P \to Q$ is **order-preserving** if for all $a, b \in P$ $a <_P b$ implies $f(a) <_Q f(b)$.

Lemma 3.1. *If $(P, <_P)$ is a wellfounded strict partial order and $f \colon P \to P$ is an order preserving function from $(<_P, P)$ to $(<_P, P)$, then $f(a) \not<_P a$ for all $a \in P$.*

Proof. Let $X = \{a \in P \colon f(a) <_P a\}$ be the set of points which are moved "downward". Assume for a contradiction that X is nonempty. Then by definition of wellfoundedness, X must have a $<_P$-minimal element a. By definition of X, $f(a) <_P a$. Since a is minimal in X, and $f(a) <_P a$, we must have $f(a) \notin X$. Now since f is order preserving, and $f(a) <_P a$, we must have $f(f(a)) <_P f(a)$. But then $f(a) \in X$ by definition of X. Contradiction! $\square$

We say that a bijection f from P to Q is an **isomorphism** from $(P, <_P)$ to $(Q, <_Q)$ if for all $a, b \in P$, $a <_P b \leftrightarrow f(a) <_Q f(b)$. Hence, both f and f^{-1} are order preserving.

Lemma 3.2. *If $(P, <_P)$ is a wellordering and $f \colon P \to P$ is an isomorphism from $(P, <_P)$ to $(P, <_P)$, then f is the identity.*

Proof. By the previous lemma, $f(a) \geq_P a$ for all $a \in P$. Since f^{-1} is also an isomorphism from $(P, <_P)$ to $(P, <_P)$, for all $b \in P$, $f^{-1}(b) \geq_P b$, so letting $b = f(a)$, we see $a \geq_P f(a)$ for all $a \in P$. Hence, f is the identity. $\square$

Corollary 3.3. *If $(P, <_P)$ and $(Q, <_Q)$ are isomorphic wellorderings, then there is a unique isomorphism between them.*

Proof. If f, g where two isomorphisms that were not equal, then $f^{-1} \circ g$ would be an isomorphism from $(P, <_P)$ to $(P, <_P)$ that is not the identity. $\square$

If $(P, <_P)$ is a wellordering and $x \in P$, then the **initial segment of** $(P, <_P)$ **below** x, noted $(P, <_P) \restriction x$ is the wellordering $(Q, <_P \cap Q \times Q)$, where $Q = \{a \in P \colon a <_P x\}$. An **initial segment of** $(P, <_P)$ is an ordering $(P, <_P) \restriction x$ for some $x \in P$. Note for example that an initial segment of a wellordering is always a wellordering.

Lemma 3.4. *No wellordering $(P, <_P)$ is isomorphic to an initial segment of itself.*

Proof. Suppose $x \in P$, and $(P, <_P)$ is isomorphic to $(P, <_P) \restriction x$ for some $x \in P$ via the function f, which is therefore an order preserving function from P to itself. Then $f(x) <_P x$ contradicting Lemma 3.1. $\square$

Lemma 3.5. *Suppose $(P, <_P)$ and $(Q, <_Q)$ are wellorderings. Then exactly one of the following holds:*

- *$(P, <_P)$ is isomorphic to $(Q, <_Q)$.*
- *$(P, <_P)$ is isomorphic to an initial segment of $(Q, <_Q)$.*
- *An initial segment of $(P, <_P)$ is isomorphic to $(Q, <_Q)$.*

Furthermore, this isomorphism is unique.

Proof. Consider the set of pairs

$$F = \{(x, y) \colon \text{there exists an isomorphism from } (P, <_P) \restriction x \text{ to } (Q, <_Q) \restriction y\}.$$

We claim that F is the function witnessing that this lemma is true.

Suppose $x \in P$, and $y, y' \in Q$ are such that $y <_Q y'$. Then we cannot have $(x, y) \in F$ and $(x, y') \in F$ since composing one isomorphism and the inverse of the other would contradict Lemma 3.4. So F is a function.

Similarly, if $(x, y), (x', y') \in F$ and $x <_P x'$, then we claim $y <_P y'$. Otherwise, suppose f is the isomorphism from $(P, <_P) \restriction x$ to $(Q, <_Q) \restriction y$, and f' is the isomorphism from $(P, <_P) \restriction x'$ to $(Q, <_Q) \restriction y'$. If $y' \leq_Q y$, then $f^{-1} \circ f'$ is an order-preserving function on P, and $f^{-1}(f'(x)) <_P x$, which is a contradiction. So F is an order preserving partial function from P to Q.

We claim that both $\mathrm{dom}(F)$ and $\mathrm{ran}(F)$ are closed downwards. That is, if $y <_Q y'$ and $y' \in \mathrm{ran}(F)$, then $y \in \mathrm{ran}(F)$. This is since if f is an isomorphism from $(P, <_P) \restriction x$ to $(Q, <_Q) \restriction y'$, then restricting f to the initial segment given by $f^{-1}(y)$ gives an isomorphism from $(P, <_P) \restriction f^{-1}(y)$ to $(Q, <_Q) \restriction y$. Similarly, if $x <_P x'$ and $x' \in \mathrm{dom}(F)$, then $x \in \mathrm{dom}(F)$.

We finally claim $P \setminus \mathrm{dom}(F)$ and $Q \setminus \mathrm{ran}(F)$ cannot both be nonempty. If so, let x be the $<_P$-minimum element of $P \setminus \mathrm{dom}(F)$ and y be the $<_Q$-minimum element of $Q \setminus \mathrm{ran}(f)$. But then $F \restriction \{x' \in P \colon x' < x\}$ is an isomorphism from $(P, <_P) \restriction x$ to $(Q, <_Q) \restriction y$. Contradiction.

Uniqueness follows from Corollary 3.3 $\square$

The significance of Lemma 3.5 is that it shows the relation "$(P, <_P)$ is isomorphic to an initial segment of $(Q, <_Q)$" is a linear ordering of the isomorphism classes of wellorderings. Lemma 3.4 already proves this is an irreflexive ordering. It is a good exercise to check this is a wellfounded ordering.

To simplify this global order of wellorderings and proofs about it, we introduce the ordinals. Isomorphism classes of wellorders are proper classes which are awkward to deal with. Instead, ordinals will give us a unique representative of each isomorphism class of wellorderings.

Remark 3.6. *All the results of this section apply more generally to setlike class wellorderings. That is, classes X with a class linear order $<_X$ so that for every $a \in X$, $\{b \in X : b < a\}$ is a set.*

4 Ordinals

In this section, we'll introduce ordinals and show that they are canonical representatives of each isomorphism class of wellorderings. This will make it much easier to deal with the global structure we found in Section 3 on all wellorderings under the relation "$(P, <_P)$ is isomorphic to an initial segment of $(Q, <_Q)$".

Our definition of ordinal will use the notion of a transitive set. We call a set x **transitive** if for every $a \in x$, if $b \in a$, then $b \in x$. Careful: the $\in$ relation on a transitive set need not be transitive. For example, the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ *is transitive*, but the $\in$ relation on this set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ is *not transitive* ($\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in \{\{\emptyset\}\}$, but $\emptyset \notin \{\{\emptyset\}\}$).

Exercise 4.1. *x is transitive iff for every $y \in x$, $y \subseteq x$.*

Exercise 4.2 (Unions and intersections of transitive sets are transitive)**.** *If X is a set and every $x \in X$ is transitive, then $\bigcup X$ and $\bigcap X$ are transitive.*

Definition 4.3. *An **ordinal** is a transitive set x so that the $\in$ relation on x is a wellordering. We let ORD denote the class of all ordinals. If α, β are ordinals, we define $\alpha < \beta$ iff $\alpha \in \beta$. We will use lowercase Greek letters $\alpha, \beta, \gamma, \lambda, \ldots$ for ordinals.*

For example, it is easy to check that the sets $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$ are ordinals. We will see eventually that these are the first three ordinals which we'll call 0, 1, and 2.

A technical detail in this section is that we will not use the axiom of foundation. This is because we'll want to use the ordinals later to prove that the axiom of foundation is equivalent to $\forall x(x \in V)$. Note for example that if α is an ordinal, $\alpha \notin \alpha$ just by the definition of ordinal: if $\alpha \in \alpha$, this would imply that $\in$ is not an irreflexive relation on α, and hence not a strict partial order. So we don't need to use the axiom of foundation and Proposition 2.1 to show that $\alpha \notin \alpha$.

Our first goal is to prove that the order $<$ on the ordinals is a wellordering.

Lemma 4.4. *If $\alpha \neq \beta$ are ordinals, and $\alpha \subsetneq \beta$, then $\alpha \in \beta$.*

Proof. Let γ be the $\in$-least element of the set $\beta \setminus \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus, $\alpha = \{\xi \in \beta \colon \xi < \gamma\} = \{\xi \in \beta \colon \xi \in \gamma\} = \gamma$, so $\alpha \in \beta$. □

Lemma 4.5. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal, and β is an initial segment of α under $<$.*

Proof. First we show that β is transitive. Suppose $b \in a \in \beta$. Then $a \in \alpha$ and $b \in \alpha$ since α is transitive. Since α is linearly ordered by $\in$ we must have that either $b \in \beta$, $b = \beta$, or $\beta \in b$. If $b = \beta$ or $\beta \in b$, then the set $\{b, a, \beta\}$ (which exists by Pairing) would have no ϵ-minimal element contradicting that $\in$ is a wellordering of α. So we must have $\beta \in b$.

Next, $\beta = \{\gamma \colon \gamma \in \beta\} = \{\gamma \in \alpha \colon \gamma \in \beta\} = \{\gamma \in \alpha \colon \gamma < \beta\}$ since every element of β is an element of α by transitivity.

Finally, $\in$ is a wellordering of β, since β is an initial segment of α. $\square$

It follows from this lemma that each ordinal is equal to the set of ordinals that are less than it.

$$\alpha = \{\beta \colon \beta \in \alpha\} = \{\beta \in \mathsf{ORD} \colon \beta \in \alpha\} = \{\beta \in \mathsf{ORD} \colon \beta < \alpha\}.$$

Now we're ready to prove the trichotomy property for the ordering $<$ on ORD.

Lemma 4.6. *If α, β are ordinals and $\alpha \neq \beta$, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

Proof. Let $\gamma = \alpha \cap \beta$. Now γ is an ordinal since it is transitive (the intersection of two transitive sets is transitive), and any subset of a wellordering is a wellordering.

Suppose γ is not equal to α or β. Then $\gamma \in \alpha$ and $\gamma \in \beta$ by Lemma 4.4. So $\gamma \in \gamma$, which contradicts the definition of an ordinal. $\square$

Applying Lemma 4.4 gives the following corollary:

Corollary 4.7. *If $\alpha \neq \beta$ are ordinals, then $\alpha < \beta$ or $\beta < \alpha$. So $<$ is a linear ordering of the class of ordinals.*

Next, we show that $<$ is a wellordering of ORD, which we've already shown is linear.

Lemma 4.8. *$<$ is a wellfounded ordering of the class of ordinals.*

Proof. Suppose $A \subseteq \mathsf{ORD}$ is a nonempty class of ordinals, and $\alpha \in A$. If α is not the least element of A, then $A \cap \alpha$ is a nonempty subset of α. Hence, it has a least element. $\square$

Definition 4.9. *If A is a nonempty class of ordinals, we let* $\inf(A)$ *denote its least element.*

We can similarly defined $\sup(A)$ for any set of ordinals. First, we show that any set of ordinals has an upper bound:

Lemma 4.10. *If X is any set of ordinals, then there is an ordinal β such that $\beta \geq \alpha$ for every $\alpha \in X$.*

Proof. Consider the set $\beta = \bigcup X$. β is transitive since it is a union of transitive sets. It is a set of ordinals, and so it is linearly ordered by $\in$ by Corollary 4.7 and every subset of β has a minimal element by Lemma 4.8. Finally, if $\alpha \in X$, then $\alpha \subseteq \beta$, and so $\alpha \leq \beta$. $\square$

Definition 4.11. *If X is a set of ordinals, we let* $\sup X = \inf(\{\beta \colon (\forall \alpha \in X)[\beta \geq \alpha]\}$ *denote the least upper bound of X.*

Next, we show every wellorder is isomorphic to a unique ordinal. First we give an exercise:

Exercise 4.12. *If α is a set of ordinals, then α is an ordinal iff α is a transitive set.*

Note that a set α of ordinals being transitive is equivalent to being closed downwards under $<$; i.e. α is transitive if $\beta \in \alpha$, and $\gamma < \beta$, implies $\gamma \in \alpha$.

Lemma 4.13. *If $(P, <_P)$ is a wellordering, then $(P, <_P)$ is isomorphic to a unique ordinal.*

Proof. Every wellordering is isomorphic to at most one ordinal by Lemma 3.4. Consider $F = \{(x, \alpha) \colon \alpha$ is isomorphic to the initial segment $(P, <_P) \restriction x\}$. It is clear that F is an order preserving map, that $\mathrm{dom}(F)$ is closed downwards, and $\mathrm{ran}(F)$ is a transitive set of ordinals. Finally, if $\mathrm{dom}(F) \neq P$, then letting x be the least element of P not in $\mathrm{dom}(F)$, we see that F is an isomorphism from $(P, <_P) \restriction x$ to a set of ordinals which is closed downwards, and thus must be an ordinal. Contradiction. Hence, F is an isomorphism from P to an ordinal. $\square$

Definition 4.14. *If P is a wellordering, we let $\mathrm{ot}((P, <_P))$ denote the unique ordinal isomorphic to P; the* ***ordertype of*** *P.*

There are two types of ordinals: successor ordinals and limit ordinals:

Definition 4.15. *If α is an ordinal, we define $\alpha + 1$ to be $\alpha \cup \{\alpha\}$. We say α is a* ***successor ordinal*** *if there is an ordinal β so that $\alpha = \beta + 1$. If α is not a successor ordinal, we call α a limit ordinal.*

We have the following easy lemma:

Lemma 4.16. *If α is an ordinal, $\alpha + 1$ is an ordinal.*

Proof. First, $\alpha + 1$ is transitive. Suppose $a \in \alpha + 1$ and $b \in a$. We want to show $b \in \alpha + 1$. Case 1: if $a \in \alpha$, then $b \in \alpha$ since α is transitive, hence $b \in \alpha + 1$. Case 2: if $a = \alpha$, then since $b \in \alpha$, $b \in \alpha \cup \{\alpha\} = \alpha + 1$.

The verification that $\alpha+1$ is wellfounded and linearly ordered by ϵ is similarly simple; since every element of $\alpha + 1$ is either α, or an element of α. $\square$

We have the following simple facts about $\alpha + 1$.

Lemma 4.17. *If α is an ordinal, then $\beta < \alpha + 1$ if and only if $\beta \leq \alpha$.*

Proof. If $\beta < \alpha + 1$, then either $\beta \in \alpha$ and so $\beta < \alpha$, or $\beta \in \{\alpha\}$ and so $\beta = \alpha$. $\square$

Next, we want to show that there are nonzero limit ordinals. The least such ordinal is ω.

Definition 4.18. *We let*

$$\omega = \bigcap \{x \colon x \text{ is inductive}\}$$

We call the elements of ω ***natural numbers****. We let 0 denote the ordinal $\emptyset$.*

To see that ω is a set, note that if we let x_0 be any inductive set (which exists by the Infinity axiom), then ω is a subset of x_0 which can be defined by the separation axiom.

Lemma 4.19 (Mathematical Induction)**.** *Suppose $A \subseteq \omega$ is such that $0 \in A$ and for all y, $y \in A \to y+1 \in A$. Then $A = \omega$.*

Proof. $A \subseteq \omega$ by assumption. A is inductive, so $\omega \subseteq A$. $\square$

Lemma 4.20. *Every natural number is an ordinal.*

Proof. Let A be the elements of ω that are ordinals. Now $\emptyset$ is an ordinal, and if α is an ordinal, then $\alpha+1$ is an ordinal. Hence by Lemma 4.19, $A = \omega$, and every natural number is an ordinal. $\square$

Lemma 4.21. *ω is an ordinal.*

Proof. It is easy to see that ω is transitive by induction. Let $A = \{\alpha \in \omega \colon (\forall \beta \in \alpha)\beta \in \omega\}$. Clearly $\emptyset \in \omega$. Next, suppose $\alpha \in A$. Then $\alpha + 1 \in A$, since every $\beta \in \alpha + 1$ has either $\beta = \alpha$, or $\beta \in \alpha$.

By Lemma 4.6, ω is linear. $\square$

Exercise 4.22. *ω is the least nonzero ordinal which is not a successor ordinal.*

An **sequence** is a function f with domain ω. We often write $\langle f_n \colon n \in \omega\rangle$ to represent sequences. Similarly, a **transfinite sequence** is a function f whose domain is an infinite ordinal α, and we use the notation $\langle f_\beta \colon \beta < \alpha\rangle$ for transfinite sequences. We will also consider ORD length sequences which will be class functions on ORD.

Exercise 4.23. *Show that a strict partial order $(P, <_P)$ is wellfounded iff there is no sequence $\langle a_n \colon n \in \omega\rangle$ such that $(\forall n \in \omega)a_{n+1} <_P a_n$.*

Exercise 4.24.

1. *For each α, $\alpha = \{\beta \in \mathsf{ORD} \colon \beta < \alpha\}$.*
2. *If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$, and $\bigcap C = \inf C$.*
3. *If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal. If X has no maximal element, then $\bigcup X = \sup X$.*
4. *If α is an ordinal, $\alpha + 1 = \inf\{\beta \colon \beta > \alpha\}$.*

Exercise 4.25.

1. *Show that the class ORD of all ordinals is not a set.*
2. *Say that an ordinal is **countable** if it is isomorphic to a wellordering on a subset of ω. Prove that the class of countable ordinals is a set. Carefully state the axioms of ZFC that you use.*

5 Transfinite induction and recursion

We can now formulate the principles of transfinite induction and recursion. Transfinite induction is a proof technique we use to prove statements about all ordinals, analogously to how we use ordinary induction to prove statements about all natural numbers. Transfinite recursion lets us recursively define functions on the ordinals, similarly to how ordinary recursion lets us recursively define functions on ω.

Theorem 5.1 (Transfinite induction)**.** *Suppose C is a class of ordinals such that*

- $0 \in C$,
- *For all α, $\alpha \in C \to \alpha + 1 \in C$,*
- *If λ is a nonzero limit ordinal $(\forall \alpha < \lambda)(\alpha \in C) \to \lambda \in C$.*

Then $C = \mathsf{ORD}$.

Proof. Suppose λ is the least ordinal such that $\lambda \notin C$. Then apply one of the three conditions above. $\square$

If X and Y are classes (perhaps proper classes), then a **class function** F from X to Y is a class that is a subclass of $X \times Y = \{(x, y) \colon x \in X \wedge y \in Y\}$, such that for every $x \in X$, there is a unique $y \in Y$ such that $(x, y) \in F$.

Theorem 5.2 (Transfinite recursion)**.** *Let G be a class function (on V). Then there is a unique class function F such that for all $\alpha \in \mathsf{ORD}$,*

$$F(\alpha) = G(F \upharpoonright \alpha) \tag{*}$$

Note that $F \upharpoonright \alpha = F \upharpoonright \{\beta \colon \beta < \alpha\}$.

Proof. First we prove uniqueness. Suppose f, f' are two class functions on ordinals, or ORD satisfying (*) for all $\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$. Then we claim $f = f'$ on $\mathrm{dom}(f) \cap \mathrm{dom}(f')$. Suppose not, and let α be the least ordinal such that $f(\alpha) \neq f'(\alpha)$. Then have a contradiction, since by choice of α, $f' \upharpoonright \alpha = f \upharpoonright \alpha$. So since (*) is true on $\mathrm{dom}(f)$ and $\mathrm{dom}(f')$,

$$f(\alpha) = G(f \upharpoonright \alpha) = G(f' \upharpoonright \alpha) = f'(\alpha).$$

Now we define F. Let F be the set of pairs (β, y) such that there exists an f such that $\mathrm{dom}(f) \in ORD$, and $f(\alpha) = G(f \upharpoonright \alpha)$ for all $\alpha \in \mathrm{dom}(f)$, $\beta \in \mathrm{dom}(f)$, and $f(\beta) = y$.

F is a function by the uniqueness we've proved above. We claim $\mathrm{dom}(F) = \mathsf{ORD}$, if not, let β be the least element not in $\mathrm{dom}(F)$. Then by definition, there is some function f with $\mathrm{dom}(f) = \beta$ such that (*) is true on $\mathrm{dom}(f)$. Now let $f' = f \cup \{(\beta, G(f))\}$. Then f' also satisfies (*), and has domain $\beta + 1$. Contradiction! $\square$

We give some examples of transfinite induction and recursion. We will start with some operations on ordinals.

Definition 5.3. *Define ordinal addition by recursion as follows:*

- $\alpha + 0 = \alpha$.
- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
- $\alpha + \lambda = \sup\{\alpha + \beta \colon \beta < \lambda\}$.

Technically for each fixed α, we're defining the function $\beta \mapsto \alpha + \beta$ by recursion.

There is a different way to conceive of ordinal addition rather than this recursive definition of "iterated successor". The ordertype of $\alpha + \beta$ is the same as the ordertype of the order of α followed by β.

Lemma 5.4. *For all ordinals α, β, the ordertype of $\alpha + \beta$ is the same as the ordertype of the set $\alpha \sqcup \beta = \{0\} \times \alpha \cup \{1\} \times \beta$ equipped with the ordering $\prec$ where $(m, \gamma) \prec (n, \delta)$ if $n < m$, or $n = m$ and $\gamma \prec \delta$.*

Proof. We prove this for each α by transfinite induction on β. Our base case is that α is isomorphic to $\alpha \sqcup \emptyset$, and if $\alpha + \beta$ is isomorphic to $\alpha \sqcup \beta$, then if we add one more point to each order, $\alpha + (\beta + 1)$ is isomorphic to $\alpha \sqcup \beta + 1$. Finally, given any two isomorphic wellorders, there is a unique isomorphism between them. So if $\alpha + \beta$ is isomorphic to $\alpha \sqcup \beta$ for each $\beta < \lambda$ via the isomorphism f_β, then $\bigcup_{\beta < \lambda} f_\beta$ is an isomorphism between $\alpha + \lambda$ and $\alpha \sqcup \lambda$.

□

Similarly, we can definition ordinal multiplication and exponentiation:

Definition 5.5. *Define ordinal multiplication by recursion:*

- $\alpha \cdot 0 = 0$.
- $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$
- $\alpha \cdot \lambda = \sup\{\alpha \cdot \beta \colon \beta < \lambda\}$ *for limit* λ.

Exercise 5.6. *Show that $\alpha \cdot \beta$ is isomorphic to "β many copies of α". That is, $\alpha \cdot \beta$ is isomorphic to the lexicographic order on $\alpha \times \beta$, where $(\gamma, \delta) <_{\text{lex}} (\lambda, \xi)$ iff $\gamma < \lambda$ or $\gamma = \lambda$ and $\delta < \xi$.*

Caution: neither + nor times are commutative. For example, $1 + \omega = \omega \neq \omega + 1$ and $2 \cdot \omega \neq \omega \cdot 2$.

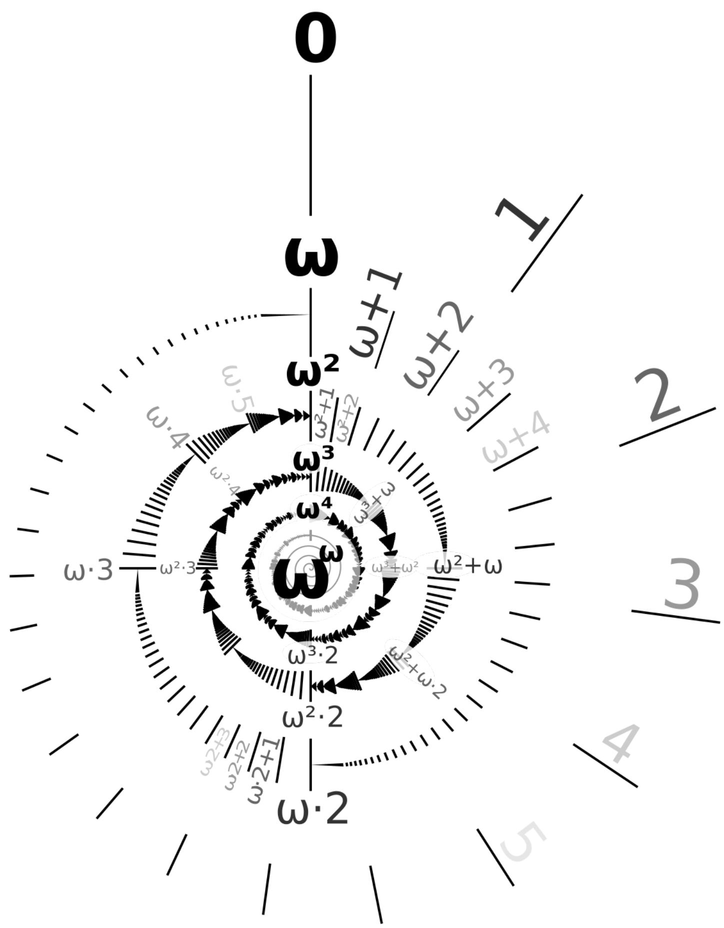

Figure 4: The ordinals up through ω^ω. Source: `https://commons.wikimedia.org/wiki/File:Omega-exp-omega-labeled.svg`

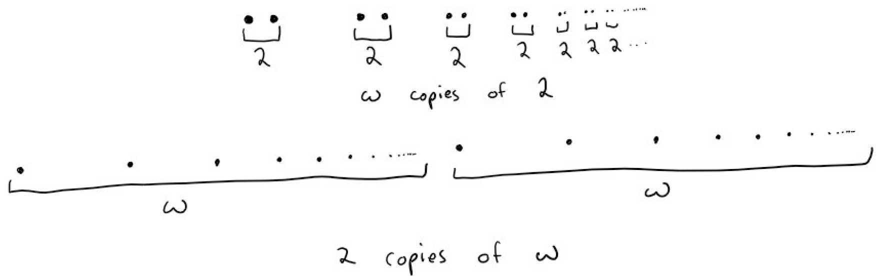

Figure 5: Ordinal multiplication is not commutative. $2 \cdot \omega \neq \omega \cdot 2$.

Definition 5.7. *Define ordinal exponentiation by recursion:*

- $\alpha^0 = 1$
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
- $\alpha^\lambda = \sup\{\alpha^\beta \colon \beta < \lambda\}$ *for limit* λ.

Cantor normal form gives a unique way of expressing every ordinal in terms of the above operations and smaller ordinals:

Exercise 5.8. *For every ordinal $\alpha > 0$, there are natural numbers $k_0, \ldots, k_n$ and ordinals $\alpha \geq \beta_0 > \beta_1 > \ldots > \beta_n$ such that*

$$\alpha = \omega^{\beta_0} \cdot k_0 + \ldots + \omega^{\beta_n} \cdot k_n.$$

Furthermore, this representation is unique.

Cantor normal form is not so useful for understanding arbitrary ordinals. For example, there are ordinals α such that $\omega^\alpha = \alpha$, and whose Cantor normal forms are "trivial". This follows from the following exercise:

Exercise 5.9. *Suppose $\langle \gamma_\alpha \rangle_{\alpha \in \mathsf{ORD}}$ is a sequence of ordinals such that if λ is a limit ordinal, then $\gamma_\lambda = \sup\{\gamma_\alpha \colon \alpha < \lambda\}$. Then this sequence contains arbitrarily large fixed points. That is, arbitrarily large α such that $\gamma_\alpha = \alpha$.*

Next, we give a characterization of when a relation is wellfounded in terms of ranks. Say that a relation R on a set X is **wellfounded** if for every set $Y \subseteq X$, there is an element $a \in Y$ that is R-minimal inside Y so for every $b \in Y$, $b \not R\, a$. (We previously defined wellfoundedness for partial orders and not for arbitrary relations). So for example, every wellfounded relation is irreflexive: if $a \mathrel{R} a$, then the set $\{a\}$ would have no R-minimal element.

Definition 5.10. *Suppose R is a relation on a set X. Define sets X_α where $\alpha \in \mathsf{ORD}$ by transfinite recursion as follows.*

- $X_0 = \{x \in X \colon x$ *is R-minimal in* $X\}$.
- $X_\alpha = \bigcup_{\beta<\alpha} X_\beta \cup \{x \in X \colon x$ *is R-minimal in* $X \setminus \bigcup_{\beta<\alpha} X_\beta\}$.

define $\mathrm{rank}_R \colon X \to \mathsf{ORD} \cup \{\infty\}$ *by* $\mathrm{rank}_R(x) = \inf\{\alpha \colon x \in X_\alpha\}$ *if there is some $\alpha \in \mathsf{ORD}$ such that $x \in X_\alpha$. Otherwise, let* $\mathrm{rank}_R(x) = \infty$.

Here ∞ should be understood as just a formal symbol. We interpret ∞ as being larger than any ordinal, and $\infty < \infty$. If $C \subseteq \mathsf{ORD}$ is empty, we define $\inf(C) = \infty$. The rank function has two key properties:

Lemma 5.11.

1. *For all $a \in X$,* $\mathrm{rank}_R(a) = \sup\{\mathrm{rank}_R(b) + 1 \colon b \mathrel{R} a\}$.
2. *If $a \mathrel{R} b$, then* $\mathrm{rank}_R(a) < \mathrm{rank}_R(b)$.

Proof. It is clear by induction that if $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$. So $\mathrm{rank}_R(a) = \alpha$ iff a is R-minimal in $X \setminus \bigcup_{\beta<\alpha} X_\beta$.

Now we prove (1). Assume a is R-minimal in $X \setminus \bigcup_{\beta<\alpha} X_\beta$. Then for all $b \mathrel{R} a$, we must have $\mathrm{rank}_R(b) < \alpha$ (otherwise a would not be R-minimal in this set), so $\mathrm{rank}_R(b) + 1 \leq \alpha$, so $\mathrm{rank}_R(a) \geq \sup\{\mathrm{rank}_R(b) + 1 \colon b \mathrel{R} a\}$. We must also have that $\mathrm{rank}_R(a) \leq \sup\{\mathrm{rank}_R(b) + 1 \colon b \mathrel{R} a\}$ since if $\alpha > \mathrm{rank}_R(b)$ for all $b \mathrel{R} a$, then clearly a is R-minimal in $X \setminus \bigcup_{\beta<\alpha} X_\beta$ if it is contained in this set.

(2) follows from (1). $\square$

Lemma 5.12. *A relation R on a set X is wellfounded if and only if* $\mathrm{rank}_R(a) \in \mathsf{ORD}$ *for every $a \in X$.*

Proof. Suppose R is wellfounded. Let $Y \subseteq X$ be $Y = \{a \in X \colon \mathrm{rank}(a) = \infty\}$. Let a be a minimal element of X. Then $\mathrm{rank}(a) = \sup\{\mathrm{rank}(b) + 1 \colon b <_P a\}$ by Proposition 5.11.(1). This is the sup of a set of ordinals, and is hence an ordinal. Contradiction.

Suppose $\mathrm{rank}_R(a) \in \mathsf{ORD}$ for every $a \in X$, and $Y \subseteq X$. Since ORD is wellfounded, let α be the minimal element of $\{\mathrm{rank}_R(a) \colon a \in Y\}$, and let $x \in X$ be such that $\mathrm{rank}(x) = \alpha$. Then by Lemma 5.11.(2), x is R-minimal in Y. $\square$

Exercise 5.13. *Consider that the set of polynomials with natural number coefficients under the eventual domination ordering: $p(x) <^* q(x)$ if $\exists x \forall x' > x(p(x) < q(x))$. Show that this is a wellordering isomorphic to ω^ω.*

Exercise 5.14. *Say an ordinal α is indecomposable iff there do not exist $\beta, \gamma < \alpha$ such that $\alpha = \beta + \gamma$. That is, α cannot be decomposed into the sum of two smaller ordinals. So ω and ω^2 are examples of ordinals that are indecomposable. Show the following are equivalent:*

1. *α is indecomposable.*

2. *α is a power of ω. That is, $\alpha = \omega^\xi$ for some ξ.*

3. *$\forall X \subseteq \alpha$, either $(X, \in)$ is isomorphic to α or $(\alpha \setminus X, \in)$ is isomorphic to α.*

Exercise 5.15. *Show that there is an ordinal ξ such that $\omega^\xi = \xi$. Describe the least such ordinal (it is called ϵ_0).*

Exercise 5.16. *Show that in ZFC − Infinity (i.e. ZFC without the axiom of infinity), the following are equivalent:*

1. *The axiom of infinity.*

2. *$\exists x(\emptyset \in x \wedge (\forall y \in x)(\{y\} \in x))$.*

Exercise 5.17. *Recall if A is a subset of a topological space X and $x \in A$, then x **is isolated in** A if there is an open set U such that $U \cap A = \{x\}$. A is **perfect** if there are no isolated points in A Prove the continuum hypothesis is true for closed subset of $\mathbb{R}$ sets:*

1. *Show that if $A \subseteq \mathbb{R}$ is closed, then there is a countable set $C \subseteq A$ so that $A \setminus C$ is perfect and closed. [Hint: by transfinite recursion, let $A_0 = A$, $A_{\alpha+1} = A_\alpha \setminus \{x\colon x \text{ is isolated in } A_\alpha\}$, and if λ is a limit, then $A_\lambda = \bigcap_{\alpha<\lambda} A_\alpha$. Show that for every α, $A \setminus A_\alpha$ is closed (it is equal to A minus an open set). Then show there is a countable ordinal α such that $A_{\alpha+1} = A_\alpha$ and hence A_α is perfect (to see this, use the fact that $\mathbb{R}$ has a countable basis: all intervals (a, b) with rational endpoints).]*

2. *Show that if $A' \subseteq \mathbb{R}$ is a nonempty closed perfect set, then there is an injection from $\mathcal{P}(\mathbb{N})$ to A.*

5.1 Goodstein's theorem*

A hereditary representation of a number in base b is that number expressed as a sum of powers of b, where the exponents are also recursively represented as sums of power of b, and the exponents of the exponents and so on. So for example:

$$537 = 2^9 + 2^4 + 2^3 + 2^0 = 2^{2^{2^1+2^0}+2^0} + 2^{2^2} + 2^{2^1+2^0} + 2^0$$

where the last representation is hereditary in base 2.

Given a number n, the Goodstein sequence $G_2(n), G_3(n), \ldots$ is defined by setting $G_2(n) = n$, and $G_{b+1}(n)$ is obtained by writing $G_b(n)$ in hereditary base b, replacing all occurrences of b with $b+1$, and then subtracting one. So for example, if we start at the number 3

- $G_2(3) = 2^1 + 2^0 = 3$
- $G_3(3) = 3^1 + 3^0 - 1 = 3^1 = 3$

- $G_4(3) = 4^1 - 1 = 3$
- $G_5(3) = 3 - 1 = 2$
- $G_6(3) = 2 - 1 = 1$
- $G_7(3) = 1 - 1 = 0$

If we start at the number 4,

- $G_2(4) = 2^{2^1} = 4$
- $G_3(4) = 3^3 - 1 = 2 \cdot 3^2 + 2 \cdot 3 + 2 = 26$
- $G_4(4) = 2 \cdot 4^2 + 2 \cdot 3 + 2 - 1 = 2 \cdot 4^2 + 2 \cdot 4 + 1 = 41$
- $G_5(4) = 2 \cdot 5^2 + 2 \cdot 5 + 1 - 1 = 2 \cdot 5^2 + 2 \cdot 5 = 60$
- $G_5(4) = 2 \cdot 6^2 + 2 \cdot 6 - 1 = 2 \cdot 6^2 + 6 + 5 = 83$
- $\vdots$
- $G_{3 \cdot 2^{402653211} - 1}(4) = 0$

Now the Goodstein sequence $G_2(n), G_3(n), \ldots$ starting at any number will eventually reach 0. To see this, replace the expressions of each number $G_b(n)$ in hereditary base b, with the ordinal $G_b^*(n)$ obtained by replacing b with ω. So for example,

- $G_2^*(4) = \omega^{\omega^1}$
- $G_3^*(4) = \omega^2 \cdot 2 + \omega \cdot 2 + 2$
- $G_4^*(4) = \omega^2 \cdot 2 + \omega \cdot 2 + 1$
- $G_5^*(4) = \omega^2 \cdot 2 + \omega \cdot 2$
- $G_6^*(4) = \omega^2 \cdot 2 + \omega + 4$
- $\vdots$

Then $G_2^*(n), G_3^*(n), G_4^*(n), \ldots$ is a decreasing sequence of ordinals, which must therefore eventually reach 0. We therefore have Goodstein's theorem:

Theorem 5.18 (Goodstein's theorem)**.** *For every natural number n, the sequence $G_2(n), G_3(n), \ldots$ eventually reaches* 0*.*

This is a famous example of a theorem about the natural numbers which cannot be proved in Peano Arithmetic PA.

Theorem 5.19 (Kirby-Paris, 1982)**.** $\mathsf{PA} \nvdash$ *Goodstein's theorem*

The crux of their proof is that while PA can formalize and prove some basic facts about transfinite induction, it only has sufficient power to handle transfinite induction through ordinals less than some finite height tower. Proving Goodstein's theorem truly requires being able to perform transfinite induction through $\epsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. Indeed, PA + Goodstein's theorem $\vdash$ Con(PA). Another famous theorem which is true but not provable in PA is the Paris-Harrington theorem in Ramsey theory.

6 The cumulative hierarchy

We define the cumulative hierarchy V of sets by transfinite iterating the powerset operation.

Definition 6.1. *For each ordinal α, we define a set V_α as follows:*

- $V_0 = \emptyset$
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ *for all* α
- $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$, *for λ a limit ordinal.*

Let V be the class $\bigcup_{\alpha\in\mathsf{ORD}} V_\alpha = \{x \colon \exists\alpha x \in V_\alpha\}$.

We have the following proposition:

Proposition 6.2.

1. *V_α is transitive.*
2. $\alpha \leq \beta \to V_\alpha \subseteq V_\beta$.
3. $\alpha \in V_{\alpha+1} \setminus V_\alpha$.
4. $V_\alpha \in V_{\alpha+1} \setminus V_\alpha$.

Proof. We prove (1) by transfinite induction. For our base case, $\emptyset$ is transitive. At successor steps suppose $a \in V_{\alpha+1}$ and $b \in a$. Then $a \subseteq V_\alpha$ by definition of $V_{\alpha+1}$, so since $b \in a$, we have $b \in V_\alpha$. Since V_α is transitive by our induction hypothesis, if $c \in b$, then $c \in V_\alpha$. Hence, $b = \{c \colon c \in b\} \subseteq V_\alpha$, so $b \in V_{\alpha+1}$. At limit steps a union of transitive sets is transitive.

We prove (2) for each α by transfinite induction on β. Clearly $V_\alpha \subseteq V_\alpha$. Assume $V_\alpha \subseteq V_\beta$. Then if $x \in V_\alpha$ and hence $x \in V_\beta$, then since V_β is transitive by (1), if $y \in x$, then $y \in V_\beta$. Hence, $\{y \colon y \in x\} \subseteq V_\beta$ so it is an element of $V_{\beta+1}$. (2) is clear for limit ordinals β.

To prove (3), note that $\{\beta \in V_\alpha \colon \beta \in \mathsf{ORD}\} = \alpha$ by our induction hypothesis and (2). So $\alpha \in V_{\alpha+1}$, and $\alpha \notin V_\alpha$.

We prove (4) without the axiom of foundation. Clearly $V_\alpha \in V_{\alpha+1}$. Suppose $V_\alpha \in V_\alpha$. Then $V_\alpha \subseteq V_\beta$ for some $\beta < \alpha$. Since $\alpha \subseteq V_\alpha$ by (3), we would have $\alpha \subseteq V_\beta$, and hence $\alpha \in V_{\beta+1}$. But then $\alpha \in V_\alpha$, contradicting (3). $\square$

We can now use the cumulative hierarchy to define rank for elements of V:

Definition 6.3. *If $x \in V$, then* $\operatorname{rank}(x)$ *is the least ordinal α such that $x \subseteq V_\alpha$. So for instance,* $\operatorname{rank}(V_\alpha) = \alpha$, *and* $\operatorname{rank}(\alpha) = \alpha$.

Exercise 6.4. *Rank can also be defined by transfinite induction:* $\operatorname{rank}(x) = \bigcup\{\operatorname{rank}(y) + 1 \colon y \in x\}$.

Our next goal is to prove that the Foundation axiom is equivalent to the axiom that every set is in V. To do this, we first define the transitive closure of a set.

Definition 6.5. *For any set x, let $x_0 = x$, and $x_{n+1} = x_n \cup \bigcup x_n$. Then define the* ***transitive closure*** *of x to be the set* $\mathrm{TC}(x) = \bigcup_{n<\omega} x_n$

Exercise 6.6. $\mathrm{TC}(x)$ *is the smallest transitive set containing x.*

Proposition 6.7. *Assume* ZF − *Foundation. Then the following are equivalent:*

- *The axiom of foundation*
- $\forall x(x \in V)$.

Proof. Note that we didn't use the axiom of foundation in our development of the ordinals or V. For example, we defined an ordinal to be a transitive set α so $\in$ is a strict wellorder of α. So the fact that $\alpha \notin \alpha$ follows by definition (otherwise the $\in$ ordering would not be strict), and not by foundation.

First, let's prove that $\forall x(x \in V)$, assuming the axiom of foundation. Let x be an arbitrary set. We claim for all $y \in \mathrm{TC}(x)$, there is an α such that $y \in V_\alpha$. Then setting $\xi = \sup\{\mathrm{rank}(y) + 1 \colon y \in \mathrm{TC}(x)\}$ (which is a set by the axiom of replacement since it is the range of the function $y \mapsto \mathrm{rank}(y) + 1$ on the set $\mathrm{TC}(x)$), $x \subseteq \mathrm{TC}(x) \subseteq V_\xi$, so $x \in V_{\xi+1}$. To prove the claim suppose y is an $\in$-minimal element of $\mathrm{TC}(x)$ such that $y \notin V$. But then every element of y is in V_α for some α, hence if $\beta = \sup\{\mathrm{rank}(z) + 1 \colon z \in y\}$, then $y \subseteq V_\beta$, hence $y \in V_{\beta+1}$, contradiction.

Conversely, assume $\forall x(x \in V)$ and suppose $x \in V$. We want to show x has an $\in$-minimal element. Let α be the least element of $\{\mathrm{rank}(y) \colon y \in x\}$ and let $y \in X$ be such that $\mathrm{rank}(y) = \alpha$. Since $\mathrm{rank}(y) = \sup(\{\mathrm{rank}(z) + 1 \colon z \in y\})$, it must be that no element of y is in x, otherwise α would not be the minimal rank of an element of x. □

One way of using ranks is that it allows us to select set-sized subsets of proper classes in a canonical way. Suppose C is a nonempty proper class. Then let $\hat{C} = \{x \in C \colon \forall y \in C\, \mathrm{rank}(x) \leq \mathrm{rank}(y)\}$ Then $\hat{C}$ is always a nonempty subset of C. This idea is known sometimes as Scott's trick and allows us to do things like formalize ultrapowers of proper class inner models in a coherent way.

We can use the above trick to show that the axiom of Replacement is equivalent to Separation and Collection. Recall that the only difference between replacement and collection is that replacement says that there is a unique y such that $\varphi(x, y, w_1, \ldots, w_n))$ whereas collection allows there to be a proper class of such y. However, we can use replacement to prove collection by taking the elements of minimal rank satisfying the formula.

Exercise 6.8. *Assume* ZF − *Separation* − *Replacement. Show the following are equivalent:*

1. *Separation + Collection*
2. *The axiom schema of replacement: for each formula φ:*

$$\forall X \forall w_1, \ldots, w_n[((\forall x \in X)(\exists! y)\varphi(x, y, w_1, \ldots, w_n)) \to \exists Z \forall y(y \in Z \leftrightarrow (\exists x \in X)\varphi(x, y, w_1, \ldots, w_n))]$$

Exercise 6.9. TC(x) *is the smallest transitive set containing x. That is, for all sets x, if y is any transitive set containing x, then* TC$(x) \subseteq y$.

Exercise 6.10. *Suppose κ is an strongly inaccessible cardinal (i.e. κ is regular, and $\lambda < \kappa$ implies $2^\lambda < \kappa$).*

1. *Show that for all $\alpha < \kappa$, $|V_\alpha| < \kappa$.*
2. *Show that $|V_\kappa| = \kappa$.*

7 The Mostowski collapse

Definition 7.1. *Suppose R is a relation on a class X. Recall R is* ***wellfounded*** *if for every nonempty set $Y \subseteq X$, there is an R-minimal element (i.e. some $y \in Y$ so that for all $z \in Y$, $(z, y) \notin R$). For each $x \in X$, we use the notation* $\mathrm{pred}_R(x)$ *to denote the class of predecessors of x, namely* $\mathrm{pred}_R(x) = \{y \in X : (y, x) \in R\}$*. We say R is* ***setlike*** *if for every $x \in X$,* $\mathrm{pred}_R(x)$ *is a set. We say R is* ***extensional*** *if* $\mathrm{pred}(x) = \mathrm{pred}(y) \rightarrow x = y$.

So for example, the axiom of extensionality says that the relation $\in$ is extensional.

Theorem 7.2. *Suppose R is an extensional wellfounded relation on a class X. Then there is a transitive class Y and an isomorphism f between (X, R) and $(Y, \in)$. That is, $f : X \to Y$ is a bijection such that for all $x, y \in X$, $(x, y) \in R \leftrightarrow f(x) \in f(y)$. Furthermore, f and Y are unique.*

One way to prove this is by transfinite recursion on the relation R; you can perform transfinite recursion along any wellfounded relation (not just on the ordinals). You'll do this as part of your homework. Then you can recursively define the function f by $f(x) = \{f(y) : (y, x) \in R\}$.

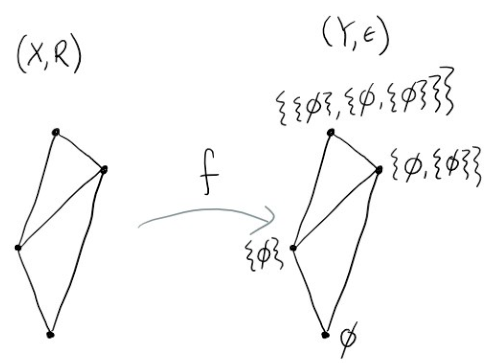

Figure 6: An example of the Mostowski collapse

We give a different proof.

Proof. Note that we are not assuming R is transitive! The first step in our proof will be handling this issue. Let R^* be the relation on X where $(x, y) \in R^*$ iff there is a finite sequence $x_0, \ldots, x_n$ such that $x = x_0$, $y = x_n$, and $x_i R x_{i+1}$ for all $i \leq n$. Now R^* is clearly transitive and since R is wellfounded, it follows that R^* is also wellfounded, and is thus a wellfounded partial order. Let $X_\alpha = \{x \in X : \mathrm{rank}_{R^*}(x) \leq \alpha\}$, where rank is as defined in Definition 5.10 for wellfounded partial orders. Note that $(x, y) \in R$ implies $\mathrm{rank}_{R^*}(x) < \mathrm{rank}_{R^*}(y)$.

We define functions $f_\alpha : X_\alpha \to V$ by transfinite induction, where $\alpha < \beta$ implies $f_\alpha \subseteq f_\beta$. Suppose we have defined f_α for all $\alpha < \beta$, such that:

- f_α is an injection,
- $\operatorname{ran}(f_\alpha)$ is a transitive
- $f_\alpha(x) \in f_\alpha(y)$ iff $(x, y) \in R$ for all $x, y \in \operatorname{dom}(f_\alpha)$, and
- $\operatorname{rank}_{R*}(x) = \operatorname{rank}(f_\alpha(x))$ for all $x \in \operatorname{dom}(f_\alpha)$.

Then for x such that $\operatorname{rank}_{R^*}(x) = \beta$, define $f_\beta(x) = \{\{f_{\operatorname{rank}_{R^*}(y)}(y) \colon (y, x) \in R\}$. Now by definition, each element of $f_\beta(x)$ is an element of $\operatorname{ran}(f_\alpha)$ for some $\alpha < \beta$. So $\operatorname{ran}(f_\beta)$ is transitive. We also have $\operatorname{rank}(f_\beta(x)) = \beta$, since $\operatorname{rank}(f_\beta(x)) = \sup\{\operatorname{rank}(y) + 1 \colon y \in f_\beta(x)\}$. Since $\operatorname{rank}(f_\beta(x)) = \beta$, to check that f_β is an injection we just need to ensure that if x, x' are distinct and $\operatorname{rank}_{R^*}(x) = \operatorname{rank}_{R^*}(x') = \beta$, then $f_\beta(x) \neq f_\beta(x')$. This follows since f_α is injective for $\beta < \alpha$, and since R is extensional.

To finish, let $f = \bigcup_{\alpha \in \mathsf{ORD}} f_\alpha$, and let $Y = \operatorname{ran}(f)$. $\square$

Suppose X is a set and E is a relation on X so that the model $M = (X; E)$ is a model of ZFC, where we interpret E as the $\in$ relation, and X as the universe of the model. Then E must be an extensional relation. If E is also wellfounded,[11] then (X, E) is isomorphic to a transitive set Y equipped with the $\in$ relation. We say a model of the form $(Y, \in\restriction Y)$ is a **transitive model**.

It is simple to see that ZFC does not prove there is a transitive model of ZFC. (Of course this is also a consequence of Gödel's theorem). If that were true, then there would exist an infinite descending sequence of transitive models under the ϵ relation. Hence, if there is a transitive model of ZFC, then there is a transitive model which does not contain any transitive model of ZFC[12] More strongly, we'll eventually see $\mathsf{ZFC}+\mathrm{Con}(\mathsf{ZFC})$ does not prove there is a transitive model of ZFC[13].

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[11] Note that assuming there is a model of ZFC, we can find a model $M = (X; E)$ so that E is an illfounded relation on X. This is by compactness. Add countably many constants $c_0, c_1, \ldots$ to our language of ZFC and let φ_n be the sentence $c_n \in c_{n-1} \in \ldots \in c_0$. Then $\mathsf{ZFC} + \{\varphi_n \colon n \in \omega\}$ is a consistent theory by the compactness theorem. So it has a model. Note that this illfounded model will still be a model of the axiom of regularity; M "thinks" that E is wellfounded. It simply does not contain a set with no E-minimal element.

[12] In fact, it is theorem of Shepherdson (which was rediscovered by Cohen) that if there is a transitive model of ZFC, then there is a minimal transitive model M in the sense that for all transitive models N of ZFC, $M \subseteq N$. This minimal model M has a simple description; it is L_α, where α is the inf of the heights of transitive models of ZFC.

[13] To see this, suppose $\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$ and let M be a transitive model of ZFC that does not contain any other model of ZFC (which must exist since $\in$ is wellfounded). Then it is easy to see that ω in M must be equal to the real ω, and since $\mathrm{Con}(\mathsf{ZFC})$ is true in the universe (i.e. there is no natural number coding a proof of a contradiction in ZFC), then $\mathrm{Con}(\mathsf{ZFC})$ is still true in M, and so $M \vDash \mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$, but M does not contain any transitive model of ZFC.

8 The axiom of choice

Recall that the axiom of choice says that if X is a set whose elements are nonempty and pairwise disjoint, then there is a choice set C so C contains exactly one element from each set in X. There are a few special cases of the axiom of choice which are true in ZF. For example, ZF proves the axiom of choice is true

1. When X is finite (by induction).
2. Sets such that there is a linear ordering of $\bigcup X$ and every $x \in X$ is finite (pick the least element in each $x \in X$).

However, ZF without choice cannot prove the following:

1. There is a choice set for $\{\{x + r : r \in \mathbb{Q}\} : x \in \mathbb{R}\}$
2. There is a choice set for $\{\{y \in \mathcal{P}(\mathbb{N}) : x \triangle y \text{ is finite}\} : x \in \mathcal{P}(\mathbb{N})\}$.
3. There is a wellordering of $\mathbb{R}$.

In this section, we'll prove some basic consequences of the axiom of choice.

Definition 8.1. *Suppose $(P, <_P)$ is a partial ordering. We say that a subset $X \subseteq P$ is a* ***chain*** *if for all $x, y \in X$, either $x \leq_P y$ or $y \leq_P x$. We say that $X \subseteq P$ is an* ***antichain*** *if for all $x, y \in X$, $x \not<_P y$ and $y \not<_P x$. If $X \subseteq P$, then we say that $a \in P$ is an* ***upper bound*** *for X if for all $b \in X$, $b \leq_P a$. We say an element $a \in P$ is* ***maximal in P*** *if there is no $b \in X$ such that $a <_P b$.*

Lemma 8.2 (Zorn)**.** *Suppose $(P, <_P)$ is a nonempty partial ordering so that for all nonempty chains X, there is an upper bound for X in P. Then P has a maximal element.*

Proof. By assumption, given any chain $X \subseteq P$, there exists an upper bound a for X. Note that a is maximal in the whole partial order P if and only if there is no upper bound a' for X so that $a' \notin X$. Hence, for each chain X, the set

$$\{(X, a) : a \text{ is an upper bound for } X \text{ and } a \notin X \text{ or } a \text{ is maximal in } P\}$$

is nonempty. So consider the collection of such sets:

$$\{\{(X, a) : a \text{ is an upper bound for } X \text{ and } a \notin X \text{ or } a \text{ is maximal in } P\} : X \text{ is a chain}\}$$

These sets are nonempty, and any two elements of this set are disjoint; they are sets of ordered pairs with different first coordinates. Hence, by the axiom of choice, there is a set G so $\mathrm{dom}(G) = \{X \subseteq P : X \text{ is a chain}\}$ and so that $G(X)$ is an upper bound for X so that $G(X) \notin X$, or $G(X)$ is maximal in P.

Now by transfinite recursion, consider the unique function F such that $F(\alpha) = G(\mathrm{ran}(F \restriction \alpha))$. Hence, F is a class function from ORD to P. We claim that for each α, $\mathrm{ran}(F \restriction \alpha) = \{F(\beta) : \beta < \alpha\}$ is a chain. This is true by transfinite induction. It is true for $\alpha = 0$ since $\emptyset$ is a chain. It is true at limit α since a union of chains is a chain. Finally, if it is true at α, it is true at $\alpha + 1$, since $F(\alpha)$ is a upper bound for the chain $\{F(\beta) \colon \beta < \alpha\}$, and the union of a chain and an upper bound for it is also a chain.

Now F cannot be an injection, since ORD is a proper class, and there cannot be an injective class function from a proper class to a set (apply the axiom of replacement to its inverse to get a contradiction). So there exists $\xi < \alpha$ such that $F(\xi) = F(\alpha)$. Then the chain $X = \{F(\beta) \colon \beta < \alpha\}$ includes $F(\xi) = F(\alpha)$. Hence, by the definition of G, since $F(\alpha)$ is an upper bound for X that is contained in X, we must have that $F(\alpha)$ is maximal in P. $\square$

The following is a special case of Zorn's lemma for the $\subseteq$ relation on subsets of $\mathcal{P}(X)$:

Corollary 8.3 (The Hausdorff Maximality principle)**.** *Suppose $X \neq \emptyset$ and $A \subseteq \mathcal{P}(X)$ is nonempty. Suppose that for all $B \subseteq A$, if $\forall x, y \in B$, $x \subseteq y$ or $y \subseteq a$, then there is some $z \in A$ so that $x \subseteq z$ for all $x \in B$. Then there exists some $a \in A$ so that there is no $b \in A$ so that $a \subsetneq b$.*

Lets prove that every set can be wellordered, assuming AC.

Theorem 8.4 (Zermelo's wellordering theorem)**.** *There is a wellordering of every set X.*

Proof. It suffices to find an injection from an ordinal to X.

Let y be such that $y \notin X$. Let $G \colon \mathcal{P}(X) \to X \cup \{y\}$ be such that $G(A) \in A$ for all nonempty $A \subseteq X$, and $G(\emptyset) = y$. Such a function G exists by the axiom of choice applied to the set $\{\{(A, a) \colon a \in A \lor (A = \emptyset \land a = y)\} \colon A \in \mathcal{P}(X)\}$. Then by transfinite recursion, we can find a function $F \colon \mathsf{ORD} \to X \cup \{y\}$ so that $F(\alpha) = G(X \setminus \operatorname{ran}(F \restriction \alpha))$.

We claim that if $\alpha < \beta$ and $F(\alpha) \in X$, then $F(\beta) \neq F(\alpha)$. This is true by definition of $F(\alpha)$, and since $G(A) \in A$ unless $A = \emptyset$ in case which $G(\emptyset) = y \notin X$.

Since ORD is a proper class, there cannot be an injection from ORD to the set $X \cup \{y\}$. Thus, there must be $\alpha < \beta$ so that $F(\alpha) = F(\beta)$. By the above claim, we must have that $F(\alpha) = y$. Let γ be the least ordinal such that $F(\gamma) = y$, and hence $\operatorname{ran}(F \restriction \alpha) = X$. Then by our claim Then $F \restriction \alpha$ is an injection from α to X. $\square$

One could also prove the wellordering theorem from the Hausdorff maximality principle by taking a maximal injection from an ordinal to a subset of X. The class of injections from ordinals to X is a set by Hartog's theorem in the next section.

The following is one of the oldest open problems in set theory:

Open Problem 8.5. *Assume ZF. Are the following equivalent:*

- *(The partition principle) for all sets X and Y, if there is a surjection from X to Y, then there is an injection from Y to X.*
- *The axiom of choice.*

8.1 Fragments of the axiom of choice*

There are small fragments of the axiom of choice which are still not provable in ZF. For example, **the axiom of countable choice** AC_ω says that if X is a countable set whose elements are nonempty and pairwise disjoint, then there is a choice set C for X. **The axiom of dependent choice** says that if R is a binary relation on X and for all $a \in X$ there exists $b \in X$ such that $a\ R\ b$ (i.e. R is **entire**), then there is a sequence $\langle x_n \colon n \in \omega \rangle$ such that for all $n \in \omega$, $x_n\ R\ x_{n+1}$.

It is easy to see that in ZF, AC implies DC which implies AC_ω. To prove DC assuming AC, let G be a function on X so that for all $a \in X$, $a\ R\ G(a)$, which exists by AC. Then use recursion to find some F so that $F(0) \in X$ and $F(n+1) = G(F(n))$. To prove AC_ω from DC, take a bijection from ω to a countably infinite set X, and relation R on $\bigcup X$ where $a\ R\ b$ if there exists n so that $a \in f(n)$ and $b \in f(n+1)$. None of the implications $\mathsf{AC} \to \mathsf{DC} \to \mathsf{AC}_\omega$ can be reversed.

ZF by itself is a pretty dismal theory. For example ZF does not prove that a countable union of countable sets is countable. Indeed, ZF cannot even prove that $\mathbb{R}$ is not a countable union of countable sets. A recent theorem of Gitik is that assuming certain large cardinal axioms, there is a model of ZF where there are no uncountable regular cardinals.

But these are scary bedtime stories; mathematicians almost always work in $\mathsf{ZF} + \mathsf{DC}$, even when they are being careful about the use of the axiom of choice. DC is absolutely essential for developing lots of basic mathematics (e.g. all of analysis), and DC does not have any of the "pathological" consequences of AC. For example, DC does not imply the Banach-Tarski paradox, or existence of a nonmeasurable subset of $\mathbb{R}$, or a wellordering of the real numbers.

9 Cardinality in ZF

We'll begin our discussion of cardinality in ZF without assuming the axiom of choice. We will sometimes emphasize that a theorem is true just in ZF by writing it in the assumptions, but all the definitions and theorems in the section are true without assuming the axiom of choice.

Definition 9.1. *If X and Y are sets, say that X and Y have the same cardinality and write $|X| = |Y|$ if there is a bijection from X to Y. Formally, cardinality is an equivalence relation, and $|X|$ denotes the equivalence class under this equivalence relation*[14]*. Say that X has cardinality less than or equal to Y and write $|X| \leq |Y|$ if there is an injection from X to Y.*

Assuming ZF, if $|X| \leq |Y|$, then there is a surjection from Y to X (but the converse does not hold). If we also assume the axiom of choice, then $|X| \leq |Y|$ if and only if there is a surjection from Y to X.

Exercise 9.2. *Suppose X and Y are nonempty sets*

1. *(ZF) If $|X| \leq |Y|$, then there is a surjection from Y to X.*
2. *(ZFC) $|X| \leq |Y|$ if and only if there is a surjection from Y to X.*

Theorem 9.3 (ZF, Cantor-Shröder-Bernstein)**.** *Assume X and Y are sets, $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to X$ be injections. We define sequences of sets $\langle A_n\colon n \in \omega\rangle$ and $\langle B_n\colon n \in \omega\rangle$ simultaneously by induction. Let $A_0 = X$, $B_n = Y \setminus f(A_n)$, and $A_{n+1} = X \setminus g(B_n)$. Then it is easy to see that the A_n are decreasing and B_n are increasing.

Let $A = \bigcap_n A_n$, and $B = \bigcup_n B_n$. Now define a function $h\colon X \to Y$ by $h = f \upharpoonright A \cup g^{-1} \upharpoonright g(B)$. To see h is a bijection note that $f(A) = f(\bigcap_n A_n) = \bigcap_n (Y \setminus B_n) = Y \setminus \bigcup_n B_n = Y \setminus B$. So since $f(A)$ and B are disjoint h is one-to-one, and since $f(A) \cup B = Y$, we see h is onto. $\square$

In ZF without assuming choice, there can be many incomparable cardinalities. For example, it is possible that $|\mathbb{R}| \not\leq |\omega_1|$ and $|\omega_1| \not\leq |\mathbb{R}|$. An important class of cardinalities are those that come from the sizes of ordinals.

Definition 9.4. *Say that an ordinal α is a **cardinal** if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. We will use Greek letters $\kappa, \lambda, \ldots$ for cardinals. By transfinite recursion, let ω_α be the αth infinite ordinal which is a cardinal. That is, ω_α is the least ordinal whose cardinality is greater than ω_β for all $\beta < \alpha$. Let $\aleph_\alpha$ denote the cardinality of ω_α.*

So for example, $\omega_0 = \omega$, and ω_1 is the least uncountable ordinal.

Exercise 9.5. *Show that if α is a limit ordinal, then $\omega_\alpha = \sup_{\beta<\alpha} \omega_\beta$.*

[14]When it is convenient to have a set representing this cardinality, we can use Scott's trick and take all elements of minimal rank that have a given cardinality

We have the following theorem relating the cardinality of arbitrary sets to ordinals.

Theorem 9.6 (ZF, Hartog's theorem)**.** *For each set X there is an ordinal which cannot be injected into X.*

Proof. Consider $\kappa = \{\alpha\colon \text{there is an injection from } \alpha \text{ to } X\}$. This is equal to $\{\alpha\colon \text{there is a wellordering of a subset of } X \text{ of ordertype } \alpha\}$, which is a set by the axiom of replacement. It is closed downward, and is hence an ordinal. Since κ is not an element of itself, it does not inject into X. $\square$

The least ordinal that cannot be injected into X is called the **Hartog's number** of X, and is denoted $h(X)$. It is always a cardinal.

Note that for all ordinals α, there is an ordinal λ of greater cardinality than α: it is $\{\beta\colon |\beta| \leq |\alpha|\} = h(\alpha)$. We have special notation for this:

Definition 9.7. *If κ is a cardinal, we let κ^+ denote the least ordinal of cardinality greater than κ.*

We've already proved:

Theorem 9.8 (Cantor, *ZF*)**.** *For every set X, $|X| < |\mathcal{P}(X)|$.*

So for every cardinality, there is a strictly larger cardinality. A different way to map a set to a set of larger cardinality is the map $X \mapsto X \cup h(X)$. In ZFC for infinite cardinals κ this is just the map $\kappa \mapsto \kappa^+$.

We have the following operations that we define on cardinalities:

Definition 9.9. *Suppose X and Y are sets. Then $|X|+|Y|$ is defined to be the cardinality of $X \sqcup Y$, the disjoint union of X and Y[15]. $|X| \cdot |Y|$ is defined to be the cardinality of the product $X \times Y$. $|Y|^{|X|}$ is defined to be the cardinality of the set of functions from X to Y.*

Exercise 9.10.

1. *The operations $+, \cdot, \exp$ are well defined on cardinals. If X, Y are sets and $|X| = |X'|$ and $|Y| = |Y'|$, then $|X|+|Y| = |X'|+|Y'|$, $|X|\cdot|Y| = |X'|\cdot|Y'|$, and $|X|^{|Y|} = |X'|^{|Y'|}$.*

2. *The operations are nondecreasing in every coordinate. Suppose X, X', Y are sets, and $|X| \leq |X'|$. Then $|X|+|Y| \leq |X'|+|Y|$, $|X|\cdot|Y| \leq |X'|\cdot|Y|$, $|X|^{|Y|} \leq |X'|^{|Y|}$, and $|Y|^{|X|} \leq |Y|^{|X'|}$.*

3. *The operations $+$ and $\cdot$ on cardinalities are commutative and associative.*

Our next goal is to show that the operations of addition and multiplication of cardinality of infinite ordinals are trivial, and $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max(\aleph_\alpha, \aleph_\beta)$. We begin with some easy facts about how cardinality behaves on ordinals:

Exercise 9.11. *For all infinite ordinals α, $|\alpha| = |\alpha + 1|$.*

[15] Recall $X \sqcup Y = \{0\} \times X \cup \{1\} \times Y$

We have the following:

Lemma 9.12. *(ZF) If α is an infinite ordinal, then $|\alpha \times \alpha| = |\alpha|$.*

Proof. We prove $|\alpha \times \alpha| = |\alpha|$ by transfinite induction on α. For our base case, $|\omega \times \omega| = |\omega|$ (e.g. via the pairing function $f((n, m)) = \frac{1}{2}(n+m)(n+m+1)+m$).

Now suppose that for all infinite $\beta < \alpha$ we have $|\beta \times \beta| = |\beta|$.

Case 1: α has the same cardinality as β for some $\beta < \alpha$. Then $|\alpha| \cdot |\alpha| = |\beta| \cdot |\beta| = |\beta| = |\alpha|$ by our induction hypothesis.

Case 2: suppose α is a cardinal. Then we claim that the following ordering $\prec$ on $\alpha \times \alpha$ is a wellordering of ordertype α. Define $(\beta, \gamma) \prec (\beta', \gamma')$ iff

- $\max(\beta, \gamma) < \max(\beta', \gamma')$, or
- $\max(\beta, \gamma) = \max(\beta', \gamma')$ and $\beta < \beta'$, or
- $\max(\beta, \gamma) = \max(\beta', \gamma')$, $\beta = \beta'$ and $\gamma < \gamma'$.

It is easy to check this is a wellordering.

Suppose $(\beta, \gamma) \in \alpha \times \alpha$. It suffices to show that the initial segment $\{(\beta', \gamma') : (\beta', \gamma') \prec (\beta, \gamma)\}$ of $\prec$ on $\alpha \times \alpha$ has ordertype less than α. Hence, the ordertype map sending each $(\beta, \gamma) \in \alpha \times \alpha$ to the ordertype of this initial segment is an injection from $\alpha \times \alpha$ to α.

Let $\lambda = \max(\beta, \gamma) + 1$, so $\lambda < \alpha$. Since $(\beta', \gamma') \preceq (\beta, \gamma)$ implies $\beta', \gamma' < \lambda$, the initial segment given by (γ, β) has cardinality $\leq |\lambda \cdot \lambda| = |\lambda| < \alpha$ by our induction hypothesis, so this initial segment of the wellordering $\preceq$ on $\alpha + \alpha$ must have ordertype less than α. Hence, every element of $\alpha \times \alpha$ is mapped to an ordinal less than α, so $|\alpha| \cdot |\alpha| \leq |\alpha|$, and hence $|\alpha| \cdot |\alpha| = |\alpha|$. $\square$

Exercise 9.13. *(ZF) Suppose X and Y are infinite sets. Then $|X| + |Y| \leq |X| \cdot |Y|$.*

Corollary 9.14. *(ZF) $|\aleph_\alpha| + |\aleph_\beta| = |\aleph_\alpha| \cdot |\aleph_\beta| = |\aleph_{\max(\alpha,\beta)}|$.*

Proof. Without loss of generality, assume $\alpha \leq \beta$. Then

$$|\aleph_\beta| \leq |\aleph_\alpha| + |\aleph_\beta| \leq |\aleph_\alpha| \cdot |\aleph_\beta| \leq |\aleph_\beta| \cdot |\aleph_\beta| = \aleph_\beta.$$

where the last equality is by the above Lemma, since $\aleph_\beta$ is an infinite ordinal. $\square$

Cardinal addition and multiplication on nonwellorderable sets in ZF can be quite interesting. Indeed, if cardinal addition and multiplication is too simple on infinite sets, then the axiom of choice must be true.

Theorem 9.15 (ZF)**.** *Suppose that for all infinite sets X and Y, $|X| + |Y| = |X| \times |Y|$. Then the axiom of choice is true.*

Proof. Suppose that for all infinite sets X and Y, $|X| + |Y| = |X| \times |Y|$. It suffices to show that every infinite set can be wellordered. Assume X is infinite, and f is an injection from $X \times h(X)$ to $X \sqcup h(X)$.

Case 1: there is some $x \in X$ so that for all $\alpha \in h(X)$, $f(x, \alpha) \in X$. Then $\alpha \mapsto f(x, \alpha)$ is an injection from $h(X)$ to X, which is a contradiction.

Case 2: For all $x \in X$, there is some $\alpha \in h(X)$ such that $f(x, y) \in h(X)$. In this case, let $g(x) = f(x, \alpha)$, where α is least such that $f(x, \alpha) \in h(X)$. Then g is an injection from X to $h(X)$, and hence X is wellorderable. $\square$

Using similar tricks with Hartog's number, one can prove the following:

Exercise 9.16 (ZF)**.** *The* GCH *in* ZF *is the statement that for all sets X, if Y is such that $|X| \leq |Y| \leq 2^{|X|}$, then $|Y| = |X|$ or $|Y| = 2^{|X|}$. Show that in* ZF*, if* GCH *is true, then* AC *is true.*

9.1 Cardinality in models of the axiom of determinacy*

The axiom of determinacy is an axiom that contradicts the axiom of choice, and paints a much different picture of the universe of sets. It is an important topic of study in modern set theory. Models of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$ in some sense contain only well behaved definable sets, have very regular behavior and no pathologies. They are compatible with large cardinals. We briefly discuss what cardinality is like in these models: they are beautiful examples of models of ZF without choice.

We've already seen two completely different proofs of the existence of uncountable sets: $\mathbb{R}$ is uncountable by Cantor's diagonal argument. ω_1 is uncountable since it is the set of all countable ordinals, and cannot be an element of itself.

In natural models of the axiom of determinacy, $\mathbb{R}$ cannot be wellordered and the cardinalities $|\mathbb{R}|$ and $|\omega_1|$ are incomparable; neither injects into the other. However, all other sets are uncountable by virtue of containing a copy of one of these two sets. Thus, in some sense, these two proofs for $\mathbb{R}$ and ω_1 are the *only* proofs that there exists an uncountable set.

Theorem 9.17 (Woodin, see [CK] and [H])**.** *In "natural" models of* AD *(such that $L(\mathbb{R})$ or $L(\mathcal{P}(\mathbb{R}))$ assuming large cardinals), if X is an uncountable set, then $|\mathbb{R}| \leq |X|$ or $|\omega_1| \leq |X|$.*

Note that assuming the axiom of choice, for nonempty X, $|X| \leq |Y|$ iff there is a surjection from Y to X. However, this is not true in ZF. If we have a surjection from Y to X, we need the axiom of choice to choose one element from each preimage to construct an injection from X to Y. Indeed, it is possible that if X is a set, and E is an equivalence relation on X, for $|X| < |X/E|$! Taking quotients can increase the cardinality of sets! In these models of AD, important cardinalities arise from equivalence relations E on the real numbers such that $|\mathbb{R}| < |\mathbb{R}/E|$.

9.2 Resurrecting Tarski's theory of cardinal algebras*

In the 1930s, Tarski investigated the theory of cardinal addition in ZF from an algebraic viewpoint. He created an axiomatic framework for investigating the addition operation on cardinalities in set theory without the full axiom of choice (which makes cardinal addition trivial).[16]

There are a few surprising and nontrivial theorems in this setting. One of the most famous is the following:

Theorem 9.18 (Lindenbaum and Tarski)**.** *Suppose X and Y are sets and $|n \times X| = |n \times Y|$, then $|X| = |Y|$.*

Even the case $n = 3$ in this theorem is not at all straightforward, and John Conway and Peter Doyle published a famous paper about this theorem called "Division by three" [CD].

Tarski published a volume on cardinal algebras in 1949 which contains many pages of tedious algebraic manipulations, but with some nice applications, such as the above theorem. Tarski's theory seems to have been largely forgotten for more than half a century. However, Kechris and Macdonald realized recently that there are many natural cardinal algebras being studied in modern set theory, such as in the theory of Borel equivalence relations, or equidecomposability in group actions with respect to a give σ-algebra [KM] (and which have a similar flavor in some ways to cardinality in models of set theory without the axiom of choice). Tarski's theory yields many new theorems in these settings which were not previously known.

[16]Precisely, a cardinal algebra is a triple $(A, +, \sum)$ where

- A is a set
- $+$ is a binary function on A, and
- $\sum \colon A^\omega \to A$ is a function

satisfying a short list of axioms (for example, one of the axioms is $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=0}^{\infty} a_{n+1}$ for every sequence $\langle a_n : n \in \omega \rangle$). Here we think of A as a set of cardinalities, and $+$ and $\sum$ as addition of two and countably many cardinalities.

10 Cofinality

Cofinality is a way of measuring how hard an ordinal is to approach from below. It plays a huge role in our understanding of cardinality.

Definition 10.1. *Suppose α is an ordinal and $C \subseteq \alpha$. Then we say that C is* ***cofinal*** *in α if for all $\beta \in \alpha$ there is some $\gamma \in C$ such that $\gamma \geq \beta$.*

Suppose $\alpha + 1$ is a successor ordinal. Then the set $\{\alpha\}$ containing just the maximal element of $\alpha + 1$ is cofinal in $\alpha + 1$. Hence, the cofinality of every successor ordinal is 1, and cofinality is only interesting for limit ordinals.

If α is a limit ordinal, then $C \subseteq \alpha$ is cofinal iff $\sup C = \alpha$.

Definition 10.2. *We define $\mathrm{cf}(\alpha)$ to be the least ordinal β such that there is a function $f\colon \beta \to \alpha$ such that $\mathrm{ran}(f)$ is cofinal in α.*

Exercise 10.3. *If α is a limit ordinal, then $\mathrm{cf}(\alpha) \geq \omega$.*

We compute a few examples:

- $\mathrm{cf}(\omega^\omega) = \omega$. This is since $\mathrm{cf}(\omega^\omega) \geq \omega$ by Exercise 10.3, and $\mathrm{cf}(\omega^\omega) \leq \omega$ since $\{\omega^n : n \in \omega\}$ is cofinal in ω^ω.
- $\mathrm{cf}(\omega_\omega) = \omega$, since $\{\omega_n : n \in \omega\}$ is cofinal in ω_ω by Exercise 9.5.
- $\mathrm{cf}(\omega_1) = \omega_1$. This is because if $C \subseteq \omega_1$ is cofinal, then $\omega_1 = \bigcup C$. However, since each ordinal in ω_1 is countable and a countable union of countable sets is countable, a countable set cannot be cofinal in ω_1.

Lemma 10.4. *Suppose α is a limit ordinal. Then $\mathrm{cf}(\alpha)$ is the least ordinal β such that there is a nondecreasing function $f\colon \beta \to \alpha$ such that $\mathrm{ran}(f)$ is cofinal in α.*

Proof. Suppose $f\colon \mathrm{cf}(\alpha) \to \alpha$ is such that $\mathrm{ran}(f)$ is cofinal in α. Now we define a nondecreasing $f'\colon \beta \to \alpha$ so that $\mathrm{ran}(f')$ is cofinal in α:

$$f'(\gamma) = \sup\{f(\delta) : \delta \leq \gamma\}.$$

Then $f'(\gamma) \geq f(\gamma)$ for all γ by definition, and $f'(\gamma) < \alpha$ for all $\gamma < \mathrm{cf}(\alpha)$, by definition of $\mathrm{cf}(\alpha)$, since $f \restriction \beta$ cannot be cofinal in α for $\beta < \mathrm{cf}(\alpha)$. Finally, f' is nondecreasing by definition. ☐

Indeed, by slightly modifying the above proof we have the following:

Exercise 10.5. *For every ordinal α, $\mathrm{cf}(\alpha)$ is equal to the shortest length λ of a strictly increasing sequence cofinal in α.*

We will often use the following to compute cofinalities:

Exercise 10.6. *Suppose α and β are limit ordinals, and there are nondecreasing functions $f\colon \alpha \to \beta$ and $g\colon \beta \to \alpha$ such that $\mathrm{ran}(f)$ is cofinal in β and $\mathrm{ran}(g)$ is cofinal in α. Then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.*

For example, if α is a limit ordinal, then $\text{cf}(\omega_\alpha) = \text{cf}(\alpha)$, since $\beta \mapsto \omega_\beta$ is a nondecreasing cofinal function from α to $\aleph_\alpha$, and $\beta \mapsto \sup\{\gamma \colon \aleph_\gamma \leq \beta\}$ is a nondecreasing cofinal function from $\aleph_\alpha$ to α.

Lemma 10.7. *For all ordinals α, $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$.*

Proof. Suppose $f \colon \beta \to \alpha$ is such that $\text{ran}(f)$ is cofinal and nondecreasing, and $g \colon \gamma \to \beta$ is such that $\text{ran}(g)$ is cofinal and nondecreasing. Then $g \circ f \colon \gamma \to \alpha$ is nondecreasing and cofinal. $\square$

Lemma 10.8. *If α is not a cardinal, then $\text{cf}(\alpha) < \alpha$.*

Proof. By definition, if α is not a cardinal, there is a $\beta < \alpha$ such that there is a bijection from β to α (and the range of this bijection is clearly cofinal). $\square$

Corollary 10.9. *For every α, $\text{cf}(\alpha)$ is a cardinal.*

Proof. If $\text{cf}(\alpha)$ was not a cardinal, then $\text{cf}(\text{cf}(\alpha)) < \text{cf}(\alpha)$, by Lemma 10.8, contradicting Lemma 10.7. $\square$

Cofinality breaks cardinals into two types that have very different behavior.

Definition 10.10. *We say a cardinal κ is* ***regular*** *if $\text{cf}(\kappa) = \kappa$. Otherwise, we say that κ is singular.*

Our first observation is that all successor cardinals are regular.

Theorem 10.11 (ZFC)**.** *Every infinite successor cardinal κ^+ is regular.*

Proof. To see this, suppose for a contradiction that $C \subseteq \kappa^+$ has cardinality κ and is cofinal. Since every element of C has cardinality $\leq \kappa$, by the axiom of choice we can pick an injection f_α from each element of C to κ. Now we can make an injection from $\bigcup C$ to $\kappa \times \kappa$ by letting $f(\beta) = (\alpha, f_\alpha(\beta))$ where $\alpha \in C$ is least such that $\beta \in \alpha$. By Lemma 9.12 this implies that $\bigcup C = \sup C$ has cardinality $\leq \kappa$, contradicting the definition of κ^+. $\square$

We give some examples of singular cardinals. The first infinite singular cardinal is $\aleph_\omega$, since ω is regular, and each $\aleph_n$ for $n > 0$ is a successor cardinal and hence regular. We computed earlier that $\text{cf}(\aleph_\omega) = \omega$. Indeed, since $\text{cf}(\aleph_\alpha) = \alpha$ for all limit α, we have that $\text{cf}(\aleph_{\omega_1}) = \omega_1$, so $\aleph_{\omega_1}$ is a singular cardinal. Another example of a singular cardinal is the first fixed point of the function $\alpha \mapsto \aleph_\alpha$. Let $\alpha_0 = \aleph_0$, $\alpha_{n+1} = \aleph_{\alpha_n}$, and let $\alpha = \sup_n \alpha_n$. Then since α_n is a strictly increasing sequence of cardinals, $\sup \alpha_n$ is a limit cardinal greater than $\aleph_{\alpha_n}$ for all n. But at limits α, $\aleph_\alpha = \sup_{\beta<\alpha} \aleph_\beta$, and hence $\aleph_\alpha = \alpha$. So we have $\text{cf}(\aleph_\alpha) = \omega$, since α_n is a cofinal sequence of ordertype ω.

If κ is a regular limit cardinal, then we say that κ is **weakly inaccessible**. We'll see soon that ZFC cannot prove uncountable weakly inaccessible cardinals exist. This is because if κ is uncountable and weakly inaccessible, then ZFC proves L_κ is a model of ZFC, and since ZFC $\nvdash$ Con(ZFC), ZFC cannot prove a weakly inaccessible cardinal exists.

We'll finish this section on cofinality by discussing König's theorem, which gives another important map sending each cardinal κ to a cardinality greater than κ:

Theorem 10.12 (König)**.** *If κ is a cardinal, then $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.*

Proof. It suffices to show that if $\langle f_\alpha \colon \alpha \in \kappa \rangle$ is a sequence of functions where $f_\alpha \colon \mathrm{cf}(\kappa) \to \kappa$, we can find some $h \colon \mathrm{cf}(\kappa) \to \kappa$ such that $h \neq f_\alpha$ for all $\alpha < \kappa$. This implies there is no surjection κ to $\kappa^{\mathrm{cf}(\kappa)}$, and hence there is no injection from $\kappa^{\mathrm{cf}(\kappa)}$ to κ.

Let $g \colon \mathrm{cf}(\kappa) \to \kappa$ be so that $\mathrm{ran}(g)$ is cofinal in κ. Now given any collection of values $\{f_\alpha(\beta) \colon \alpha \leq g(\beta)\}$, since this collection has cardinality $\leq |g(\beta)|$ which is less than κ, there is some element of κ not in this set. Let $h \colon \mathrm{cf}(\kappa) \to \kappa$ be defined by

$$h(\beta) = \inf \left(\kappa \setminus \{f_\alpha(\beta) \colon \alpha \leq g(\beta)\}\right).$$

Then for every $\alpha \in \kappa$, there is some $g(\beta)$ such that $\alpha \leq g(\beta)$, and hence $h(\beta) \neq f_\alpha(\beta)$. $\square$

One corollary of this theorem is that it tells us something about the cofinality of 2^κ, assuming ZFC.

Corollary 10.13 (ZFC)**.** $\mathrm{cf}(2^\kappa) > \kappa$.

Proof. $(2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa$. However, $(2^\kappa)^{\mathrm{cf}(2^\kappa)} > 2^\kappa$ by König's theorem. So we must have $\mathrm{cf}(2^\kappa) > \kappa$. $\square$

For example we cannot have $|\mathbb{R}| = \aleph_\omega$. This is because $|\mathbb{R}| = 2^\omega$, so $\mathrm{cf}(|R|) > \omega$ by König's theorem. However, $\mathrm{cf}(\aleph_\omega) = \omega$. Set theorists have shown using forcing that $\mathrm{cf}(|\mathbb{R}|) \neq \omega$ is essentially the only restriction ZFC imposes on the cardinality of $|\mathbb{R}|$.

Definition 10.14. *The **gimel function** is defined to be $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$ for κ an infinite cardinal.*

Note that if κ is regular, then $\gimel(\kappa) = \kappa^\kappa = 2^\kappa$. We will see that in ZFC the values of this function determine κ^λ for all κ, λ.

11 Cardinal arithmetic in ZFC

In this section, we further develop the basics of cardinal arithmetic. We emphasize that we heavily are using the axiom of choice in this section. Since AC implies every set can be wellordered, this implies every infinite set has the cardinality of some $\aleph_\alpha$. So for example, if $\kappa > \aleph_\alpha$, then $\kappa \geq \aleph_{\alpha+1}$.

Our eventual goal in this section is to get an inductive understanding of the value of κ^λ for infinite cardinals κ, λ. ZFC doesn't decide even the value of $\aleph_0^{\aleph_0}$. However, we'll give a (recursive) formula for the value of κ^λ just in terms of the gimel function $\gimel(\kappa) = \kappa^{\operatorname{cf} \kappa}$. The big question then becomes what possible values are there for the $\gimel$ function in ZFC. This is still partially an open question, although much is known.

In Section 9, we defined sums and products for pairs of cardinals. More generally, we can add or multiply any set of cardinals.

Definition 11.1. *Suppose κ_i is a cardinal for all $i \in I$. Then*

$$\sum_{i \in I} \kappa_i = \left| \bigcup_{i \in I} \{i\} \times \kappa_i \right|$$

and

$$\prod_{i \in I} \kappa_i = |\{f \colon \operatorname{dom}(f) = I \wedge (\forall i \in I) f(i) \in \kappa_i\}|$$

Exercise 11.2. *These operations are well defined on cardinalities. If $|x_i| = |y_i|$ for all $i \in I$, then $|\sum_{i \in I} x_i| = |\sum_{i \in I} y_i|$ and $|\prod_{i \in I} x_i| = |\prod_{i \in I} y_i|$.*

Computing infinite sums of cardinals is easy by the following lemma:

Lemma 11.3. *If $\kappa_i > 0$ for all $i < \lambda$, and $\lambda \geq \aleph_0$, then*

$$\sum_{i < \lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i = \max(\lambda, \sup_{i<\lambda} \kappa_i).$$

Proof. We clearly have $\sum_{i<\lambda} \kappa_i \leq \sum_{i<\lambda}(\sup_{i<\lambda} \kappa_i) = \lambda \cdot \sup_{i<\lambda} \kappa_i$. For the other inequality we have $\lambda = \sum_{i<\lambda} 1 \leq \sum_{i<\lambda} \kappa_i$ and $\sup_{i<\lambda} \kappa_i = |\bigcup_{i<\lambda} \kappa_i| \leq \sum_{i<\lambda} \kappa_i$. So $\max(\lambda, \sup_{i<\lambda} \kappa_i) \leq \sum_{i<\lambda} \kappa_i$. $\square$

Computing products of cardinals is more difficult. One inequality we will often use is that if $\kappa_i \leq \kappa$ for all i, then $\prod_{i<\lambda} \kappa_i \leq \kappa^\lambda$.

Exercise 11.4. *Suppose κ and λ are cardinals. Then $\prod_{i<\lambda} \kappa = \kappa^\lambda$.*

Summing cardinals gives us a different way of understanding singular cardinals.

Lemma 11.5. *An infinite cardinal κ is singular iff it is a sum of fewer than κ cardinals smaller than κ. That is, κ is singular iff $\exists \lambda < \kappa$ and a sequence $\langle \kappa_i : i < \lambda \rangle$ where $\kappa_i < \kappa$ for all $i < \lambda$ such that $\kappa = \sum_{i<\lambda} \kappa_i$.*

Proof. Suppose $\langle \alpha_i \colon i \in \lambda \rangle$ is a sequence of ordinals with $\alpha_i < \kappa$ for all i and $\lambda < \kappa$. Then if the sequence α_i is cofinal in κ, then $\sup_{i<\lambda} \alpha_i = \kappa$, so $\sum_{i<\lambda} |\alpha_i| = \lambda \cdot \kappa = \kappa$. However, if $\sup_{i<\lambda} \alpha_i < \kappa$, then $\sum_{i<\lambda} \alpha_i = \lambda \cdot \sup_{i<\lambda} \alpha_i = \max(\lambda, \sup_{i<\lambda} \alpha_i) < \kappa$. $\square$

We have the following theorem relating cardinal sums and exponentiation, which generalizes the two different diagonalization proofs that we've already discussed: Cantor's theorem that $\kappa < 2^\kappa$ and König's theorem that $\kappa < \kappa^{\operatorname{cf} \kappa}$.

Theorem 11.6 (König)**.** *Suppose $\kappa_i < \lambda_i$ for all $i \in I$. Then*

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

Proof. Suppose we have a function $f \colon \sum_{i \in I} \kappa_i \to \prod_{i \in I} \lambda_i$. We need to show f is not a surjection.

Define $h \in \prod_{i \in I} \lambda_i$ as follows. We need to define $h(i) \in \lambda_i$ for each $i \in I$. Given i, consider the set $\{f((i,\alpha))(i) \colon \alpha \in \kappa_i\}$. This set is a κ_i size subset of λ_i. Since $\kappa_i < \lambda_i$, the complement of $\{f((i,\alpha))(i) \colon \alpha \in \kappa_i\}$ inside λ_i is nonempty and so we can define

$$h(i) = \inf(\lambda_i \setminus \{f((i,\alpha))(i) \colon \alpha \in \kappa_i\}).$$

Since $h(i) \neq f((i,\alpha))(i)$ for all $i \in I$ and $\alpha \in \kappa_i$, we have $h \neq f((i,\alpha))$. So $h \notin \operatorname{ran}(f)$. $\square$

Cantor's theorem is the special case of König's theorem where we let $\kappa_i = 1$ and $\lambda_i = 2$ for all $i < \kappa$:

$$\kappa = \sum_{i<\kappa} 1 < \prod_{i<\kappa} 2 = 2^\kappa$$

König's theorem that $\kappa < \kappa^{\operatorname{cf}(\kappa)}$ is the special case where $I = \operatorname{cf}(\kappa)$, $f \colon I \to \kappa$ is cofinal, $\kappa_i = f(i)$ and $\lambda_i = \kappa$:

$$\kappa = \sum_{i<\operatorname{cf}(\kappa)} f(i) < \prod_{i<\operatorname{cf}(\kappa)} \kappa = \kappa^{\operatorname{cf}(\kappa)}.$$

Our next goal is to describe a little of what ZFC proves about cardinal exponentiation κ^λ. First, we have the following lemma that says we can determine the value of κ^λ recursively if we know the values of the function $\kappa \mapsto 2^\kappa$ and $\kappa \mapsto \kappa^{\operatorname{cf}(\kappa)}$.

Theorem 11.7. *Fix an infinite cardinal λ. Then for all infinite cardinals κ, the value of κ^λ is the following:*

1. *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
2. *If $\kappa > \lambda$ and $(\exists \mu < \kappa)\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$.*

3. *If $\kappa > \lambda$ and $(\forall \mu < \kappa)\mu^\lambda < \kappa$, then*

   (a) *if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$.*

   (b) *if $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$.*

Proof. For (1),
$$2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^\lambda.$$

For (2),
$$\mu^\lambda \leq \kappa^\lambda \leq (\mu^\lambda)^\lambda = \mu^\lambda$$

For (3a), since $\mathrm{cf}(\kappa) > \lambda$ every function $f\colon \lambda \to \kappa$ is bounded, so $\kappa^\lambda = |\bigcup_{\alpha<\kappa} \alpha^\lambda| = \sum_{\alpha<\kappa} |\alpha|^\lambda$. But
$$\kappa = \sum_{\alpha<\kappa} 1 \leq \sum_{\alpha<\kappa} |\alpha|^\lambda \leq \sum_{\alpha<\kappa} \kappa = \kappa \cdot \kappa = \kappa.$$

For (3b), if $\mathrm{cf}(\kappa) \leq \lambda$, then first write $\kappa = \sum_{i<\mathrm{cf}(\kappa)} \kappa_i$ where $1 < \kappa_i < \kappa$ for each i. Then since $\sum_{i<\mathrm{cf}(\kappa)} \kappa_i \leq \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$, we have
$$\kappa^\lambda \leq \left(\prod_{i<\mathrm{cf}(\kappa)} \kappa_i\right)^\lambda = \prod_{i<\mathrm{cf}(\kappa)} (\kappa_i^\lambda) \leq \prod_{i<\mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda.$$

□

Recall Cantor's continuum hypothesis is the statement that there is no cardinal κ with $|\mathbb{N}| < \kappa < |\mathbb{R}|$. Since assuming AC, every set has the same cardinality as an ordinal, and the next cardinal after $\aleph_0 = |\mathbb{N}|$ is $\aleph_1$, we can reformulate this as $2^{\aleph_0} = \aleph_1$. This statement is abbreviated CH
$$\mathsf{CH} : 2^{\aleph_0} = \aleph_1.$$

The generalized continuum hypothesis or GCH is the statement
$$\mathsf{GCH} : \text{ for all infinite } \kappa,\ 2^\kappa = \kappa^+.$$

We will show eventually that $L \vDash \mathsf{GCH}$.

If GCH holds, then we can simplify Theorem 11.7.

Exercise 11.8. *Assuming GCH, then for κ, λ infinite,*
$$\kappa^\lambda = \begin{cases} \kappa & \text{if } \lambda < \mathrm{cf}(\kappa) \\ \kappa^+ & \text{if } \mathrm{cf}(\kappa) \leq \lambda < \kappa \\ \lambda^+ & \text{if } \kappa \leq \lambda \end{cases}$$

Theorem 11.7 describes the value of κ^λ in terms of values of 2^λ and $\kappa^{\mathrm{cf}\,\kappa}$. Next, we turn to describing the value of 2^λ. First we need another definition.

Definition 11.9. *If κ and λ are cardinals, then $\kappa^{<\lambda} = \bigcup_{\mu<\lambda} \kappa^\mu$*

For example, $\kappa^{<\omega} = \kappa$ for all infinite cardinals κ, since $\kappa^n = \kappa$ for all n.

We also have the following exercise:

Exercise 11.10. *For all cardinals κ, $2^\kappa \leq (2^{<\kappa})^{\mathrm{cf}(\kappa)}$. [Hint: choose a sequence κ_i cofinal in κ, and find an injection from 2^κ into $\prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i}$.]*

Now we have the following theorem which recursively computes the value of 2^κ in terms of smaller values 2^λ for $\lambda < \kappa$, and the gimel function:

Theorem 11.11. *Suppose κ is an infinite cardinal. Then*

1. *If κ is regular, then $2^\kappa = \gimel(\kappa)$.*
2. *If κ is singular and 2^λ is eventually constant as $\lambda \to \kappa$ (so $2^{<\kappa}$ is this constant value), then $2^\kappa = 2^{<\kappa}$.*
3. *If κ is singular and 2^λ is not eventually constant as $\lambda \to \kappa$, then $2^\kappa = \gimel(2^{<\kappa})$.*

Proof. (1) If κ is regular, $\mathrm{cf}(\kappa) = \kappa$, and $2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cf}(\kappa)}$ by Theorem 11.7.1

(2) First, note $2^\kappa \geq 2^{<\kappa}$. Next, take $\mathrm{cf}(\kappa) \leq \lambda < \kappa$ such that $2^{<\kappa} = 2^\lambda$. Then $(2^{<\kappa})^{\mathrm{cf}(\kappa)} = (2^\lambda)^{\mathrm{cf}(\kappa)} = 2^{\lambda\cdot\mathrm{cf}(\kappa)} = 2^\lambda = 2^{<\kappa}$. So

$$2^{<\kappa} \leq 2^\kappa \leq (2^{<\kappa})^{\mathrm{cf}(\kappa)} = 2^{<\kappa}$$

(3) In this case $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}(\kappa)$ since $2^{<\kappa} = \bigcup_{\lambda<\kappa} 2^\mu$, and by Exercise 10.6. Hence, $\gimel(2^{<\kappa}) = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \geq 2^\kappa$ by Exercise 11.10. On the other hand, $\gimel(2^{<\kappa}) = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \leq (2^\kappa)^{\mathrm{cf}\,\kappa} = 2^\kappa$. □

11.1 Some equivalents of CH

The continuum hypothesis has many implications and equivalences in of areas of mathematics quite far from set theory. We briefly give a couple examples. Our first example in Wetzel's problem in complex analysis.

> Wetzel's problem: Let $\{f_i\}_{i\in I}$ be a family of pairwise distinct analytic functions on the complex numbers such that for each $z \in \mathbb{C}$, the set of values $\{f_i(z) \colon i \in I\}$ is countable. Does it follow that the set of functions $\{f_i\}_{i\in I}$ is countable?

Theorem 11.12 (Erdős, 1963)**.** *Wetzel's problem has a positive solution if and only if* CH *is true.*

Proof. Assume first that CH is false. We will then show that for any family $\{f_i\}_{i\in I}$ of analytic functions of size $\aleph_1$, there exists a complex number z so that the values $\{f_i(z) \colon i \in I\}$ are distict. Hence, Wetzel's problem has a positive solution.

We claim that for any i, j the set $A(i, j) = \{z \in \mathbb{C} \colon f_i(z) = f_j(z)\}$ is countable. This is because an analytic function is uniquely determined by its values on any infinite set that has an accumulation point. Hence, if there are infinitely many points of $A(i,j)$ inside the bounded set $B_n = \{z\colon |z| = n\}$ so that $f_i(z) = f_j(z)$, then $f_i(z) - f_j(z)$ must be the constant zero function. So there are only finitely many points z in $A(i,j) \cap B_n$ for each n, and so $|A(i,j)| \leq \aleph_0$. Hence $|\bigcup_{i,j \in I} A(i,j)| \leq |\aleph_0 \cdot \aleph_1| < |\mathbb{R}|$.

Now assume that CH is true. Let $\{z_\alpha\colon \alpha < \omega_1\}$ be a wellordering of $\mathbb{C}$. Let D be the set $\{p + qi\colon p, q \in Z\}$ which is dense in $\mathbb{C}$. We will construct a family $\{f_\alpha : \alpha < \omega_1\}$ of functions such that for all $\alpha < \beta$:

1. $f_\alpha(z_\beta) \in D$, and
2. $f_\alpha(z_\beta) \neq f_\beta(z_\beta)$.

Given α, let $\beta_0, \beta_1, \ldots$ be an enumeration of all the ordinals less than α.

Let $f_\alpha(x) = \epsilon_0 + \epsilon_1(x - z_{\beta_0}) + \epsilon_2(x - z_{\beta_0})(x - z_{\beta_1}) + \ldots$.

We will choose the values of ϵ_n successively in order to make f_α analytic, and to ensure that conditions (1) and (2) are satisfied. If $|\epsilon_n| \to 0$ sufficiently fast, then f_α will be analytic. Suppose we have already chosen $\epsilon_0, \ldots, \epsilon_{n-1}$, so that $f_\alpha(z_{\beta_i}) \neq f_{\beta_i}(z_{\beta_i})$ for $i < n$. (Note that the value of $f_\alpha(z_{\beta_i})$ depends only on the first n values of ϵ_i, since all subsequent terms are 0 at z_{β_i}). Then there is only one choice of ϵ_n that would make $f_\alpha(z_{\beta_n}) = f_{\beta_n}(z_{\beta_n})$. Hence, since D is dense, we can choose a sufficiently small value for ϵ_n so that $f_\alpha(z_{\beta_n}) \neq f_{\beta_n}(z_{\beta_n})$ and $f_\alpha(z_{\beta_{n)}}) \in D$. Hence, f_α will be analytic and satisfy (1) and (2) above. $\square$

Another beautiful equivalent of CH is the axiom of symmetry whose study dates back to Sierpiński.

> The axiom of symmetry: Suppose $f\colon \mathbb{R} \to \mathcal{P}_{\omega_1}(\mathbb{R})$ assigns to each real number $x \in \mathbb{R}$ a countable set of real numbers. Then there exists $x, y \in \mathbb{R}$ such that $x \notin f(y)$ and $y \notin f(x)$.

Theorem 11.13. *The axiom of symmetry holds iff* CH *is false.*

Proof. Assume CH. Let $\prec$ be a wellordering of $\mathbb{R}$ of ordertype ω_1. Then let $f(x) = \{y \in \mathbb{R}\colon y \preceq x\}$. Then given any x, y either $x \preceq y$ or $y \preceq x$, and hence $x \in f(y)$ or $y \in f(x)$.

Now assume $\neg$CH. Let $\{x_\alpha\colon \alpha < \omega_1\}$ be a sequence of ω_1 many distinct reals. Then $A = \bigcup\{f(x_\alpha)\colon \alpha < \omega_1\}$ has size $\aleph_1$, so there exists some $y \in R$ so that $y \notin A$. Since $f(y)$ is countable, there also exists some $\alpha < \omega_1$ so that $x_\alpha \notin f(y)$. $\square$

There is a thought experiment due to Freiling (built on ideas of Stuart Davidson) that claims that the axiom of symmetry should be intuitively true. Suppose we randomly throw two darts at the interval $[0, 1]$ and they land at the points x and y.

If we throw the first dart and it lands at x, then the second dart should land in the set $f(x)$ with probability 0. Indeed, in the sense of Lebesgue measure, if we randomly pick $y \in [0, 1]$ the chance it is in the countable set $f(x)$ is precisely 0.

Since the chance that y lands in $f(x)$ is 0 no matter how we pick x, we should be able to make this prediction before we throw the first dart and it hits x: "$y \notin f(x)$" almost surely. So if we threw two darts at the same time, then we should similarly have $y \notin f(x)$ almost surely. But then if we throw these two darts simultaneously, then by the symmetry of the situation, we should have $x \notin f(y)$ and $y \notin f(x)$.

The reason that this is an intutive argument and not a precise proof is that the switch from randomly picking x and then y to picking x and y at the same time requires a result like the Fubini theorem from analysis to justify, and Futini's theorem is not true for all functions f. Indeed, the kind of Lebesgue measurability that Freiling's intution relies on can be pushed futher to give an "intuitive" proof that the axiom of choice should be false; you shouldn't be able to wellorder the real numbers.

A much more detailed discussion of Freiling's philosophical arguments, their connections to precise mathematics, and their relation to measure theory, Baire category, GCH and the axiom of choice are in Freiling's paper [F].

11.2 The singular cardinals hypothesis*

An early theorem proved using forcing was Easton's theorem that on regular cardinals, powerset function $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$ can be any function which is nondecreasing and has $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$ (which is necessary by Corollary 10.13). However, the possible behavior of the powerset function on singular cardinals remained open.

In the mid sixties, Solovay asked whether it is possible to have $2^{\aleph_n} = \aleph_{n+1}$ for every n, and $2^{\aleph_\omega} = \aleph_{\omega+2}$. This ended up being a deep question which anticipated part of the singular cardinals problems, and requires large cardinals to answer.

As we've already seen, the gimel function $\gimel(\kappa)$ is the key we need to understand all of cardinal exponentiation. One trivial case is that if $2^{\mathrm{cf}\,\kappa} \geq \kappa$, then $2^{\mathrm{cf}(\kappa)} = (2^{\mathrm{cf}\,\kappa})^{\mathrm{cf}\,\kappa} \geq \kappa^{\mathrm{cf}\,\kappa} \geq 2^{\mathrm{cf}\,\kappa}$, so $\kappa^{\mathrm{cf}\,\kappa} = 2^{\mathrm{cf}(\kappa)}$, and we can understand $\kappa^{\mathrm{cf}(\kappa)}$ in terms of the powerset function at a smaller regular cardinal ($\mathrm{cf}(\kappa)$ is regular).

So the real interesting case is what happens to the value $\kappa^{\mathrm{cf}(\kappa)}$ when $2^{\mathrm{cf}\,\kappa} < \kappa$. Since $\kappa^{\mathrm{cf}(\kappa)} > \kappa$, the smallest value it could possibly take is κ^+, and this is the singular cardinals hypothesis, abbreviated SCH.

$$\mathsf{SCH}: \text{ if } \kappa \text{ is singular and } 2^{\mathrm{cf}(\kappa)} < \kappa, \text{ then } \kappa^{\mathrm{cf}(\kappa)} = \kappa^+$$

In a surprising development at the time, Jensen showed that the failure of SCH has large cardinal strength; if SCH is not true, then certain large cardinals exist. Eventually set theorists were able to prove from large cardinals there are models in which SCH fails. For example, it is a result of Magidor that assuming large cardinals, there is a positive answer to Solovay's question: a model of ZFC where $2^{\aleph_n} = \aleph_{n+1}$ for every n, and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

The possible behavior of the cardinal exponentiation at singular cardinals is still a topic of current research in set theory, and is intimately tied to large cardinals. For a longer introduction to this topic and pcf theory (which provides stunning limitations on the possible values of 2^κ), see [J].

12 Filters and ultrafilters

Filters and ideals are an important way of measuring when sets are "large" in many areas of mathematics.

Definition 12.1. *A* ***filter*** *F on a set X is a collection of subsets of X such that $X \in F$, $\emptyset \notin F$, F is closed under finite intersections (if $A \in F$ and $B \in F$, then $A \cap B \in F$), and F is closed upward under $\subseteq$ (if $A \in F$, $B \subseteq X$, and $A \subseteq B$, then $B \in F$).*

An ***ideal*** *I on a set X is a set of subsets of X such that $\emptyset \in I$, $X \notin I$, I is closed under finite unions (if $A \in I$ and $B \in I$, then $A \cup B \in I$), and I is closed downward under $\subseteq$ (if $A \in I$ and $B \subseteq A$, then $B \in I$).*

These definitions are dual to each other. If F is a filter on X, $\{X \setminus A \colon A \in F\}$ is an ideal called the **dual ideal** of F. Similarly, if I is an ideal, then the set of complements of elements of I forms a dual filter.

Here are some examples of ideals and filters:

- The collection of subsets of $[0, 1]$ having Lebesgue measure 1 are a filter. The dual ideal is the collection of nullsets.

- If $A \subseteq \mathbb{N}$, we say A has **asymptotic density** d if

$$\lim_{n\to\infty} \frac{|A \cap n|}{|n|} = d.$$

  For example, the even numbers have asymptotic density 1/2. The collection of subsets of $\mathbb{N}$ of asymptotic density 1 are a filter.

- If X is a set, κ is an infinite cardinal and $|X| \geq \kappa$, then $\mathcal{P}_\kappa(X) = \{A \subseteq X \colon |A| < \kappa\}$ is an ideal. For example, the collection of finite subsets of ω form an ideal.

You should think of a filter as a collection of "large" sets, and an ideal as a collection of "small" sets. If I is an ideal on X and F is its dual filter, then a set $A \subseteq X$ is **I-positive/F-positive** if $A \notin I$. You should think of I-positive sets as being "not small". For example if I is the ideal of subsets of $\mathbb{N}$ of asymptotic density 0, then the I-positive sets are those with positive upper density (i.e. $\limsup \frac{|A\cap n|}{|n|} > 0$).

Exercise 12.2. *Suppose X is a set, and $S \subseteq \mathcal{P}(X)$ is nonempty, S is closed under finite intersections, and $\emptyset \notin S$. Let $F = \{A \subseteq X \colon (\exists B \in S)(B \subseteq A)\}$. Show that F is a filter on X, called the* ***filter generated by*** *S.*

The following is an important property of filters, which we will use in subsequent sections:

Definition 12.3. *We say that a filter F on X is* ***κ-complete*** *if it is closed under intersections of size less than κ. That is, for all $\lambda < \kappa$, if $\langle A_\alpha \colon \alpha < \lambda\rangle$ is a sequence of elements of F, then $\bigcap_{\alpha<\lambda} A_\alpha \in F$.*

For example, the Lebesgue conull filter is $\aleph_1$-complete (the intersection of countably many conull sets is conull) and the dual filter of $\mathcal{P}_\kappa(X)$ is $\mathrm{cf}(\kappa)$-complete. Every filter is $\aleph_0$-complete.

A filter F on X is **maximal** if there is no filter F' on X such that $F' \supsetneq F$. A filter F is an **ultrafilter** if for all $A \subseteq X$, either $A \in F$ or $(X \setminus A) \in F$. These two notions are actually the same:

Lemma 12.4. *Suppose F is a filter on X, and $A \subseteq X$. Then either we can find a filter $F' \supseteq F$ on X so that $A \in F'$, or we already have $(X \setminus A) \in F$.*

Proof. Consider $S = \{B \cap A \colon B \in F\}$. Clearly S is closed under finite intersections, since F is. If $\emptyset \notin S$, then the ultrafilter F' generated by S is a filter containing A, by Exercise 12.2. If $\emptyset \in S$, then there is some $B \in F$ such that $B \cap A = \emptyset$, so $B \subseteq (X \setminus A)$, so $(X \setminus A) \in F$. □

Lemma 12.5. *A filter F on X is a maximal filter iff it is an ultrafilter.*

Proof. $\Rightarrow$: Suppose F is a maximal filter, and $A \subseteq X$. We cannot have $X \setminus A$ and A both in F, since then $\emptyset$ would be in F since F is closed under finite intersections. If neither $X \setminus A$ nor A are in F, then F is not maximal by Lemma 12.4.

$\Leftarrow$: Suppose F is an ultrafilter, $F' \supseteq F$ is a filter, and $A \in F' \supsetneq F$. Then since $A \notin F$, we must have $X \setminus A \in F$, so both $X \setminus A$ and A are in F'. Contradiction. □

A very useful theorem of Tarski is that every filter can be extended to an ultrafilter:

Theorem 12.6 (Tarski, ZFC)**.** *If F is a filter on X, then F can be extended to an ultrafilter. That is, there is an ultrafilter $F' \supseteq F$.*

Proof. Suppose $\mathcal{F}$ is a chain of filters. That is, for every $F, F' \in \mathcal{F}$, either $F \subseteq F'$ or $F' \subseteq F$. Then it is easy to check that $\bigcup \mathcal{F}$ is a filter on X. Hence, by the Hausdorff maximality principle applied to the set of all filters F' on X so that $F' \supseteq F$, there is maximal filter F' on X so that $F' \supseteq F$. This maximal filter is an ultrafilter by Lemma 12.5. □

One trivial example of an ultrafilter is a principal ultrafilter.

Definition 12.7. *Say that an filter F on X is **principal** if there is some $A \subseteq X$ such that $B \in F$ iff $A \subseteq B$.*

If X is a set and $x \in X$, then $\{A \subseteq X \colon \{x\} \subseteq A\}$ is a principal ultrafilter on X. It is consistent with ZF that every ultrafilter is principal [B]. However, in ZFC, there are many nonprincipal ultrafilters. For example, let F be the filter of cofinite subsets of ω. Then by Theorem 12.6, there is an ultrafilter F' extending F which cannot be principal, since for every $n \in \omega$, $\omega \setminus \{n\} \in F$, so $\omega \setminus \{n\} \in F'$ and $\{n\} \notin F'$.

Ultrafilters have many uses beyond the scope of this course. For example, given any topological space X, there is a "largest" compact Hausdorff space βX

containing X, which is largest in the sense that any map from X to a compact Hausdorff space K factors through X. This universal βX is called the Stone-Čech compactification of X. βX is just the space of all ultrafilters on X with an appropriate topology, and each $x \in X$ is mapped to the principal ultrafilter containing x. We give some more examples of applications:

Exercise 12.8 (The Stone representation theorem)**.** *Suppose B is a boolean algebra (a structure having constants $\top$ and $\bot$, binary function $\wedge$, $\vee$ and a unary function $\neg$, and satisfying the same theory as the standard two-element boolean algebra $\{True, False\}$). Then B is isomorphic to a field of sets. That is, there is a set X and a bijection $f\colon B \to \mathcal{P}(X)$ such that*

- $f(a \wedge b) = f(a) \cap f(b)$.
- $f(a \vee b) = f(a) \cup f(b)$.
- $f(\neg a) = X \setminus f(a)$.
- $f(\bot) = \emptyset$.

[Hint: Define a filter on B to be a subset F of B such that $\top \in B$, $\bot \notin B$, if $a, b \in F$, then $a \wedge b \in F$, and if $a \in B$ and $a \to b$, then $b \in F$ (where $a \to b$ iff $\neg a \vee b = \top$). Define an ultrafilter on B to be a maximal filter. Let $f(a) = \{U\colon U$ is an ultrafilter on B and $a \in U\}$.]

One use of ultrafilters is in taking ultralimits. First, we define limits with respect to a filter:

Definition 12.9. *Suppose F is a filter on ω. If $\langle a_n\colon n \in \omega\rangle$ is a sequence of real numbers, we say that $\lim_F a_n = x$ iff for all $\epsilon > 0$, $\{n\colon |a_n - x| < \epsilon\} \in F$.*

For example, if F is the filter of cofinite subsets of ω, then $\lim_F$ is the usual limit. More generally, for any filter F, $\lim_F$ satisfies all the usual limit laws.

Exercise 12.10. *Suppose F is a filter on ω.*

1. *Show that for every sequence of real numbers $\langle a_n\colon n \in \omega\rangle$, $\lim_F a_n$ has at most one value.*
2. *Show that if $\lim_F a_n$ and $\lim_F b_n$ exist, then $\lim_F a_n + \lim_F b_n = \lim_F (a_n + b_n)$.*
3. *Show that if $\lim_F a_n$ exists and c is a constant, then $\lim_F c a_n = c \lim_F a_n$.*
4. *Let U be an ultrafilter on ω. Suppose $\langle a_n\colon n \in \omega\rangle$ is a bounded sequence of real numbers. Show that $\lim_U a_n$ exists.*

Note that in Definition 12.9 we could have more generally taken a filter F on any index set I and then defined the filter limit $\lim_F a_i$ of sequences $\langle a_i\colon i \in I\rangle$ indexed by I. In this case all the parts of would still be true.

One way of using ultralimits is to define a finitely additive measure on all subsets of $\mathbb{N}$:

Exercise 12.11. *Suppose U is a nonprincipal ultrafilter on $\mathbb{N}$. Define $\mu\colon \mathcal{P}(\mathbb{N}) \to [0,1]$ by $\mu(A) = \lim_U \frac{|A\cap n|}{|n|}$. Show that μ is a finitely additive, μ is defined on all subsets of $\mathbb{N}$, $\mu(\mathbb{N}) = 1$, and $\mu(\{n\}) = 0$ for all $n \in \omega$.*

An important problem in the history of set theory was whether one could similarly find a countably additive measure on all subsets of $[0,1]$.

12.1 Measurable cardinals*

A cardinal κ is called **measurable** if there is a κ-complete ultrafilter on κ. The name measurable cardinal comes from the connection of this concept with the *measure problem* studied by Lebesgue, Banach, Ulam, Tarski, and others. In the wake of the construction of Vitali sets, set theorists became interested in the problem of whether there is *any* **measure** on $[0,1]$, a function $\mu\colon \mathcal{P}([0,1]) \to [0,1]$ such that:

- $\mu([0,1]) = 1$,
- $\mu(\{x\}) = 0$ for every $x \in [0,1]$,
- μ is countably additive: if $\langle A_n : n \in \omega\rangle$ are disjoint, then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$.

Since we are dropping here any geometric requirements like translation invariance (which is ruled out by the existence of Vitali sets), we may as well replace $[0,1]$ with any set X and ask if there is a measure μ on X (a countably additive function $\mu\colon \mathcal{P}(X) \to [0,1]$ so that $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for every $x \in X$). This clearly only depends on the cardinality of X.

It is easy to prove that if κ is the least cardinal such that there is a countably additive measure on κ, then every measure on κ is much more than countably additive, it is κ-additive, and a cardinal κ such that there is a κ-additive measure on κ is called a **real-valued measurable cardinal**. If μ is a κ-additive measure on the smallest cardinal κ admitting a measure, then it is easy to see that $\mu(A) = 0$ if $|A| < \kappa$, and hence by κ-additivity, κ is regular. Ulam proved in 1930 (using a technique now called an Ulam matrix) that if κ is real-valued measurable, then κ must be a limit cardinal and hence is weakly inaccessible. Thus, the existence of real-valued measurable cardinals cannot be proved in ZFC and is a large cardinal property.

Now if there exists a κ-complete ultrafilter U on κ, then U gives a κ-additive measure on κ by letting $\mu(A) = \begin{cases} 1 & \text{if } A \in U \\ 0 & \text{if } A \notin U \end{cases}$. So a measurable cardinal (as we've defined it above) is also real-valued measurable. A completely different kind of measure μ on κ is an **atomless** measure, where for every $A \subseteq \kappa$ with $\mu(A) > 0$, there is some $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$.

Ulam proved the following dichotomy:

Theorem 12.12 (Ulam)**.** *If κ is a real-valued measurable cardinal, then either*

- *$\kappa \leq 2^{\aleph_0}$, there is an atomless measure μ on κ, and there is also is a measure on the full powerset of $\mathcal{P}([0,1])$ extending Lebesgue measure.*

- *κ is measurable (i.e. there is a κ-complete ultrafilter on κ), $\kappa > 2^{\aleph_0}$ (in fact κ is a strong limit), and every measure μ on κ has an atom which yields a κ-additive κ-complete ultrafilter on κ.*

Hence, there are really two completely different kinds of real-valued measurable cardinals. It is the latter type, measurable cardinals, which have become by far more important.

One source of the utility of measurable cardinals is the ultrapower construction. If κ is a measurable cardinal, then let U be a κ-complete ultrafilter on κ. Then we can take the universe V, and take its ultrapower by this ultrafilter to get an inner model M of V, and an elementary embedding $j\colon V \to M$. These types of elementary embedding from the universe into inner models are fundamental tools in the study of large cardinals.

13 Ultraproducts

Ultraproducts were introduced independently in the 1950s in both logic and operator algebras. Our goal in this section is to describe the ultraproduct construction which has a myriad of applications in algebra, analysis, combinatorics, as well as in model theory and set theory.

There are many examples where we would like to take a product of structures $\langle M_i \colon i \in I\rangle$, but mod out by phenomena that occur on only a small (e.g. finite) subset of the structures. The ultraproduct construction gives a precise way of doing this. The utility of using ultrafilters is that it ensures convergence in a very general setting when we take limits. It is also key to getting the logical properties that we would like. For example, negation will work the way we want because if U is an ultrafilter on I then for each formula φ, $\{i \colon M_i \vDash \varphi\} \in U$ iff $\{i \colon M_i \vDash \neg\varphi\} \notin U$ by the ultrafilter property of U.

Definition 13.1 (Ultraproducts)**.** *Suppose $\mathcal{L}$ is a language, $\langle M_i \colon i \in I\rangle$ are $\mathcal{L}$ structures, and U is an ultrafilter on I. Let*

$$X = \prod_{i \in I} M_i$$

So X is the set of functions with domain I so that $f(i) \in M_i$ for all $i \in I$. Let $\sim$ be the equivalence relation on X where $f \sim g$ if $\{i \colon f(i) = g(i)\} \in U$ (the fact that this is an equivalence relation uses that U is a filter). Now let $\prod_U M_i$ be the structure whose universe is $X/\sim$ and where we interpret the constants, relations, and functions of $\mathcal{L}$ as follows:

- *For each constant symbol c of $\mathcal{L}$ we let $c^{\prod_U M_i} = [i \mapsto c^{M_i}]_\sim$*
- *For each relation symbol R of $\mathcal{L}$, we let $R^{\prod_U M_i}([f_1]_\sim, \dots [f_n]_\sim)$ be true if $\{i \colon R^{M_i}(f_1(i), \dots, f_n(i))\} \in U$, and false otherwise.*
- *For each function symbol g of $\mathcal{L}$, we let $g^{\prod_U M_i}([f_1]_\sim, \dots [f_n]_\sim) = [i \mapsto g^{M_i}(f_1(i), \dots, f_n(i))]_\sim$.*

That fact that the function $g^{\prod_U M_i}$ is well defined uses the fact that U is a filter. If $f_1 \sim f_1', \dots f_n \sim f_n'$, then $\{i \colon g^{M_i}(f_1(i), \dots, f_n(i)) = g^{M_i}(f_1'(i), \dots, f_n'(i))\} \in U$ since U is closed under finite intersections.

The reason we require U to be an ultrafilter is to obtain the following theorem:

Theorem 13.2 (Løs' theorem)**.** *Suppose $\langle M_i \colon i \in I\rangle$ are $\mathcal{L}$-structures, U is an ultrafilter on I, and φ is an $\mathcal{L}$-sentence. Then $\prod_U M_i \vDash \varphi$ iff $\{i \colon M_i \vDash \varphi\} \in U$.*

Proof. Suppose $\varphi(x_1, \dots, x_n)$ is a formula with free variables $x_1, \dots, x_n$. We prove by induction on formula complexity that

$$(\forall f_1, \dots, f_n \in X/\sim) \prod_U M_i \vDash \varphi(f_1, \dots, f_n) \leftrightarrow \{i \colon M_i \vDash \varphi(f_1(i), \dots, f_n(i)\} \in U. \tag{*}$$

This is true for atomic formulas by definition of $\sim$ and our definition of the functions and relations in the ultraproduct. Now we add logical connectives and quantifiers:

$\neg$: Suppose we've proven (*) for the formula φ. We would like to prove (*) for the formula $\neg\varphi$. Then

$$\prod_U M_i \vDash \neg\varphi(f_1,\ldots,f_n) \leftrightarrow \neg \prod_U M_i \vDash \varphi(f_1,\ldots,f_n)$$
$$\leftrightarrow \{i\colon M_i \vDash \varphi(f_1(i),\ldots,f_n(i))\} \notin U \leftrightarrow \{i\colon M_i \vDash \neg\varphi(f_1(i),\ldots,f_n(i))\} \in U$$

where the second-last step is by (*) for φ, and the last step is since U is an ultrafilter, so $A \notin U$ iff $I \setminus A \in U$.

$\wedge$: Suppose (*) holds for the formulas φ and ψ. We would like to show it holds for $\varphi \wedge \psi$. Then

$$\prod_U M_i \vDash \varphi(f_1,\ldots,f_n) \wedge \psi(f_1,\ldots,f_n)$$
$$\leftrightarrow (\prod_U M_i \vDash \varphi(f_1,\ldots,f_n)) \wedge (\prod_U M_i \vDash \psi(f_1,\ldots,f_n))$$
$$\leftrightarrow \{i\colon M_i \vDash \varphi(f_1(i),\ldots,f_n(i))\} \in U \wedge \{i\colon M_i \vDash \psi(f_1(i),\ldots,f_n(i))\} \in U$$
$$\leftrightarrow \{i\colon M_i \vDash \varphi(f_1(i),\ldots,f_n(i)) \wedge \psi(f_1(i),\ldots,f_n(i))\} \in U$$

where the second-last step is by (*) holding for φ and ψ and the last step is since for any filter U, $A \cap B \in U$ iff $A \in U$ and $B \in U$.

$\exists$: Suppose we've already shown that for all $f \in X/\sim$, $\prod_U M_i \vDash \varphi(f) \leftrightarrow \{i\colon M_i \vDash \varphi(f(i))\} \in U$, and now we want to show this for the formula $\exists x\varphi(x)$. Then

$$\prod_U M_i \vDash \exists x\varphi(x) \leftrightarrow \exists f \prod_U M_i \vDash \varphi(f)$$
$$\leftrightarrow (\exists f\{i\colon M_i \vDash \varphi(f(i))\} \in U) \leftrightarrow \{i\colon M_i \vDash \exists x\varphi(x)\} \in U$$

(where the last step uses the axiom of choice). □

An easy corollary of Løs's theorem is the compactness theorem of first order logic.

Corollary 13.3. *Suppose $\mathcal{L}$ is a language, and T is a set of sentences in $\mathcal{L}$ such that for every finite subset of T, there is a model M of T. Then there is a model of T.*

Proof. We may assume T is infinite. Let $[T]^{<\infty}$ be the set of finite subsets of T. Let F be the filter on $[T]^{<\infty}$ generated by the sets $\{\{R \in [T]^{<\infty}\colon S \subseteq R\}\colon S \in [T]^{<\infty}\}$. Let U be an ultrafilter extending F.

For each finite $S \subseteq T$, let M_S be a structure such that $M_S \vDash S$, and let $\mathcal{M}$ be the ultraproduct of the M_S with respect to the ultrafilter U. Since for each $S \in [T]^{<\infty}$ we have that $\{R\colon S \subseteq R\} \in U$, we have $\mathcal{M} \vDash S$ by Løs's theorem. □

Another important type of ultraproduct is an **ultrapower**

Definition 13.4. *If M is a structure, and U is an ultrafilter on a set I, the ultrapower of M by U is the structure $\prod_U M$.*

Ultrapowers of structures are very nontrivial and interesting objects that are quite different than the original structure (even though they will be elementarily equivalent to it). For example, phenomena that happen "mod ϵ" for arbitrarily small ϵ inside M will happen exactly inside the ultraproduct $\prod_U M$.

By Lós's theorem, an ultrapower $\prod_U M$ is elementarily equivalent to M. Hence, if M and N are structures and $\prod_U M$ and $\prod_V N$ are isomorphic, then clearly M and N are elementarily equivalent. A beautiful theorem of Kiesler and Shelah is the the converse is true:

Theorem 13.5 (Kiesler-Shelah)**.** *Two structures M, N in a language $\mathcal{L}$ are elementarily equivalent if and only if there are ultrafilters U and V so that $\prod_U M$ and $\prod_V N$ are isomorphic.*

Kiesler first proved this theorem assuming the GCH, and then Shelah showed that the theorem is true in all models of ZFC. We give the special case of Keisler's theorem for countable structures as an exercise. Recall a structure M is κ-saturated if every 1-type over $A \subseteq M$ with $|A| < \kappa$ is realized.

Exercise 13.6. *Fix a countable language $\mathcal{L}$ and a nonprincipal ultrafilter U on ω.*

1. *Show that if M_1, and M_2 are elementarily equivalent κ-saturated $\mathcal{L}$-structures of cardinality κ, then M_1 and M_2 are isomorphic.*

2. *Show that no countably infinite $\mathcal{L}$-structure is ω_1-saturated.*

3. *Show that if $\langle M_i \colon i \in \omega \rangle$ are countable $\mathcal{L}$-structures then their ultraproduct $\prod_U M_i$ is ω_1-saturated.*

4. *Show that if $\langle M_i \colon i \in \omega \rangle$ are countable $\mathcal{L}$-structures, then either $\prod_U M_i$ is finite or uncountable.*

5. *Assume CH is true. Show that if and M_1, M_2 are countable elementarily equivalent $\mathcal{L}$-structures, then their ultrapowers $\prod_U M_1$ and $\prod_U M_2$ are isomorphic.*

13.1 Ultraproducts of metric spaces and asymptotic cones*

Some very nice examples of ultraproducts come from metric spaces. Unfortunately, metric spaces are not such natural first-order (discrete) structures in the sense of model theory; they exist more naturally in the model theory of metric structures. If we were determined, we could make a metric space a first-order structure by having countably many relations $D_q(x, y)$ expressing that the distance between x and y is less than q for each rational number q, and then we could recover the original metric d by $d(x,y) = \inf_c \neg D_c(x,y)$. (So we are essentially using Dedekind cuts to represent real numbers). Instead of doing the above (which is ugly), we'll make a special definition of what an ultraproduct of pointed metric spaces is.

Definition 13.7. *Suppose $\langle (X_n, x_n, d_n,) : n \in \omega\rangle$ are triples where each pair (X_n, d_n) is a metric space (so X_n is a set, $d_n : X_n^2 \to [0,\infty)$ is a metric on X_n), and $x_n \in X_n$ is a point in X_n which we call a "base point". If U is an ultrafilter on ω, we define the ultraproduct $\prod_U (X, x_n, d_n)$ as follows. Let X be the set of sequences $\langle a_n : n \in \omega\rangle$ where $a_n \in X_n$, and the sequence $\langle d(x_n, a_n) : n \in \omega\rangle$ is bounded. Let $\sim$ be the equivalence relation on X where $\langle a_n : n \in \omega\rangle \sim \langle b_n : n \in \omega\rangle$ if $\lim_U d_n(a_n, b_n) = 0$, and let d be the metric on $X/\sim$ where $d([\langle a_n : n \in \omega\rangle]_\sim, [\langle b_n : n \in \omega\rangle]_\sim) = \lim_U d_n(a_n, b_n)$. We define the ultraproduct $\prod_U (X, x_n, d_n)$ to be the triple $(X, [n \mapsto x_n], d)$.*

To be clear, we are not using Definition 13.1 here because we are not viewing metric spaces as a first order structures.

An important example of this type of ultraproduct of metric spaces is the following:

Definition 13.8. *If (X, d) is a metric space with a given basepoint p, the **asymptotic cone** of (X, p, d) is equal to $\prod_U (X, p, d/n)$.*

So we take the product of countably many copies of the same set and base point X and p, but we keep "zooming out" as n increases by dividing the metric d by n. Intuitively, the asymptotic cone is a way of roughly viewing (X, d) from "infinitely far away". For example,

Exercise 13.9. *The asymptotic cone of $(\mathbb{Z}, 0, d_{\mathbb{Z}})$ where $d_{\mathbb{Z}}$ is the usual metric on $\mathbb{Z}$ is isomorphic to $(\mathbb{R}, 0, d_{\mathbb{R}})$ where $d_{\mathbb{R}}$ is the usual metric on $\mathbb{R}$. [Hint: show that the map $[\langle a_n : n \in \omega\rangle]_\sim \mapsto\mapsto \lim_U a_n/n$ is a bijection between the asymptotic cone and $(\mathbb{R}, d_{\mathbb{R}})$.]*

Asymptotic cones are very useful in fields like geometric group theory. For instance, we can view a finitely generated group as a discrete metric space (using the word metric metric). After taking an asymptotic cone of this metric space along with the group structure coming from the ultraproduct, we get a larger group which is an important tool for studying it. For example, if our original group is nilpotent, its asymptotic cone will be a Lie group. See e.g. [DK] for an introduction to asymptotic cones, and an application of them for proving Gromov's theorem on polynomial growth.

13.2 The Ax-Grothendieck theorem*

A nice application of ultraproducts is the Ax-Grothendieck theorem. A simple case of the theorem says that if p is a polynomial on $\mathbb{C}$, and p is injective, then it is surjective. Lets give a short proof of this using ultraproducts and the completeness of the theory of algebraically closed fields of characteristic 0.

For each n, we can write a formula φ_n in the language of fields that says that for all polynomials p of degree n, if p is injective it is surjective. Now each finite field $\mathsf{GF}(k)$ of order k satisfies every sentence φ_n since any injective map on a finite set is surjective. Now if n divides m and p is prime, then $\mathsf{GF}(p^n)$ is a subfield of $\mathsf{GF}(p^m)$, and it is not hard to see that for each prime p, $\bigcup_m \mathsf{GF}(p^{m!})$ is the algebraic closure of $\mathsf{GF}(p)$. Since the sentences φ_n are $\forall\exists$ sentences and $\mathsf{GF}(p^{m!})$ satisfies φ_n for all n, it is easy to see that $\bigcup_m \mathsf{GF}(p^{m!})$ satisfies φ_n for all n.

Now consider the ultraproduct of these algebraically closed fields of characteristic p. By Løs's theorem, this ultraproduct is an algebraically closed field, it has characteristic 0, and it satisfies φ_n for each n. Hence, by completeness of the theory ACF_0 every algebraically closed field of characteristic 0 (including $\mathbb{C}$) has the property that every injective polynomial is surjective.

The last step of this proof is a more generally a type of Lefschetz principle for algebraically closed fields. If a first order formula in the language of fields is true for a sequence of algebraically closed fields of arbitrarily large characteristic, it is true for all algebraically closed fields of characteristic 0. Just take their ultraproduct to see this.

14 Clubs and stationary sets

In set theory, the club filter and stationary sets are perhaps the most important largeness notions for subsets of cardinals κ. They have myriad uses in set theory.

Definition 14.1. *Suppose λ is a limit ordinal and $C \subseteq \lambda$. Then a set $C \subseteq \lambda$ is **closed** in λ if for all limit $\nu < \lambda$, if $C \cap \nu$ is cofinal in ν, then $\nu \in C$. We say C is **unbounded** in λ if for all $\alpha \in \lambda$ there exists $\beta \in C$ so that $\beta \nleq \alpha$. We call a closed unbounded subset of λ a **club set** in λ.*

Alternately, $C \subseteq \lambda$ is closed if it contains the suprema of increasing sequences in C.

Exercise 14.2. *Show that $C \subseteq \lambda$ is closed if and only if for every increasing sequence $\langle \alpha_\xi : \xi < \beta \rangle$ with $\alpha_\xi \in C$ and $\sup_\xi \alpha_\xi < \lambda$, we have $\sup_\xi \alpha_\xi \in C$.*

For example, let $C \subseteq \omega_1$ be the set of limit ordinals in ω_1. Then C is closed, and any C' with $C \subseteq C' \subseteq \omega_1$ is closed.

There is a more topological way of understanding closed subsets of ordinals. If λ is an ordinal, consider the **order topology** on λ. That is, the topology where a subbasis consisting of all rays $\{\gamma \in \lambda : \beta < \gamma\}$ and $\{\gamma \in \lambda : \gamma < \delta\}$ for every $\beta, \delta < \lambda$. So a basis for this topology is all the above rays together with the open intervals $(\beta, \delta) = \{\gamma : \beta < \gamma < \delta\}$.

Exercise 14.3. *Show that a set $C \subseteq \lambda$ is closed in the sense of Definition 14.1 if and only if it is closed in the order topology.*

Unfortunately tools from topology are not so useful for understanding ordinals and club sets, and so we won't take this topological viewpoint. Instead interactions between set theory and topology (that is, what is called the field of set-theoretic topology) mostly focus on using set theoretic techniques to construct interesting topological spaces, and prove independence results in topology.

Exercise 14.4. *Show that ω_1 equipped with the order topology is sequentially compact, but not compact.*

We'll begin by showing that an intersection of clubs is a club. Hence, we will be able to use the club sets to generate a filter.

Lemma 14.5. *Suppose λ is a limit ordinal with $\mathrm{cf}(\lambda) > \omega$, and $C, C' \subseteq \lambda$ are clubs in λ. Then $C \cap C'$ is a club.*

Proof. It is clear that $C \cap C'$ is closed. To see that $C \cap C'$ is unbounded, suppose $\beta \in \lambda$. Let $\alpha_0 \in C$ be such that $\alpha_0 > \beta$. Then let α_{n+1} be the least element of C' that is greater than α_n, and α_{n+2} be the least element of C greater than α_{n+1}. These ordinals exist since C, C' are unbounded. Then $\sup\{\alpha_{2n} : n \in \omega\} = \sup\{\alpha_{2n+1} : n \in \omega\}$ is in $C \cap C'$ $\square$

Indeed, a stronger version of this lemma is true.

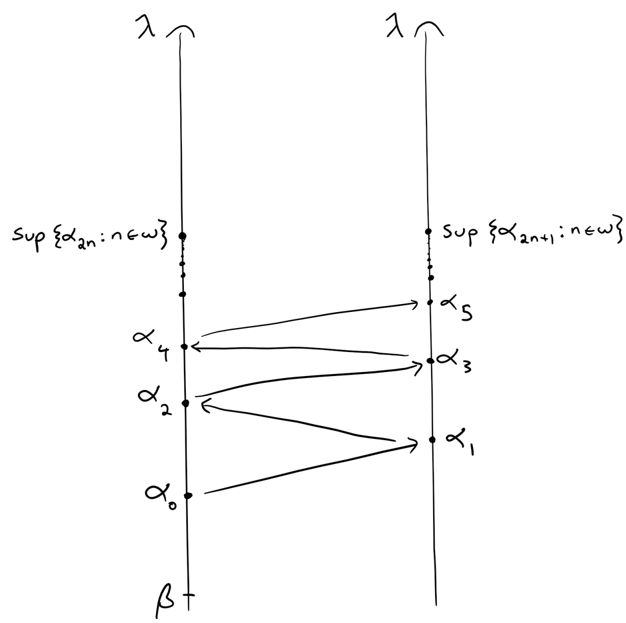

Figure 7: The proof that the intersection of two clubs is unbounded.

Lemma 14.6. *If λ is a cardinal with $\mathrm{cf}(\lambda) > \omega$, and $\langle C_\alpha \colon \alpha < \beta \rangle$ is a sequence of clubs of length $\beta < \mathrm{cf}(\lambda)$, then $\bigcap_{\alpha<\beta} C_\alpha$ is a club.*

Proof. It is clear that $\bigcap_{\alpha<\beta} C_\alpha$ is closed. To show it is unbounded, similarly to Lemma 14.5, we can make an increasing sequence of length $\beta \cdot \omega$ where the (α, n)th element is an ordinal in C_α for every n, and hence the sup of this sequence is in $\bigcap_{\alpha<\beta} C_\alpha$. □

This lemma is best possible:

Exercise 14.7. *If $\mathrm{cf}(\lambda) = \omega$, there exist disjoint clubs in λ. If $\mathrm{cf}(\lambda) > \omega$, then there is a sequence of $\mathrm{cf}(\lambda)$ many clubs whose intersection is empty.*

Since the intersection of two clubs is a club, the clubs generate a filter. We will be most interested in this filter on regular cardinals:

Definition 14.8 (The club filter)**.** *The filter F generated by the club sets on a regular cardinal κ is called the* ***club filter*** *on κ. That is, $C \subseteq \kappa$ is in the club filter on κ if and only if C contains a club set in κ. Note that the club filter on κ is κ-complete by Lemma 14.6.*

Caution: an element of the club filter is not a club. It is a set that contains a club.

One common source of clubs is that they are the closure points of operations on ordinals:

Exercise 14.9. *Suppose $\kappa > \omega$ is a regular cardinal, $\lambda < \kappa$ and $\langle f_\alpha \colon \alpha < \lambda \rangle$ are functions where $f_\alpha \colon \kappa \to \kappa$. Let $C = \{\gamma \colon (\forall \alpha < \lambda)(\forall \beta < \gamma) f_\alpha(\beta) < \gamma\}$ be the set of ordinals γ so that γ is closed under all the functions f_α. Show that C is club in κ.*

If we think of an element of the club filter as being a "large set", then a "not small" set is a set not in the dual ideal (i.e. an I-positive set wrt the dual ideal I). That is, a set S is "not small" if its complement does not contain a club, which is true iff S intersects every club.

Definition 14.10. *If $\kappa > \omega$ is a regular cardinal, then $S \subseteq \kappa$ is **stationary** if S intersects every club in κ.*

The dual ideal to the club filter is called the **nonstationary ideal**.

Exercise 14.11. *Suppose $\kappa > \omega$ is a regular cardinal.*

1. *Show that if $C \subseteq \kappa$ is a club and $S \subseteq \kappa$ is stationary, then $C \cap S$ is stationary.*
2. *Suppose $S \subseteq \kappa$ is a stationary. Then show S is unbounded.*
3. *Suppose $S \subseteq \kappa$ is stationary. Then show $\{\lambda \in S \colon \lambda$ is a limit ordinal$\}$ is stationary.*

Exercise 14.12. *Suppose $\kappa > \omega$ is a regular cardinal. Show that if $S \subseteq \kappa$ is stationary, and we partition S into $\lambda < \kappa$ many sets $\langle S_\alpha \colon \alpha < \lambda \rangle$ where the S_α are pairwise disjoint and $\bigcup_{\alpha<\lambda} S_\alpha = S$, then there is some $\alpha < \lambda$ such that S_α is stationary. [Hint: a union of fewer than κ many nonstationary sets is nonstationary since an intersection of fewer than κ many clubs is a club.]*

Our next goal is to prove Fodor's lemma. To do this, we'll first discussed a more refined type of intersection which is very useful when dealing with club sets.

Definition 14.13 (Diagonal intersection)**.** *Let $\langle X_\alpha \colon \alpha < \kappa \rangle$ be a sequence of κ many subsets of κ. Their diagonal intersection is defined to be*

$$\triangle_{\alpha<\kappa} X_\alpha = \{\beta < \kappa \colon \beta \in \bigcap_{\alpha<\beta} X_\alpha\}.$$

Theorem 14.14. *Suppose κ is a regular cardinal. Then the diagonal intersection of κ many clubs in κ is club in κ.*

Proof. Suppose $\langle C_\alpha \colon \alpha < \kappa \rangle$ is a sequence of clubs. We may assume that the C_α are decreasing under $\supseteq$; letting $C'_\alpha = \bigcap_{\beta \leq \alpha} C_\alpha$ it is easy to see that $\triangle_{\alpha<\kappa} C_\alpha = \triangle_{\alpha<\kappa} C'_\alpha$. The sets C'_α are still club since the club filter is κ-complete. Let $C = \triangle_{\alpha<\kappa} C_\alpha$.

First we show that C is closed. Suppose $\nu \in C$ is such that $C \cap \nu$ is cofinal in ν. Fix $\alpha < \nu$. We want to show $\nu \in C_\alpha$. This is true since for all $\beta \in C \cap \nu$

such that $\beta > \alpha$, we have $\beta \in \bigcap_{\xi<\beta} C_\xi$, hence $\beta \in C_\alpha$. Thus, $\nu \in C_\alpha$ since C_α is closed, and C_α contains an increasing sequence whose limit is ν. So $\nu \in \bigcap_{\alpha<\nu} C_\alpha$, and $\nu \in C$.

Next we show that C is unbounded. Let $\beta_0 \in C_0$ be an arbitrarily large ordinal. Now let $\beta_{n+1} \in C_{\beta_n}$ be the least element of C_{β_n} larger than β_n. We claim $\beta = \sup\{\beta_n : n \in \omega\}$ is in C. This is because for all $\alpha < \beta$, there is some n so that $\alpha \leq \beta_n$, and hence $\beta_{m+1} \in C_\alpha$ for all $m \geq n$ since $\beta_{m+1} \in C_{\beta_m} \subseteq C_\alpha$ for all $m \geq n$, since the C_α are decreasing. Thus, $\beta \in C_\alpha$ for all $\alpha < \beta$ and so $\beta \in C$. □

A filter on κ is called **normal** if it is closed under diagonal intersections. Hence, Theorem 14.14 states that the club filter is normal.

A very important fact about stationary sets is Fodor's lemma.

Lemma 14.15 (Fodor's lemma)**.** *Suppose $\kappa > \omega$ is a regular cardinal,k$S \subseteq \kappa$ is stationary and $f : S \to \kappa$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is some $\gamma \in \kappa$ and some stationary set $T \subseteq S$ such that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

Proof. For a contradiction, suppose that for each $\gamma < \kappa$, the set $\{\alpha \in S : f(\alpha) = \gamma\}$ is nonstationary. Hence, there is a club C_γ such that $f(\alpha) \neq \gamma$ for all $\alpha \in C_\gamma$. Then the diagonal intersection $\triangle_{\gamma<\kappa} C_\gamma$ is club, and hence $S \cap \triangle_{\gamma<\kappa} C_\gamma$ is nonempty. Let α be a nonzero element of $S \cap \triangle_{\gamma<\kappa} C_\gamma$. Then $f(\alpha) = \gamma$ for some $\gamma < \alpha$, so by our choice of C_γ, $\alpha \notin C_\gamma$. But then $\alpha \notin \bigcap_{\gamma<\alpha} C_\gamma$, contradicting the fact that α is in the diagonal intersection. □

Fodor's lemma for a filter F on κ is actually equivalent to normality of F.

Exercise 14.16. *Suppose F is a filter on κ. Then the following are equivalent:*

1. *F is normal (i.e. closed under diagonal intersections).*

2. *If $S \subseteq \kappa$ is F-positive and $f : \kappa \to \kappa$ is such that $f(\alpha) < \alpha$, then there is some $\gamma < \kappa$ and F-positive $T \subseteq S$ so that $f(\alpha) = \gamma$ for all $\alpha \in T$.*

In fact, every normal filter on κ must extend the club filter.

Exercise 14.17. *Suppose F is a nontrivial normal κ-complete filter on a regular cardinal κ. Then every club set is in F.*

15 Applications of Fodor: Δ-systems and Silver's theorem

Anytime in set theory we can make an interesting function f on ordinals such that $f(\alpha) < \alpha$, we can often gain a great deal of insight by applying Fodor's lemma.

We give a couple applications to illustrate this. Our first application is the Δ-system lemma. The Δ-system lemma is the key combinatorial principle behind Cohen's proof of the consistency of $\neg\mathsf{CH}$.

Definition 15.1. *Suppose X is a collection of sets, r is a set, and for all distinct $A, B \in X$, $A \cap B = r$. Then we call X a* ***Δ-system with root*** *r.*

For example, if the elements of X are pairwise disjoint, then X is a Δ-system with root $\emptyset$.

Lemma 15.2 (The Δ-system lemma)**.** *Suppose X is an uncountable set of finite sets. Then there are an uncountable subset $X' \subseteq X$ and a finite set r such that X' is a Δ-system with root r.*

Proof. We may assume $|X| = \omega_1$. Let $\langle X_\alpha \colon \alpha < \omega_1\rangle$ be an enumeration of the elements of X. Since each X_α is finite, $|\bigcup X| \leq \omega_1$, so by relabeling the elements of $\bigcup X$ with countable ordinals we may also assume that $\bigcup X \subseteq \omega_1$. That is, we may assume each X_α consists of finitely many ordinals less than ω_1.

We will use Fodor's lemma to begin refining our collection of X_α. Let $f \colon \omega_1 \to \omega_1$ be the function

$$f(\alpha) = \begin{cases} \sup(X_\alpha \cap \alpha) & \text{if } X_\alpha \cap \alpha \text{ is nonempty} \\ 0 & \text{otherwise.} \end{cases}$$

Since $X_\alpha \cap \alpha$ is a finite set of ordinals less than α, its sup is less than α if it is nonempty, and so $f(\alpha) < \alpha$ for $\alpha > 0$. Hence, by Fodor's lemma, there are some γ and some stationary set S such that for every $\alpha \in S$, $f(\alpha) = \gamma$.

There are only countably many finite sets of ordinals less than or equal to γ which could be possible values for $X_\alpha \cap \alpha$. Hence, if we partition S into countably many sets $\{\alpha \in S \colon X_\alpha \cap \alpha = r'\}$ for each such $r' \subseteq \gamma$, one of these sets is stationary by Exercise 14.12. Fix a finite r and stationary $S' \subseteq S$ so that $X_\alpha \cap \alpha = r$ for every $\alpha \in S'$.

Let $C = \{\alpha \colon (\forall \beta < \alpha) X_\beta \subseteq \alpha\}$. Then C is a club subset of ω_1 by Exercise 14.9 since it is the set of closure points of the function $\alpha \mapsto \sup \bigcup_{\beta<\alpha} X_\beta$.

We claim $\{X_\alpha \colon \alpha \in C \cap S' \wedge \alpha > \gamma\}$ is a Δ-system with root r. Consider X_β, X_α in this set where $\beta < \alpha$. Then $X_\beta \cap X_\alpha = X_\beta \cap X_\alpha \cap \alpha = r$ since $X_\beta \subseteq \alpha$ by definition of C. $\square$

Exercise 15.3. *Give a different proof of the Δ-system lemma that does not use Fodor's lemma. First, show it suffices to prove the Δ-system lemma in the case where every element of X has size n. Then prove the lemma by induction on n.*

More generally, we have the following Δ-system lemma for κ size collections of sets of size $< \lambda$ where $\kappa > \lambda$ are infinite regular cardinals:

Exercise 15.4. *Suppose $\kappa > \lambda$ are infinite regular cardinals. Assume that for all $\delta < \kappa$, $\delta^{<\lambda} < \kappa$. Let X be a collection of κ many sets of cardinality less than λ. Then there is a subset of X' of size κ so that X' forms a Δ-system.*

The possible values of the powerset function 2^κ on singular cardinals κ is a deep problem in set theory (see Section 11.2). One early theorem which gives limitations on its possible values is Silver's theorem from 1974 that GCH cannot fail first at a singular cardinal of uncountable cofinality. The essential ingredient in proving this theorem is stationary sets. Our proof below is due to Baumgartner and Prikry [BP]. We begin with a few exercises:

Exercise 15.5. *Suppose $<_P$ is a partial order on the set P. Then there is a linear order $<_L$ on P extending $<_P$. That is, for all $a, b \in P$, $a <_P b \to a <_L b$. [Hint: Use Zorn's lemma]*

Exercise 15.6. *Suppose $<_L$ is a linear order on a set L so that for each $a \in L$, $|\{b \in L \colon b < a\}| \leq \kappa$. Then $|L| \leq \kappa^+$. [Hint: Suppose that f is an injection from κ^+ to L. Show that $\operatorname{ran}(f)$ must be unbounded in L. Hence, $L = \bigcup_{y \in \operatorname{ran}(f)} \{b \in L \colon b < y\}$ is a union of κ^+ many sets of size κ.]*

Theorem 15.7. *Suppose κ is a singular cardinal of uncountable cofinality, and $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. Then $2^\kappa = \kappa^+$.*

Proof. To show that $2^\kappa = \kappa^+$ we'll put a linear order on 2^κ so that each point has at most κ predecessors, and then apply Exercise 15.6. First, however, we'll replace 2^κ by a set of the same cardinality which is easier to work with.

Let $\langle \mu_\alpha \colon \alpha < \operatorname{cf}(\kappa) \rangle$ be a strictly increasing sequence of length $\operatorname{cf}(\kappa)$ cofinal in κ such that if α is a limit ordinal, then $\mu_\alpha = \sup_{\beta < \alpha} \mu_\beta$. For each $\alpha < \kappa$, let $g_\alpha \colon \mathcal{P}(\mu_\alpha) \to \mu_\alpha^+$ be a bijection (which exists since $2^{\mu_\alpha} = \mu_\alpha^+$). For each $A \in \mathcal{P}(\kappa)$, let $f_A \colon \operatorname{cf}(\kappa) \to \kappa$ be the function $f_A(\alpha) = g_\alpha(A \cap \mu_\alpha)$. Note that $f_A(\alpha) < \mu_\alpha^+$ for every α. Let $\mathcal{F} = \{f_A \colon A \in \mathcal{P}(\kappa)\}$ so the function $A \mapsto f_A$ is a bijection from $\mathcal{P}(\kappa)$ to $\mathcal{F}$.

The point of "coding" elements $A \in \mathcal{P}(\kappa)$ in terms of these functions f_A, is that we only need to remember a small amount of information about f_A to recover what A is[17]. In particular, if $S \subseteq \operatorname{cf}(\kappa)$ is unbounded, then we can recover A just from the values of $f_A(\alpha)$ for $\alpha \in S$. This is since $A = \bigcup_{\alpha \in S} A \cap \mu_\alpha = \bigcup_{\alpha \in S} g_\alpha^{-1}(f_A(\alpha))$.

Claim. *For $A \in \mathcal{P}(\kappa)$, $|\{B \in \mathcal{P}(\kappa) \colon \{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ is stationary$\}| \leq \kappa$.*

Proof of Claim. Fix A. We'll prove this by using Fodor's lemma to show that if $\{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ is stationary, then f_B is determined by a small amount of information, and there are only κ many possibilities for what it is.

[17] These types of robust codings are commonly used in set theory and computability theory. For example, if $A \subseteq \omega$ then the function $f_A(n) = A \cap n$ has the property that we can recover A just from knowing infinitely many values of f_A. This is a trick used often in computability theory.

Now $f_A(\alpha) < \mu_\alpha^+$ for every $\alpha \in \mathrm{cf}(\kappa)$. Hence, for each $\alpha \in \mathrm{cf}(\kappa)$, we can choose some injection $h_\alpha \colon \{\beta \colon \beta < f_A(\alpha)\} \to \mu_\alpha$. So if $f_B(\alpha) < f_A(\alpha)$, then $h_\alpha(f_B(\alpha)) < \mu_\alpha$.

Now suppose $B \in \mathcal{P}(\kappa)$ is such that $\{\alpha \in \mathrm{cf}(\kappa) \colon f_B(\alpha) < f_A(\alpha)\}$ is stationary. Consider the following function f'_B with domain

$$\{\alpha \colon \alpha \text{ is a limit ordinal and } f_B(\alpha) < f_A(\alpha)\},$$

which is a stationary subset of $\mathrm{cf}(\kappa)$ by Exercise 14.11.3. Let $f'_B(\alpha)$ be the least β such that $h_\alpha(f_B(\alpha)) < \mu_\beta$. Then $h_\alpha(f_B(\alpha)) < \mu_\alpha$, and if α is a limit (so $\mu_\alpha = \sup_{\beta<\alpha} \mu_\beta$) we must have $f'_B(\alpha) < \alpha$. Hence, by Fodor's lemma, there is a stationary set S_B and an ordinal γ_B such that $f'_B(\alpha) = \gamma_B$ for all $\alpha \in S_B$.

By our discussion above, we can recover all the values of f_B from the function $f_B \restriction S_B$, which is determined by the values $h_\alpha \circ f_B \restriction S_B$, where $\mathrm{ran}(h_\alpha \circ f_B \restriction S_B) \subseteq \mu_{\gamma_B}$. Now there are $\mathrm{cf}(\kappa) < \kappa$ many choices of γ_B, there are at most $2^{\mathrm{cf}(\kappa)} = \mathrm{cf}(\kappa)^+ < \kappa$ different stationary subsets S_B of $\mathrm{cf}(\kappa)$, and letting $\lambda = \max(|\mu_{\gamma_B}|, \mathrm{cf}(\kappa)) < \kappa$ there are at most $\lambda^\lambda = 2^\lambda = \lambda^+ \leq \kappa$ many functions from S_B to μ_{γ_B} which could be equal to $h_\alpha \circ f_B \restriction S_B$. Hence, there are at most κ many B such that $\{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ is stationary. □ Claim.

Now consider the partial ordering on $\mathcal{F}$ where $f_B < f_A$ if $\{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ contains a club. This is a partial order since it is clearly irreflexive, and it is transitive since the intersection of two clubs is a club. So by Exercise 15.5, there is a linear order $<^*$ on $\mathcal{F}$ extending $<$. Now if $A \neq$ B, then if $\{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ is not stationary, then $\{\alpha \colon f_B(\alpha) > f_A(\alpha)\}$ is in the club filter (since the set $\{\alpha \colon f_A(\alpha) = f_B(\alpha)\}$ is bounded), so $f_A <^* f_B$. Taking the contrapositive, $f_B <^* f_A$ implies that $\{\alpha \colon f_B(\alpha) < f_A(\alpha)\}$ is stationary, and hence by the claim, $\{f_B \colon f_B <^* f_A\}$ has size $\leq \kappa$. Hence, by Exercise 15.6, $|\mathcal{F}| = \kappa^+$. □

Exercise 15.8. *Suppose κ is a singular limit cardinal and $2^\kappa = \kappa$. Suppose F is a set of functions from κ to κ such that:*

- *For all $f \in F$, $f(\alpha) < \alpha$ for all $\alpha > 0$. (The functions in F are regressive).*
- *For all $f, g \in F$, if $f \neq g$, then there exists $\beta < \kappa$ such that $f(\alpha) \neq g(\alpha)$ for all $\alpha > \beta$ (F is an eventually different family of functions).*

Show that $|F| \leq \kappa$.

We remark that this proof actually gives the following stronger result:

Theorem 15.9 (Silver, 1974)**.** *If κ is a singular cardinal of uncountable cofinality and $2^\lambda = \lambda^+$ for a stationary set of $\lambda < \kappa$, then $2^\kappa = \kappa^+$.*

16 Trees

A **tree** is a partial order $(<_T, T)$ so that for all $x \in T$, $\{y \colon y \leq_T x\}$ is wellordered by $<_T$. For each $x \in T$, $\mathrm{rank}_T(x)$ is therefore the ordertype of $\{y \colon y <_T x\}$. The α**th level of** T is defined to be $\{x \in T \colon \mathrm{rank}_T(x) = \alpha\}$. This is an example of an **antichain** in T, a set $A \subseteq T$ so that for all distinct $x, y \in A$, $x \not<_T y$ and $y \not<_T x$. A **chain** in T is a subset of T that is linearly ordered by $<_T$. A **branch** in T is a chain which is closed downwards. The **height** of the tree T is $\sup\{\mathrm{rank}_T(x) + 1 \colon x \in T\}$; the least ordinal greater than all the levels in T.

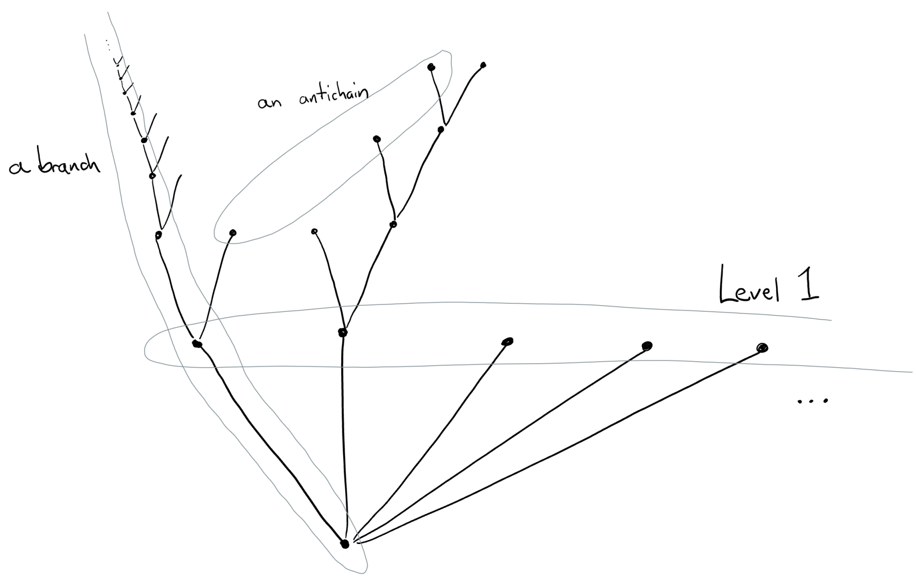

Figure 8: A tree

An example of a tree of height κ is the following. Let X be a set, and $X^{<\kappa}$ be the set of functions from ordinals less than κ to X. Then $X^{<\kappa}$ is a tree under the ordering $\subsetneq$.

Lemma 16.1 (König's lemma)**.** *Suppose T is a tree of height ω and every level of T is finite. Then T has an infinite branch.*

To prove König's lemma, we'll repeatedly use the "pigeonhole principle" that ω is regular: if X is an infinite set and we write it as a union of finitely many sets X_i, so $X = \bigcup_{i<n} X_i$, then some X_i is infinite.

Proof. We define an infinite branch $\{x_n \colon n \in \omega\}$ where x_n is at level n. Let x_0 be a node at level 0 such that $\{y \colon y >_T x_0\}$ is infinite (which exists by the above pigeonhole principle).

Inductively, let $x_{n+1} >_T x_n$ be an element of T at level $n+1$ where $\{y \in T \colon y >_T x_{n+1}\}$ is infinite. Such an x_{n+1} must exist by the pigeonhole principle. $\square$

König's lemma is a type of compactness phenomenon for ω. It has a precise relationship to topological compactness:

Exercise 16.2. *Suppose T is a tree of height ω and every level of T is finite. Let X be the set of infinite branches in T. For each $t \in T$ let $N_t = \{x \in X \colon t \in x\}$ be the set of all infinite branches that include t. Show that the topology on X generated by the basic open sets N_t is compact.*

The tree we considered in König's lemma is called an ω-tree. More generally, a tree T is a κ-tree iff the height of T is κ and every level has size less than κ. Our next theorem is that the analogue of König's lemma fails for ω_1.

Theorem 16.3 (Aronszajn)**.** *There is an ω_1-tree with no branches of order type ω_1.*

Proof. The tree T of all injections in $\omega^{<\omega_1}$ has height ω_1. However, it has no branches of length ω_1, since there is no injection from ω_1 to ω. We will construct a subtree of T whose levels are countable which has height ω_1.

Let $=^*$ be the equivalence relation of equality mod finite on $\omega^{<\omega_1}$. That is $s =^* t$ if $\mathrm{dom}(s) = \mathrm{dom}(t)$ and $\{\alpha \colon s(\alpha) \neq t(\alpha)\}$ is finite. Note that for every $s \in \omega^{<\omega_1}$ there are countably many $t \in \omega^{<\omega_1}$ so that $s =^* t$.

We construct a sequence $\langle s_\alpha \colon \alpha < \omega_1\rangle$ by transfinite recursion so that for every α,

1. $\omega \setminus \mathrm{ran}(s_\alpha)$ is infinite, and
2. $\beta < \alpha \to s_\alpha \restriction \beta =^* s_\beta$.

Here condition 2 is what we really want at the end, and condition 1 is just an extra hypothesis to ensure that our construction will work. Condition 1 makes sure that as we construct larger and larger s_α, there is enough empty space to inject larger ordinals into ω, while satisfying 2.

At successor ordinals, let $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$ for some $n \notin \mathrm{ran}(s_\alpha)$. For limit α, choose an increasing sequence $\langle \alpha_n \colon n \in \omega\rangle$ cofinal in α. To define s_α we first will inductively define a sequence t_n of partial functions from α_n to ω, total functions $t'_n \colon \alpha_n \to \omega$ so that $t'_n \supseteq t_n$, and finite sets $x_n \subseteq \omega$ where $|x_n| = n$ and x_n is disjoint from $\mathrm{ran}(t'_n)$. We'll ensure that $t'_n =^* s_{\alpha_n}$. Let $t_0 = t'_0 = s_{\alpha_0}$ and $x_0 = \emptyset$. Let

$$t_{n+1} = t'_n \cup s_{\alpha_{n+1}} \restriction \{\beta \colon \beta \geq \alpha_n \wedge s_{\alpha_{n+1}}(\beta) \notin (\mathrm{ran}(t'_n) \cup x_n)\}.$$

Since $t'_n =^* s_{\alpha_n} =^* s_{\alpha_{n+1}} \restriction \alpha_{n+1}$, we have that Note that $\mathrm{dom}(t_{n+1})$ is α_{n+1} minus a finite set, and t_{n+1} is an injection by definition. Let t'_{n+1} be any extension of t_{n+1} so $\mathrm{dom}(t'_{n+1}) = \alpha_{n+1}$ and $\mathrm{ran}(t'_{n+1})$ is disjoint from x_n. Finally, let $x_{n+1} = x_n \cup \inf \omega \setminus \mathrm{ran}(t'_{n+1})$. Now let $s_\alpha = \bigcup_n t_n$. Now s_α satisfies (1), since $\mathrm{ran}(s_\alpha)$ is disjoint from $\bigcup_n x_n$. Finally S_α satisfies (2) since each t'_n does.

Finally, let S be the set of injections t so that there exists $\alpha < \omega_1$ so $t =^* s_\alpha$. Then $t \upharpoonright \beta =^* s_\alpha \upharpoonright \beta =^* s_\beta$ by (2). Hence, $t \upharpoonright \beta \in S$. Hence, S is closed downwards in $\omega^{<\omega_1}$, and so levels in S agree with levels in $\omega^{<\omega_1}$. So S is a tree that has countable levels, and height ω_1. $\square$

The sequence of s_α we constructed in the above proof is a type of "coherent sequence", which are an important in many part of set theory.

In general, a cardinal κ has the **tree property** if every κ-tree has a branch of ordertype κ. So we have proved ω has the tree property and ω_1 does not. A κ-tree with no branch of ordertype κ (which is a counterexample to the tree property at κ) is called a κ-Aronszajn tree. So we have proved that an ω_1-Aronszajn tree exists.

Exercise 16.4. *Assuming* CH *show that there is an ω_2-tree with no branches of length ω_2. [Hint: use the fact that $\omega_1^\omega = 2^\omega$ to help control the number of countable subsets of ω_1.]*

It is a result of Mitchell that assuming large cardinals, it is consistent that ω_2 has the tree property.

16.1 Compactness and incompactness in set theory*

Compactness and incompactness phenomena in set theory are important topics of modern research. Compactness here refers to a much broader notion than just topological compactness. It refers to reflection-type principles which roughly state that if every "smaller subobject" of some object has a property, then the object has this property. For instance, we can rephrase a topological space X being compact as follows: X is compact iff for all collections $\mathcal{U}$ of open sets in X, if every finite subset of $\mathcal{U}$ does not cover X, then $\mathcal{U}$ does not cover X. So for topological compactness, the types of objects we are considering are open covers, and "smaller" means finite.

Just as there are many interesting both compact and non-compact topological spaces, there are many different examples of compactness and incompactness phenomena in set theory which come from considering different types of objects and notions of size. Myriad interesting open problems come from asking to what extent compactness phenomena can hold throughout the universe of sets, and especially when they can coexist with other incompactness phenomena.

We've already seen some incompactness that happens at the cardinal ω_1: there is an ω_1-Aronszajn tree. At the level of ω_1 there are many other interesting examples of incompactness. Much research is motivated by trying to understand to what extent types of incompactness that we find at ω_1 can hold at other cardinals.

We give some examples of compactness phenomena:

- The compactness theorem for first order logic: if every finite subset of a first-order theory is satisfiable, then the theory is satisfiable.

- Silver's theorem that if κ is a singular cardinal of uncountable cofinality and GCH holds below κ, then GCH holds at κ.

- König's lemma.

And some examples of incompactness phenomena:

- The failure of compactness for the infinitary logic $\mathcal{L}_{\omega_1,\omega}$: there are $\mathcal{L}_{\omega_1,\omega}$ theories all of whose finite subsets are consistent, but where the theory is not[18].

- Magidor's theorem that GCH can first fail at $\aleph_\omega$.

- ω_1-Aronszajn trees

- Kurosh monsters: uncountable groups all of whose proper subgroups are countable. These were first constructed by Shelah.

- The $\square$ principle which follows from $V = L$.

Many large cardinal axioms are related to compactness. For example, some simple large cardinal axioms are compactness principles which say that phenomena which happen in V must happen at a particular V_α. More sophisticated examples come from the type of reflection we get from elementary embeddings of the universe into inner models.

Large cardinals are also an important way of measuring the strength of other compactness principles. One important dividing line is whether a given compactness principle at κ implies that κ is strongly inaccessible. One example of a compactness principle for a cardinal κ is a higher analogue of Ramsey's theorem: does every function $f\colon [\kappa]^2 \to 2$ have a homogeneous set of size κ? If κ has this property, we say κ is **weakly compact**. Weakly compact cardinals are strongly inaccessible.

Even when is not the case that a compactness principle at κ implies that κ is strongly inaccessible, it is often true that compactness principles at κ imply that κ has large cardinal properties in canonical inner models. For example, if κ has the tree property, then κ is strongly inaccessible in L.

[18] Consider the language with countably many constant symbols $\langle c_n : n \in \omega\rangle$, the theory containing the sentences $c_0 \neq c_n$ for each $n \geq 1$ and the sentence $\forall x(\bigvee_{n\geq 1} x = c_n)$. Every finite subset of this theory is consistent and has a model, but the whole theory is inconsistent. (Note, however, that "higher" versions of compactness principles for $\mathcal{L}_{\omega_1,\omega}$ are consistent from large cardinal axioms.)

17 Suslin trees and $\diamondsuit$

The real numbers have a simple order-theoretic characterization. Recall that a linear order $<$ on X is **dense** if for all $a, b \in X$ with $a < b$, there exists $c \in X$ such that $a < c < b$. A subset $A \subseteq X$ is **dense in** X if for all $a < b$ in X there is a $c \in A$ such that $a < c < b$. A linear order $<$ on X is **complete** if any set with an upper bound has a least upper bound, and any set with a lower bound has a greatest lower bound. An **endpoint** of a linear order is an element that is either greater than every other element, or less than every other element.

Exercise 17.1.

1. *Show that any two countable dense linear orders without endpoints are isomorphic. [Hint: back-and-forth]*
2. *Show that $\mathbb{R}$ is order-isomorphic to any complete dense linear order without endpoints that has a countable dense subset.*

Note that if we drop the requirement of having a countable dense subset, then there are complete dense linear orders without endpoints that have arbitrarily large cardinality. This is because there are dense linear orders without endpoints of arbitrarily large cardinality (e.g. by Lowenheim-Skolem), and we can then take their Dedekind completions.

In 1920, Suslin asked whether we can weaken the hypothesis in Exercise 17.1.2 of having a countable dense subset to instead say that any disjoint collection of open intervals is countable Recall that an open interval is a set of the form $(a, b) = \{c : a < c < b\}$. In modern terminology, Suslin asked if there is a Suslin line:

Definition 17.2. *Say that a linear order $<$ on a set X is a **Suslin line** if $<$ is a complete dense linear order without endpoints such that every set of disjoint open intervals is countable, but $<$ has no countable dense set (hence it is not isomorphic to $\mathbb{R}$).*

There is a related type of object called a **Suslin tree**.

Definition 17.3. *A **Suslin tree** is a tree T of height ω_1 so that every chain and antichain in T is countable.*

So a Suslin tree is an ω_1-Aronszajn tree with the extra property that every antichain is countable.

Definition 17.4. *Say that a tree T is **Hausdorff** if for all limit ordinals λ, if $\operatorname{rank}_T(x) = \operatorname{rank}_T(y) = \lambda$ and $\{z : z <_T x\} = \{z : z <_T y\}$, then $x = y$. In particular, if a tree is Hausdorff, then T has a unique least element (called the root).*

Exercise 17.5. *Show that if there is a Suslin tree, then there is a Suslin tree T so that T is Hausdorff and if $x \in T$ is not maximal and x is on level α, then there is a countable infinite set of y at level $\alpha + 1$ such that $y > x$.*

Lemma 17.6. *There is a Suslin tree iff there is a Suslin line.*

Proof. $\Leftarrow$: Suppose $(X, <)$ is a Suslin line. By transfinite recursion, construct a set $T = \{(a_\alpha, b_\alpha) : \alpha < \omega_1\}$ of ω_1 many open intervals such that for all $I, J \in T$, either I and J are disjoint, or $I \subsetneq J$. For each $\alpha < \omega_1$, since $\{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ is countable and therefore not dense in X (since X is a Suslin line), we can find an interval (a_α, b_α) in X which does not contain any of the endpoints a_β or b_β for $\beta < \alpha$.

Now we claim the set T under the relation $\subsetneq$ is a Suslin tree. Every chain in T under $\subsetneq$ is wellfounded. This is since the rank of (a_α, b_α) is $\leq \alpha$ by construction. T cannot have any uncountable antichain, since this would be an uncountable set of disjoint open intervals in X. T cannot have a countable chain $\langle (c_\alpha, d_\alpha) : \alpha < \omega_1 \rangle$ since then the intervals $(c_\alpha, c_{\alpha+1})$ and $(d_\alpha, d_{\alpha+1})$ would be an uncountable set of open intervals.

$\Rightarrow$: Suppose T is a Suslin tree. By Exercise 17.5, assume T is a Suslin tree with the two properties given in that exercise. For each $x \in T$ of level α that is not maximal, since $\{y : y >_T x \wedge y \text{ is at level } \alpha + 1\}$ is countably infinite, we can define a dense linear order without endpoints $<_x$ on this set (using some bijection with $\mathbb{Q}$).

Now let X be the set of maximal branches through T. Let $<$ be the linear order on X defined as follows. Let $a = \{a_\gamma : \gamma < \alpha\}$ and $b = \{b_\gamma : \gamma < \beta\}$ be maximal branches in X. Since a and b are both maximal branches neither is a subset of the other, so there is a least γ such that $a_\gamma \neq b_\gamma$. By the exercise, γ cannot be a limit ordinal, so $\gamma = \xi + 1$ for some ξ, where $a_\xi = b_\xi$. Now define $a < b$ iff $a_\gamma <_{a_\xi} b_\gamma$ using the ordering $<_{a_\xi}$ on the successor of a_ξ.

It is clear that this ordering is dense and has no endpoints (since each ordering $<_x$ is dense and has no endpoints), and there is no countable dense set (since the tree T has height ω_1). Finally, if $\{(a_i, b_i) : i \in I\}$ is any set of disjoint open intervals in X, then choose $x_i \in T$ such that $N_{x_i} \subseteq (a_i, b_i)$, where $N_{x_i} = \{c \in X : x_i \in c\}$. Then $\{x_i : i \in I\}$ forms an antichain in T, since the intervals (a_i, b_i) are pairwise disjoint. Hence, since T has no uncountable antichains, there is no uncountable set of disjoint open intervals. $\square$

If $V = L$, then there is a Suslin line. In 1970, Jensen isolated a combinatorial principle called $\diamondsuit$ that follows from $V = L$ which implies that there is a Suslin tree.

Definition 17.7. *A $\diamondsuit$-sequence is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ where $A_\alpha \subseteq \alpha$ such that for all sets $X \subseteq \omega_1$, $\{\alpha : X \cap \alpha = A_\alpha\}$ is stationary. $\diamondsuit$ is the statement that there exists a $\diamondsuit$-sequence.*

$\diamondsuit$ is a powerful principle for constructing objects in ω_1 many steps. Obviously we cannot list all subsets of ω_1 in ordertype ω_1 since $2^{\omega_1} > \omega_1$. However, a $\diamondsuit$ sequence lets us understand and anticipate all subsets of ω_1 (via their bounded subsets) in an ω_1-length construction.

CH is an easy consequence of $\diamondsuit$:

Proposition 17.8. *$\diamondsuit$ implies CH.*

Proof. Fix a diamond sequence $\langle A_\alpha \colon \alpha < \omega_1 \rangle$. We claim $\mathcal{P}(\omega) \subseteq \{A_\alpha \colon \alpha < \omega_1\}$. This is since for all $X \subseteq \omega$, there must be some $\alpha > \omega$ such that $X = X \cap \alpha = A_\alpha$. □

Next, we'll prove the following theorem of Jensen.

Theorem 17.9 (Jensen)**.** $\diamondsuit$ *implies there is a Suslin tree.*

The proof of this theorem is based on the following lemma:

Lemma 17.10. *Suppose T is a Suslin tree. Define for each $\alpha < \omega_1$ $T_{<\alpha} = \{x \in T \colon \mathrm{rank}_T(x) < \alpha\}$. If A is a maximal antichain in T, then the set $C = \{\alpha \colon A \cap T_{<\alpha}$ is a maximal antichain in $T_\alpha\}$ is club in ω_1.*

Proof. Define a function $f \colon T \to \omega_1$ as follows. Given $x \in T$, let $f(x)$ be the least β such that there is a y so y is compatible with x ($y \leq x$ or $x \leq y$) and $y \in A$. Now let $C = \{\alpha \colon \forall x \in T_{<\alpha} f(x) < \alpha\}$. Then if $\alpha \in C$, then A is a maximal antichain in T_α, and C is closed. □

Proof of Theorem 17.9. Our tree T will be on the set ω_1. Our tree will have the property

if $x \in T$ as a successor in T, then x has infinitely many successors in T. ((*))

Hence, if there is an uncountable chain C in T, then the set of $x \notin C$ such that x is a successor of some $y \in C$, is also an uncountable antichain. Thus, it suffices to show there are no uncountable antichains in T.

For each countable limit ordinal α, we will define the ordering on α. Let $\langle A_\alpha \colon \alpha \in \omega_1 \rangle$ be a $\diamondsuit$ sequence.

Given that we have defined $T_{<\alpha}$, if A_α is a maximal antichain in $T_{<\alpha}$, define $T_{<\alpha+1}$ by letting the only nodes at height α be computable with elements of A_α. Otherwise, extend $T_{<\alpha}$ to $T_{<\alpha+1}$ so that $T_{<\alpha+1}$ has height $\alpha + 1$, and subject to condition (*).

Now suppose A is a maximal antichain. Then $C = \{\alpha \colon A \cap T_{<\alpha}$ is a maximal antichain in $T_\alpha\}$ is club, and since $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ is a $\diamondsuit$ sequence, $S = \{\alpha \colon A \cap \alpha = A_\alpha\}$ is stationary. Hence, $C \cap S$ is stationary, and is hence nonempty. Suppose $\alpha \in C \cap S$. Then every node of rank α in T is above $A \cap \alpha$. Hence, there can be no elements of A that are greater than α, else this would contradict that A is an antichain. Hence, A is countable. □

18 Models of set theory and absoluteness

The majority of questions of set theory are independent from the ZFC axioms, so much of modern set theory deals with building and studying models of ZFC with diverse properties. In this section we'll prove some basic facts about models of ZFC before we move to the study of Gödel's constructible universe L. Precisely, by a **model of** ZFC we mean a set M and a relation E on M (which we interpret as the $\in$ relation) so that (M, E) satisfies the ZFC axioms. There are lots of important definable sets, classes, and functions in ZF (e.g. ω, ORD, $\alpha \mapsto V_\alpha$). If M is a model of ZF (or some weaker theory) and x is a definable set which ZF proves exists and is unique, we let x^M be the interpretation of x inside M. For example, ω^M denotes the least nonzero ordinal in M that is not a successor ordinal in M.

Theorem 18.1 (Skolem's paradox)**.** *If* ZFC *is consistent, then there is a countable model of* ZFC*.*

Proof. By completeness and the Löwenheim-Skolem theorem. □

This is often called a paradox because ZFC proves there are uncountable sets. So if (M, E) is a countable model of ZFC, then there must be $x \in M$ so that $(M, E) \vDash$ "there is no injection from x to ω". However, since M is countable, we know that x has countably many elements in M. Of course, the resolution to this paradox is that (M, E) doesn't contain an injection from x to ω^M, but this doesn't mean that such an injection doesn't exist outside the model. In generally, any set-size model of ZFC is of size κ for some cardinal κ, and thus there is a proper class of sets that this model is "missing".

Models of ZFC can in general be very strange. For example, even though the axiom of foundation asserts that the $\in$ relation is wellfounded, this only means that if (M, E) is a model of ZFC, then $(M, E) \vDash$ "E is wellfounded"; that is, M will not contain any sets that don't have any E-minimal elements. It is quite possible that from outside the model, we see that E is actually illfounded.

Theorem 18.2. *If* ZFC *is consistent, then there is a model* (M, E) *of* ZFC *such that* E *relation is illfounded.*

Proof. Let $\langle c_n : n \in \omega \rangle$ be new constant symbols we add to the language of set theory, and let φ_n be the formula $c_{n+1} \in c_n$. Then $\mathsf{ZFC} + \bigcup_n \varphi_n$ is consistent by the compactness theorem, since any finite subset is consistent. To see this, let M be a model of ZFC. Note that given any number m, we can let $c_k = (m-k)^M$ for $k \leq m$. Then $M \vDash \mathsf{ZFC} + \varphi_0 \wedge \ldots \wedge \varphi_{m-1}$.

Now take a model (M, E) of $\mathsf{ZFC} + \bigcup_n \varphi_n$. The set $\{c_n^M : n \in \omega\}$ has no E-minimal element, so E is illfounded. □

The Mostowski collapse lemma tells us some interesting information about wellfounded models of ZFC.

Theorem 18.3. *Suppose there is a model* (M, E) *of* ZFC *such that the relation* E *on* M *is wellfounded. Then* (M, E) *is isomorphic to a model of the form* $(N, \in \restriction N)$*.*

Proof. Apply the Mostowski collapse lemma to the relation E on M. □

Illfounded models of ZFC are useful, but they can also think that all sorts of crazy things are true. For example, by Gödel's incompleteness theorem there are models of $\mathsf{ZFC} + \neg\mathrm{Con}(\mathsf{ZFC})$. These models must be illfounded, and in fact the $\in$ relation on ω in such a model must be illfounded. In these notes we will try to avoid illfounded models.

A model of the language of set theory of the form $(N, \in\restriction N)$ where N is a transitive set is called a **transitive set model**. We will also deal with models of set theory that are proper classes. In order to formalize this, we first introduce some notation. If N is a class, then we will write φ^N to denote the sentence φ where we replace the quantifiers $\forall x$ and $\exists x$ in φ with $\forall x \in N$ and $\exists x \in N$ respectively.

Exercise 18.4. *If $(N, \in\restriction N)$ is a transitive set model, then $(N, \in\restriction N) \vDash \varphi$ if and only if φ^N is true.*

A **transitive class model** is a transitive class N, equipped with the $\in$ relation. A technical issue here is that the satisfaction relation $\vDash$ is only defined for models which are sets. So if N is a transitive class, if we write $N \vDash \varphi$, we mean φ^N. (There can't be a single first-order formula[19] defining the satisfaction relation for the class V by Tarski's undefinability of truth)

There is a some meta-mathematical subtlety when dealing with transitive class models. For example, when we prove that for a transitive class M, $M \vDash \mathsf{ZFC}$, what we really mean that we can prove a theorem schema that for each axiom φ of ZFC, φ^M is true. Another thing to be aware of is that Gödel's completeness theorem (that $\mathrm{Con}(T)$ iff T has a model) is not true for class models, just set models.

The reason that we will spend so much time studying transitive models of ZFC is that they have a nice sort of Goldilocks property. Arbitrary models of ZFC are too hard to work with: it is hard to determine what is true in these models, and they can disagree with V about almost anything (for example, there are models of $\mathsf{ZFC} + \neg\mathrm{CON}(\mathsf{ZFC})$). Transitive models are just right: they agree with V about some basic facts, but on more complicated questions like CH or "there exists an inaccessible cardinal" they may differ. Strong types of agreement with V (e.g. for each $x \in M$, $\mathcal{P}(x)^M = \mathcal{P}(x)^V$) are too restrictive and are too similar to V to be an interesting type of model to study.

There are important facts which will remain true between the real universe and transitive models. These are called absoluteness results. We say that a formula $\varphi(x_1, \ldots, x_n)$ is absolute for transitive models if for all transitive class models N, for all $x_1, \ldots, x_n \in N$, we have $\varphi^N(x_1, \ldots, x_n)$ is true if and only

[19] though for each n, and each fixed transitive class model M (including V), the satisfaction relation $M \vDash^n \varphi(x_1, \ldots, x_n)$ for Σ_n formulas φ is definable. For $n = 0$, $M \vDash^0 \varphi(x_1, \ldots, x_n)$ where φ is Σ_0 and $x_1, \ldots, x_n \in M$ iff there exists a transitive set N such that $x_1, \ldots x_n \in N$ and $N \vDash \varphi(x_1, \ldots, x_n)$. This is by Δ_0 absoluteness. Now inductively, $M \vDash^{n+1} \exists y_1, \ldots, y_m \neg\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where ψ is Σ_n iff $\exists y_1, \ldots, y_m \in M \neg M \vDash^n \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$.

if $\varphi(x_1, \dots, x_n)$ is true. Absoluteness is generally true for formulas with low logical complexity.

Definition 18.5. *The Δ_0 formulas are the smallest class of formulas containing the atomic formulas, and which are closed under $\wedge, \vee, \neg$ and bounded quantifiers $(\forall x \in y)$ and $(\exists x \in y)$.*

Equivalently, a formula is a Δ_0 formula if the only quantifiers it contains are bounded quantifiers.

Proposition 18.6 (Absoluteness for Δ_0 formulas)**.** *If M is a transitive class, and φ is a Δ_0 formula with n free variables, then for all $x_1, \dots, x_n \in M$, we have that $\varphi(x_1, \dots, x_n)$ is true iff $\varphi^M(x_1, \dots, x_n)$ is true.*

Proof. By induction on formula complexity. Atomic sentences of the form $x \in y$ or $x = y$ are clearly absolute, and a conjunction or negation of an absolute formula is clearly absolute.

Finally, suppose $\varphi(x_1, \dots, x_n)$ is absolute and now consider the formula $\theta(y, x_2, \dots, x_n) := \exists x_1 \in y \varphi(x_1, x_2, \dots, x_n)$. If $\theta^N(y, x_2, \dots, x_n)$ is true, then the $x_1 \in N$ that witnesses this formula in N also witnesses it in V, since $\varphi(x_1, \dots, x_n)$ is absolute. Similarly, if $\exists x_1 \in y \varphi(x_1, x_2, \dots, x_n)$ is true in V, then the x_1 witnessing this formula must be in N, since N is transitive. $\square$

Exercise 18.7 (ZF)**.** *The following are all equivalent to Δ_0 formulas:*

1. *$x = \emptyset$, $x = \bigcup y$, x is a singleton, x is an ordered pair, $x = \{y, z\}$, $x = (y, z)$, $x \subseteq y$, x is transitive, x is an ordinal, x is a limit ordinal, x is a natural number, $x = \omega$.*

2. *$z = x \times y$, $z = x \setminus y$, $z = x \cap y$,*

3. *R is a relation, f is a function, $x = \operatorname{dom}(f)$, $y = \operatorname{ran}(f)$, $y = f(x)$, $g = f \restriction x$.*

More precisely, the definitions we gave of these concepts earlier in the notes are not themselves all Δ_0, but we can prove that they are *equivalent* to Δ_0 formulas. It is clear that if ψ is a formula, and $\vdash \psi \leftrightarrow \varphi$, then if φ is absolute, then ψ is absolute.

Recall that using prenex normal form, we can write any formula with all the quantifiers at its beginning. The Levy hierarchy measures the complexity of formulas in prenex normal form by the number of alternations between existential and universal quantifiers.

Definition 18.8 (The Levy Hierarchy)**.** *A formula is Σ_0 or Π_0 if it is Δ_0. A Σ_{n+1} formula is a formula of the form $\exists x_1 \exists x_2 \dots \exists x_k \psi$ where ψ is Π_n. A Π_{n+1} formula is a formula of the form $\forall x_1 \dots \forall x_k \theta$ where θ is Σ_n. A formula is Δ_n if it is equivalent to both Σ_n and Π_n formulas. More generally, we say that a formula φ is Δ_n in a theory T if there are Σ_n and Π_n formulas ψ and θ such that $T \vdash \varphi \leftrightarrow \psi$ and $T \vdash \varphi \leftrightarrow \theta$.*

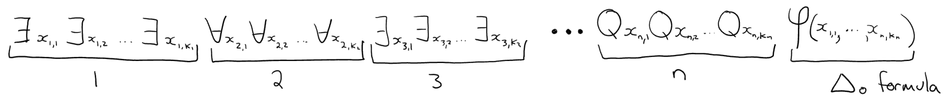

Figure 9: A Σ_n formula has n blocks of alternating existential and universal quantifiers. The last block of quantifiers is $\exists$ if n is odd, and $\forall$ if n is even.

The Levy hierarchy counts the number of alternations of unbounded quantifiers in formulas.

It is clear that every Σ_n formula is Π_{n+1}, and every Π_n formula is Σ_{n+1}.

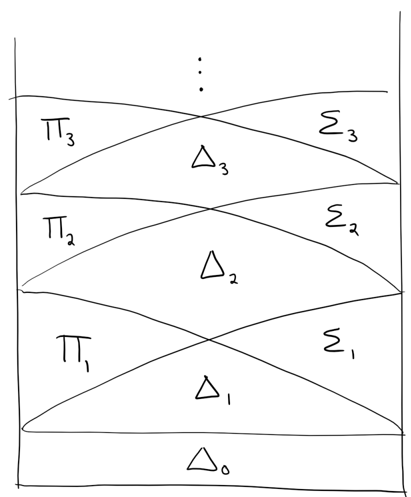

Figure 10: A picture of the Levy hierarchy

Exercise 18.9. *Show that if φ, ψ are Σ_n formulas, then*

- $\neg\varphi$ *is equivalent to a Π_n formula.*
- $\varphi \wedge \psi$ *and* $\varphi \vee \psi$ *are equivalent to Σ_n formulas.*
- $(\forall x \in y)\varphi$ *is equivalent to a Σ_n formula, assuming* ZF.

If x is a set or class in V, we say that x is Σ_n/Π_n definable (without parameters) if there is a Σ_n/Π_n formula φ with a single free variable such that $y \in x \leftrightarrow \varphi(y)$.

Lemma 18.10. *Suppose $G\colon V \to V$ is a Σ_n function. Then the function F on* ORD *such that $F(\alpha) = G(F \restriction \alpha)$ is a Σ_n function.*

Proof.

$$y = F(\alpha) \leftrightarrow \alpha \in \mathsf{ORD} \wedge \exists f(f \text{ is a function} \wedge \operatorname{dom}(f) = \alpha \\ \wedge (\forall \beta < \alpha)(f(\beta) = G(f \restriction b) \wedge y = G(f))).$$

Now apply Exercise 18.9. □

Lemma 18.11. *Suppose $F\colon V \to V$ is a Σ_n function with a Π_n domain. Then F is Δ_n.*

Proof. Let φ be a Σ_n formula defining F. We give a Π_n definition for F. $F(x) = y \leftrightarrow x \in \operatorname{dom}(F) \wedge \forall y'(y' \neq y \to \neg\varphi(x, y'))$. □

Here are some examples of concepts at various levels of the Levy hierarchy.

Exercise 18.12. *The following are Δ_1 in ZF:*

- *R is a wellfounded relation on X.*
- $\operatorname{rank}(x) = \alpha$.
- *x is finite.*
- $\operatorname{TC}(x) = y$.

The following are Σ_1:

- *x is countable.*
- *There is an injection from x to y.*

The following are Π_1:

- *κ is a cardinal.*
- *κ is a regular cardinal.*
- *κ is a limit cardinal.*

The following are Σ_2:

- *There exists a strongly inaccessible cardinal.*

The following can be written as Π_2 formulas:

- *The continuum hypothesis $2^{\aleph_0} = \aleph_1$.*

Suppose $N \subseteq M$ are transitive classes. We say that a formula φ with n free variables is **upwards absolute** if for all $x_1, \ldots, x_n \in N$, if $N \vDash \varphi(x_1, \ldots, x_n)$, then $M \vDash \varphi(x_1, \ldots, x_n)$. Similarly, we say that a formula φ is **downwards absolute** if for $x_1, \ldots, x_n \in N$, if $M \vDash \varphi(x_1, \ldots, x_n)$, then $N \vDash \varphi(x_1, \ldots, x_n)$. An important class of formulas that are upwards absolute are the Σ_1 formulas. If a Σ_1 formula is true in N, then the witness that this formula is true is also included in M, and verifying that it is actually a witness is a Δ_0 statement which is absolute between N and M. By taking negations, we similarly see that Π_1 formulas are **downwards absolute**.

Proposition 18.13. *Suppose N, M are transitive classes and $N \subseteq M$. If φ is a Σ_1 formula, then φ^N implies φ^M. If ψ is a Π_1 formula then ψ^M implies ψ^N.*

Proof. Suppose φ is a Σ_1 formula. Then take a witness $\overline{x}$ for φ in N. Since $\overline{x} \in M$, we have that φ holds in M by Δ_0 absoluteness. Downwards Π_1 absoluteness then holds by contraposition since the negation of a Σ_n formula is Π_n. $\square$

Similar types of results for absoluteness, and upwards and downwards absoluteness exist in many other fields of mathematics. In the study of models of PA if M is an end extension of N, then we have upwards absoluteness for Σ^0_1 formulas, and downwards absoluteness for Π^0_1 formulas in the arithmetical hierarchy.

Another analogy comes from topology. Suppose we have a topological space (X, τ) and we enlarge the topology by adding more open sets. In this setting, sufficiently simple topological properties of sets will be upwards/downwards absolute.

Exercise 18.14. *Suppose τ is a topology on a set X, and $\tau' \supseteq \tau$ is a topology with more open sets (i.e. a finer topology).*

1. *If $A \subseteq X$ is open in (X, τ), then A is also open in (X, τ') (i.e. being an open set is upwards absolute).*
2. *If $A \subseteq X$ is dense in (X, τ'), then $A \subseteq X$ is dense in (X, τ) (i.e. being dense is downwards absolute).*
3. *Show that $A \subseteq X$ being nowhere dense is neither upwards nor downwards absolute.*

An important consequence of Proposition 18.13 is that if a theory T proves that θ is a Δ_1 formula, and T holds in two models M and N, then $\theta^N \leftrightarrow \theta^M$. Wellfoundedness is a very important examples of Δ_1 properties which is therefore absolute for models of ZFC.

Exercise 18.15. $\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$ *does not prove there is a wellfounded model of* ZFC. *[Hint: First show that if* $\mathrm{Con}(\mathsf{ZFC})$ *is true, then if N is any wellfounded model of* ZFC*, then* $N \vDash \mathrm{Con}(\mathsf{ZFC})$*. Then argue that if* $\mathsf{ZFC} + \mathrm{Con}(\mathsf{ZFC})$ *implied there was a wellfounded model of* ZFC*, we could find an infinite descending chain of models of* ZFC*]*

Theorem 18.16 (ZF)**.** *If α is a limit ordinal, then V_α is a model of Extensionality, Foundation, Pairing, Union, Nullset, Separation, and Powerset. If $\alpha > \omega$, then V_α is model of the axiom of Infinity. If κ is a strongly inaccessible cardinal, then V_κ is a model of Replacement, and hence V_κ is a model of* ZFC.

Proof. The extensionality and foundation axioms are Π_1, hence downwards absolute and hold in all transitive classes. $\emptyset$ is an element of V_α and witnesses that the nullset axiom is true.

The rest of the axioms of ZFC are set existence axioms. To show that they are true, we check that the sets the axioms state exist are in V_α.

Union: suppose $y \in V_\alpha$. Then $x = \bigcup y$ is in V_α, since $\mathrm{rank}(\bigcup y) < \mathrm{rank}(y)$. Now $x = \bigcup y$ is Δ_0, and hence it holds in V_α.

Separation: suppose $x, w_1, \ldots, w_n \in V_\alpha$, and φ is a formula. Let $y = \{z \in x \colon \varphi^{V_\alpha}(z, w_1, \ldots, w_n)\}$. Then $y \in V_\alpha$, since $\mathrm{rank}(y) \leq \mathrm{rank}(x)$, and y witnesses this instance of the separation axiom. (Note that it important that we use φ^{V_α} and not the formula φ here).

Infinity: if $\alpha > \omega$, then $\omega \in V_\alpha$, and $x = \omega$ is Δ_0.

Powerset: if $\mathrm{rank}(x) = \beta$, then $\mathrm{rank}(\mathcal{P}(x)) = \beta + 1$. So if α is a limit ordinal and $x \in V_\alpha$, then $\mathcal{P}(x) \in V_\alpha$. Now $y = \mathcal{P}(x)$ is Π_1, so since it holds in V, it is true in V_α by downwards absoluteness.

Pairing: since $\mathrm{rank}(\{x, y\}) = \sup(\mathrm{rank}(x), \mathrm{rank}(y)) + 1$, if $x, y \in V_\alpha$ and α is a limit ordinal, then $\{x, y\} \in V_\alpha$, and $z = \{x, y\}$ is absolute.

Now suppose κ is a strongly inaccessible cardinal. We verify that the axiom of replacement is true in V_κ. Since κ is strongly inaccessible, every element of κ has cardinality less than κ. Suppose $X \in V_\kappa$ and F is a class function in V_κ (i.e. there is some formula φ so that $V_\kappa \vDash \forall x \exists! y \varphi(x, y)$). Let $F[X]^{V_\kappa} = \{y \in V_\kappa \colon (\exists x \in X) V_\kappa \vDash \varphi(x, y)\}$. Then $\mathrm{rank}(F[X]^{V_\kappa})$ is the sup of fewer than κ ordinals less than κ, so $F[X]^{V_\kappa} \in V_\kappa$, and this set witnesses this instance of the axiom of replacement. $\square$

Note that it is not true that $V_\kappa \vDash \mathsf{ZFC}$ if and only if κ is inaccessible. We'll show this in the next section.

Recall that if κ is a cardinal, then H_κ is the collection of sets whose transitive closure has size less than κ. Clearly H_κ is transitive.

Exercise 18.17. *For every regular cardinal κ, H_κ is a model of* $\mathsf{ZFC} -$ *Powerset*

Two important questions we will often study about models of ZFC are the following:

1. Does the *same definition* in two different transitive models of ZFC define the same set?

2. Does the *same set* in two different transitive models of ZFC share the same properties?

Much of what we prove in subsequent chapters will require detailed analysis of both of the above kinds.

Here an example of the first type of question. Recall that if M is a model of ZFC, we let ω_1^M indicate the least uncountable ordinal in M (which ZFC proves exists and is unique). This first exercise shows that a model of the form V_α must computes ω_1 to be the same ordinal as ω_1^V.

Exercise 18.18. *Suppose α is an ordinal such that $V_\alpha \vDash \mathsf{ZFC}$. Then $\omega_1^{V_\alpha} = \omega_1^V$. [Hint: First note that the ordinals in V_α are a subset of the ordinals in the real V. Use the fact that "β is countable" is Σ_1 definable to first show that $\omega_1^{V_\alpha} \leq \omega_1^V$. Next, show that $\alpha \geq \omega + 3$ and hence $\mathcal{P}(\omega \times \omega)^V \in V_\alpha$. Finally, show that for every ordinal $\beta < \omega_1^V$, $V_\alpha \vDash$ "β is countable" by using the absoluteness of wellfoundedness and the absoluteness of the Mostowski collapse.]*

Here is an example of the second type of question.

Exercise 18.19. *Suppose $V \models$ "κ is a cardinal", and $M \subseteq V$ is a transitive model so that $\kappa \in M$. Then show that $M \models$ "κ is a cardinal".[Hint: being a cardinal is Π_1.]*

In a later section we will show that if a measurable cardinal exists, then $\omega_1^L \neq \omega_1^V$. Hence, by the above exercise, letting $\kappa = \omega_1^V$, we must have $L \models$ "κ is a cardinal", and $L \models$ "κ is uncountable" since uncountability is downwards absolute. Hence, since $\kappa \neq \omega_1^L$, we must have $\omega_1^L < \kappa$, and so ω_1^L must be countable. So the converse of the above exercise is not true.

19 The reflection theorem

Before we we move to studying Gödel's constructible universe L, we will prove the reflection theorem. It is a general fact about all cumulative hierarchies like V or L. The following is a theorem schema; it is a theorem of ZFC that is provable for each formula φ:

Theorem 19.1. *Suppose $\langle M_\alpha \colon \alpha \in \mathsf{ORD}\rangle$ is a* ***cumulative hierarchy****, so*

- *M_α is transitive,*
- *If $\alpha < \beta$, $M_\alpha \subseteq M_\beta$, and*
- *if λ is a limit, then $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$.*

Let M be the class $M = \bigcup_{\alpha \in \mathsf{ORD}} M_\alpha$. If φ is a formula, then there is a closed and unbounded class of ordinals α such that

$$\text{for all } x_1, \dots, x_n \in M_\alpha,\ M_\alpha \vDash \varphi(x_1, \dots, x_n) \leftrightarrow M \vDash \varphi(x_1, \dots x_n). \qquad (*)$$

Proof. We prove this by induction on the complexity of φ. The theorem is true for Δ_0 formulas for the club of all ordinals by Δ_0 absoluteness, since M and M_α are transitive.

Logical connectives are easy to handle. If (*) is true for φ, then it is also true for $\neg\varphi$. Suppose φ and φ are formulas and $C, C' \subseteq \mathsf{ORD}$ are clubs such that (*) holds for φ and φ' on the clubs C and C'. Then (*) holds for the formulas $\varphi \wedge \varphi$ and $\varphi \vee \varphi'$ on the club $C \cap C'$.

Finally, suppose $\varphi(x_1, \dots, x_n)$ is a formula and (*) holds for all $\alpha \in C$ for some club C. We want to show that (*) holds for the formula $\psi := \exists x_1 \varphi(x_1, \dots, x_n)$. Note that for all $\alpha \in C$ and all $x_2, \dots, x_n \in M_\alpha$ if $M_\alpha \vDash \exists x_1 \varphi(x_1, \dots, x_n)$, then $M \vDash \exists x_1 \varphi(x_1, \dots, x_n)$ using the same witness $x_1 \in M_\alpha$ since (*) holds for φ. It is the other direction of the implication which will require careful work (and refining our club).

First we show that the class of $\alpha \in C$ such that (*) holds for ψ is closed. Suppose $\langle \alpha_\xi \colon \xi < \beta\rangle$ is an increasing sequence of ordinals in C with limit $\lambda = \sup\{\alpha_\xi \colon \xi < \beta\}$, and for all $x_2, \dots, x_n \in M_{\alpha_\xi}$, if $M \vDash \exists x_1 \varphi(x_1, \dots, x_n)$, then $M_{\alpha_\xi} \vDash \exists x_1 \varphi(x_1, \dots, x_n)$. Then we claim that for all $x_2, \dots, x_n \in M_\lambda$, if $M \vDash \exists x_1 \varphi(x_1, \dots, x_n)$, then $M_\lambda \vDash \exists x_1 \varphi(x_1, \dots, x_n)$. This is because if $x_2, \dots, x_n \in M_\lambda$, then there is some α_ξ such that $x_2, \dots, x_n \in M_{\alpha_\xi}$, and hence $\exists x_1 \in M_{\alpha_\xi} \varphi(x_1, \dots, x_n)$, so this same x_1 witnessing this statement is in M_λ.

Finally, we show that the class of $\alpha \in C$ such that (*) holds for ψ is unbounded. Our basic idea is that if $M \vDash \exists x_1 \varphi(x_1, \dots, x_n)$, then the witness $x_1 \in M$ must be in M_β for some β. We will find a closure points for these β witnessing these statements. For all $\alpha \in C$, define the function $f_\alpha(x_2, \dots, x_n)$ to be the least $\beta \in C$ such that $M_\beta \vDash \exists x_1 \varphi(x_1, \dots, x_n)$ if $M \vDash \exists x_1 \varphi(x_1, \dots, x_n)$. Otherwise, let $f_\alpha(x_2, \dots, x_n) = \alpha$. Hence, if $M \vDash \exists x_1 \varphi(x_1, x_2, \dots, x_n)$ then $M_\beta \vDash \exists x_1 \varphi(x_1, x_2, \dots, x_n)$ for $\beta = f_\alpha(x_2, \dots, x_n)$. Now let $g(\alpha) = \sup\{f_\alpha(x_2, \dots, x_n) \colon x_2 \dots, x_n \in M_\alpha\}$. So if $x_2, \dots, x_n \in M_\alpha$, then if $M \vDash \exists x_1 \varphi(x_1, x_2, \dots, x_n)$, then $M_{g(\alpha)} \vDash \exists x_1 \varphi(x_1, x_2, \dots, x_n)$.

Fix $\alpha_0 \in C$, and let $\alpha_{n+1} = g(\alpha_n)$. For all $x_2, \ldots, x_n \in M_{\alpha_n}$, if $(\exists x_1 \in M)\varphi^M(x_1, x_2, \ldots, x_n)$ is true, then $(\exists x_1 \in M_{\alpha_{n+1}})\varphi^{M_{\alpha_{n+1}}}(x_1, x_2, \ldots, x_n)$. Hence, if $\lambda = \sup\{\alpha_n \colon n \in \omega\}$, then (*) holds for the formula ψ for $\alpha = \lambda$.

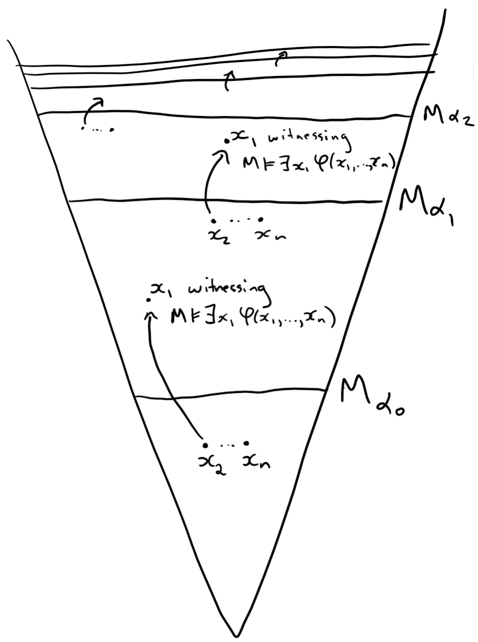

Figure 11: A proof of the reflection theorem.

□

Note that as the formula φ gets larger, the club C_φ where (*) holds requires a larger and larger formula to define it. Note that we cannot say at the end that the intersection of the countably many club classes C_φ is a club because there won't be a formula defining this intersection $\bigcap_\varphi C_\varphi$. The formula would be the conjunction of the infinitely many formulas defining the classes C_φ and would be infinitely long.

If M is a transitive model of ZFC, then from outside this model, it may indeed be the case that the intersection of the countably many clubs C_φ is empty (when M has a height that has countable cofinality). Indeed, we will soon prove that if there is a transitive model of ZFC, then there is a minimal transitive model of ZFC (one contained in all the others). This model will must exhibit this phenomenon.

To take a simpler example with the same flavor, since PA is consistent[20] we have that for every $n \in \omega$, PA $\vdash$ "n does not code a proof of $\neg\,\mathrm{Con}(\mathsf{PA})$".

[20] ZFC proves that $\mathbb{N}$ is a model of PA

However, it is not true that $\mathsf{PA} \vdash (\forall n)$"$n$ does not code a proof of $\neg\,\mathrm{Con}(\mathsf{PA})$". This would contradict the second incompleteness theorem.

We have the following corollary of the completeness theorem:

Corollary 19.2. *Given a finite set of axioms of* ZFC*, there is an α such that V_α is a model of these finitely many axioms. There is also a countable transitive model M of these finitely many axioms.*

Proof. Apply the reflection theorem to the conjunction φ of these finitely many axioms, to find some V_α such that $V_\alpha \vDash \varphi$. We can obtain a countable transitive model by taking a countable elementary submodel of V_α and then taking its transitive collapse. □

Indeed, by taking the transitive collapse of this V_α we can find a countable elementary submodel The above corollary is also a "corollary schema": we can prove it for each finite set of axioms of ZFC. Of course ZFC does not prove "for all finite subsets $S \subseteq \mathsf{ZFC}$ there is a model of S"; by the completeness theorem this would contradict Gödel's incompleteness theorem.

Here is another nice exercise which combines reflection, absoluteness, and countable elementary submodels:

Exercise 19.3. *Suppose φ is a Σ_1 formula. Show that if there is a unique set x such that $\varphi(x)$ holds, then x is countable.*

Corollary 19.2 is one way of formalizing some matters to do with forcing. In general, it is most convenient to develop forcing over countable transitive models of ZFC. However, ZFC cannot prove such models exists. However, we can instead work with a countable transitive model of a "large enough" finite subset of ZFC (large enough to prove the finitely many theorems that we use in our forcing proof). If we show this way that if φ is a sentence of set theory (e.g. $\neg\mathsf{CH}$) and for every large enough finite subset S of ZFC, there is a model of $S + \varphi$, then by the compactness theorem φ is consistent with ZFC.

Corollary 19.4. ZFC *is not finitely axiomatizable.*

Proof. By Gödel's incompleteness theorem, ZFC cannot prove there is a model of ZFC. But we have just shown that ZFC proves that for any finitely many axioms of ZFC, ZFC proves there is some V_α such that V_α models theses axioms. □

The proof of Theorem 19.1 is really just constructing appropriate closure points to witnesses the Tarski-Vaught test for being a Σ_n elementary substructure (it is a little delicate to formalize the Tarski-Vaught test for class models, so we haven't done the above proof this way).

Exercise 19.5 (The Tarksi-Vaught test)**.** *Suppose M is a structure, and $N \subseteq M$ is a substructure. Then N is an elementary substructure of M iff for every formula $\varphi(x_1, \dots, x_n, y)$ and $x_1, \dots, x_n \in N$, if $M \vDash \exists y \varphi(x_1, \dots, x_n, y)$, then there exists some $y \in N$ so that $M \vDash \varphi(x_1, \dots x_n, y)$.*

For cumulative hierarchies that are set-length (instead of length all the ordinals), we have the following version of the reflection theorem using the full Tarski-Vaught test:

Theorem 19.6. *Suppose κ is a limit ordinal and $\langle M_\alpha \colon \alpha \in \kappa \rangle$ is a sequence of sets such that*

- *M_α is transitive,*
- *If $\alpha < \beta$, $M_\alpha \subseteq M_\beta$, and*
- *if $\lambda \leq \kappa$ is a limit, then $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$.*
- $\mathrm{cf}(\kappa) > |M_\alpha| \cdot \omega$ *for all $\alpha < \kappa$.*

Then for every formula φ, there is a closed and unbounded set of ordinals α such that M_α is an elementary substructure of M_κ.

Proof. It is clear that the set of such α is closed. To see that it is unbounded, suppose $\alpha_0 < \kappa$. Then recursively define α_{n+1} to be the least ordinal β such that for all formulas φ and all $x_1, \ldots, x_n \in M_{\alpha_0}$, if $M_\kappa \vDash \exists y \varphi(x_1, \ldots, x_n)$, then there exists $y \in M_\beta$ such that $M \vDash \varphi(x_1, \ldots, x_n)$. Since $\mathrm{cf}(\kappa) > |M_{\alpha_n}| \cdot \omega$, note that this β is the sup of $|M_{\alpha_n}| \cdot \omega$ ordinals less than κ, and is hence less than κ. Now let $\alpha = \sup_n \alpha_n$. Then by the Tarski-Vaught test, we have that M_α is an elementary substructure of M_κ. $\square$

For example, the above theorem applies in the case where κ is strongly inaccessible and $M_\alpha = V_\alpha$ by Exercise . So if V_κ is strongly inaccessible, then there is a club of α in κ so that V_α is an elementary substructure of V_κ. (And so it is not true that $V_\kappa \vDash \mathsf{ZFC}$ iff κ is strongly inaccessible).

Exercise 19.7. *Suppose there exists some α such that $V_\alpha \vDash \mathsf{ZFC}$. Show that the last such α such that $V_\alpha \vDash \mathsf{ZFC}$ must have* $\mathrm{cf}(\alpha) = \omega$.

In general, we expect that the least ordinal α satisfying some "closure property" will have $\mathrm{cf}(\alpha) = \omega$.

20 Gödel's constructible universe L

In this section we'll define Gödel's constructible universe L, and prove L is a model of ZFC (more precisely, that for every axiom $\varphi \in \mathsf{ZFC}$, φ^L is true). We will do this carefully assuming only ZF without choice, since one of our goals is to show that if there is a model M of ZF, then L^M is a model of ZFC. Hence, $\mathrm{Con}(\mathsf{ZF}) \to \mathrm{Con}(\mathsf{ZFC})$, and so while the axiom of choice can be used to construct strange objects like nonmeasurable set, we can prove it doesn't create any inconsistencies.

Gödel's L is built up in a cumulative hierarchy where at each level, the next level is the sets definable using parameters over the previous levels. We will begin by carefully calculating the complexity of the map sending each set to its definable subsets.

It is easy to check that the set of all formulas is a Δ_1 definable subset of H_ω, it has a Δ_1 definition, the functions $\wedge$, $\neg$, and $\varphi \mapsto \exists x \varphi$ are Δ_1 definable, and the relation "φ is a subformula of ψ" is wellfounded.[21]

Exercise 20.1 (ZF)**.** *The relation $M \vDash \varphi(x_1, \dots, x_n)$ is a Σ_1 definable relation between transitive sets M, the set of formulas, and $M^{<\omega}$. [Hint: Define this relation using recursion on the relation of being a subformula,*

- $M \vDash x \in y$ *iff* $x \in y$.
- $M \vDash x = y$ *iff* $x = y$.
- $M \vDash \varphi \wedge \psi$ *iff* $M \vDash \varphi \wedge M \vDash \psi$
- $M \vDash \neg\varphi$ *iff* $\neg M \vDash \varphi$
- $M \vDash \exists x \varphi(x)$ *iff* $(\exists x \in M) M \vDash \varphi(x)$.

Each clause of this definition is Δ_0 (note that definition for $\exists x\varphi$ only uses a bounded quantifier over M). Now apply the same ideas as Lemmas 18.10 and 18.11 to finish.]

If M is a set, then let

$$\mathrm{Def}(M) = \{y \colon (\exists \varphi \exists z_1, \dots, z_n \in M) y = \{x \in M \colon M \vDash \varphi(x, z_1, \dots, z_n)\}\},$$

So $\mathrm{Def}(M)$ is all subsets of M that are definable using parameters in M. From Lemmas 18.10, 18.11 and Exercise 20.1, we see $M \mapsto \mathrm{Def}(M)$ is a Δ_1 function.

Definition 20.2 (Gödel's constructible universe L)**.**

- $L_0 = \emptyset$.
- $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$.

[21] If we were being excruciatingly pedantic, we'd begin here defining what a formula is, and engaging in other trivial and tedious syntactic discussions. If you want to engage in this kind of pedantry, you're on your own.

- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ *if* λ *is a limit.*
- $L = \bigcup_\alpha L_\alpha$.

Note that the map $\alpha \mapsto L_\alpha$ is Δ_1 by Lemmas 18.10 and 18.11.

Lemma 20.3 (ZF)**.** *For all ordinals* α*,*

1. $L_\alpha \subseteq V_\alpha$.
2. L_α *and* L *are transitive*
3. $\beta \leq \alpha$ *implies* $L_\beta \subseteq L_\alpha$
4. $L_\alpha \cap \mathsf{ORD} = \alpha$*. So,* L *contains all the ordinals.*

Proof. These are all proved by transfinite induction on α. They are all easy to check at limit steps.

(1) follows at successor stages since if $L_\alpha \subseteq V_\alpha$, then $L_{\alpha+1} \subseteq \mathcal{P}(L_\alpha) \subseteq \mathcal{P}(V_\alpha) \subseteq V_{\alpha+1}$.

(2) is true at successor steps since if L_α is transitive, if $x \in y \in L_{\alpha+1}$, then $y \in L_{\alpha+1}$ implies $y \subseteq L_\alpha$, so if $x \in y$, then $x \in L_\alpha$ and so $x = \{a : a \in x\} \in L_{\alpha+1}$. It follows that L is a transitive class.

(3) follows since each $x \in L_\alpha$ is definable by using itself as a parameter.

For (4), if $L_\alpha \cap \mathsf{ORD} = \alpha$, then $\{\beta \in L_\alpha : \beta \text{ is an ordinal}\} = \alpha$ since being an ordinal is Δ_0 and hence absolute. Hence $L_{\alpha+1} \cap \mathsf{ORD} = \{\beta : \beta \leq \alpha\} = \alpha+1$. ☐

The first few levels of L are the same as V, but they quickly become very different.

Exercise 20.4 (ZFC)**.** *For each infinite* α*,* $|L_\alpha| = |\alpha|$.

Now we show that L is a model of ZF.

Lemma 20.5 (ZF)**.** *For every axiom* φ *of* ZF*,* φ^L.

Proof. The extensionality and foundation axioms are true in all transitive classes. $\emptyset$ is an element of L and witnesses that the nullset axiom is true.

The rest of the axioms of ZFC are set existence axioms. To show that they are true, we check that the sets the axioms state exist are in L.

Union: suppose $x \in L$ where $x \in L_\alpha$. Then $\bigcup x$ is in $L_{\alpha+1}$ since it has a Δ_0 definition using the parameter x.

Pairing: if $x, y \in L_\alpha$, then $\{x, y\} \in L_{\alpha+1}$ since it has a Δ_0 definition from the parameters x and y.

Infinity: $\omega \in L_{\omega+1}$ and hence $\omega \in L$, and so L witnesses the axiom of infinity since the formula "$x = \omega$" is Δ_0.

Separation: Suppose, $x, w_1, \ldots, w_n \in L$, and φ is a formula. By the reflection theorem, we can find some α such that $x, w_1, \ldots, w_n \in L_\alpha$ and $L_\alpha \models \varphi(z, w_1, \ldots, w_n) \leftrightarrow L \models \varphi(z, w_1, \ldots, w_n)$ for all $z \in L_\alpha$. Then clearly $\{z \in x : L_\alpha \models \varphi(z, w_1, \ldots, w_n\}$ is in $L_{\alpha+1}$, and witnesses the separation axiom is true in L for this formula and these parameters since φ reflects at α.

Powerset: given x, let $\beta = \sup_{y \subseteq x \wedge y \in L} \inf\{\alpha\colon y \in L_\alpha\}$. Now the set $z = \mathcal{P}(x) \cap L$ is in $L_{\beta+1}$ since it has a Δ_0 definition over L_β; it is the set of y such that $y \subseteq x$. This set witnesses the powerset axiom is true in L since $y \in z \leftrightarrow y \subseteq x$ is Δ_0 and hence true for every $y \in L$.

Replacement: suppose F is a class function in L and $X \in L$. Let $\beta = \sup_{x \in X} \inf\{\alpha\colon F[x] \in L_\alpha\}$. By the reflection theorem reflecting the definition of F to some $\gamma > \beta$, we can define $F[X]$ in L_γ, and this will witness the replacement axiom is true. $\square$

Definition 20.6. *Let $V = L$ abbreviate the sentence $\forall x \exists \alpha x \in L_\alpha$.*

Lemma 20.7 (The absoluteness of constructibility)**.**

1. *If M is a transitive class model of* ZF *which contains all the ordinals, then $L^M = L$.*
2. *$(V = L)^L$.*

Proof. (1) follows from the fact that the map $\alpha \mapsto L_\alpha$ is Δ_1 and hence absolute, so $L_\alpha^M = L_\alpha$. (2) is since $(V = L)^L$ translates to $(\forall x \in L)(\exists \alpha \in L)[x \in L_\alpha^L]$ which is true since L contains all the ordinals, and the definition of L is absolute. $\square$

The above lemma may seem trivial but it definitely isn't. We're heavily using the nontrivial fact that the definition of L is Δ_1 and that Δ_1 formulas are absolute. For example, if M is a transitive class inner model of ZF, then in general $V^M \neq V$.

We say that a transitive class M is a **transitive class model** of a theory T if for every $\varphi \in T$, φ^M is true.

Corollary 20.8. *L is the smallest transitive class inner model of* ZF *which contains all the ordinals.*

Proof. If M is any transitive class model of ZF, then $L = L^M \subseteq M$. $\square$

If φ is a formula, let $\mathrm{Def}_\varphi(M)$ be the subsets of M definable using the formula φ and parameters from M.

Theorem 20.9 (ZF)**.** *$L \vDash \mathsf{AC}$.*

Proof. Since $L \vDash V = L$, we may assume $V = L$. We define a single class wellordering of the whole universe by transfinite recursion which we note $<_L$. Let $\prec$ be a Δ_1 wellordering of all formulas, and for every x, let $\mathrm{rank}_L(x) = \inf\{\alpha\colon x \in L_\alpha\}$.

Let $x <_L y$ iff $\mathrm{rank}_L(x) < \mathrm{rank}_L(y)$ or $\alpha = \mathrm{rank}_L(x) = \mathrm{rank}_L(y)$ and $\inf\{\varphi\colon x \in \mathrm{Def}_\varphi(L_\alpha)\} \prec \inf\{\varphi\colon y \in \mathrm{Def}_\varphi(L_\alpha)\}$ under our ordering $\prec$ on formulas, or the least formula ψ defining x and y over L_α is the same, and the $<_L$-least parameters in L_α in lexicographic order defining x using ψ are less than the $<_L$-least parameters in L_α in lexicographic order defining y.

Exercise 20.10. *This is a wellordering, and is Δ_1 definable.*

$\square$

20.1 Gödel operations and fine structure*

When Gödel first defined L, he was worried that the logical nature of his definition would be viewed with suspicion. To assuage such fears, he defined 8 simple functions $F_1, \ldots, F_8$, such as $F_1(X,Y) = \{X,Y\}$, $F_2(X) = \{(a,b) \in X : a \in b\}$, $F_3(X,Y) = X - Y$, $F_4(X,Y) = \{(a,b) \in X : b \in Y\}$, and $F_5(X) = \{b \in X : \exists a(a,b) \in Y\}$, (the remaining three operations permute the order of tuples). Gödel then defined $L_{\alpha+1}$ to be the subsets of L_α one could obtain from a composition of finitely many Gödel operations applied to L_α or its elements.

The above definitions yields the same set as $\mathrm{Def}(L_\alpha)$ and the proof is quite easy. In one direction, each of the Gödel operations is first-order definable. In the other direction, given any formula, we can write it in prenex normal form $\varphi(y_1, \ldots, y_n) = \exists x_1 \ldots Q x_m \psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ where ψ is quantifier free. Then it is easy to show by induction on formula complexity that the set of tuples $\{(x_1, \ldots, x_m, y_1, \ldots, y_n) \in X_1 \times \ldots \times X_m \times Y_1, \ldots \times Y_n\} : \psi(x_1, \ldots, x_m, y_1, \ldots, y_n)$ satisfying the quantifier free formula ψ is given by a composition of Godel operations applied to $X_1, \ldots, X_m, Y_1, \ldots, Y_n$. But then we can obtain the set of $y_1, \ldots, y_n$ satisfying φ since existential quantifiers correspond to projecting (using operation F_5), and universal quantifiers are $\neg\exists\neg$ (and we can perform complementation using operation F_3).

From a modern perspective, the Gödel operations aren't useful; it's easier just to talk directly about first order definability. However, Gödel's idea anticipated an even more refined way of building up L using very simple operations due to Jensen[22]. Jensen introduced "rudimentary functions", and defined a hierarchy J_α where at each step we close under applying rudimentary functions. Jensen's J hierarchy still constructs the same universe $L = \bigcup_\alpha J_\alpha$, in fact, the hierarchies coincide at limit stages. The issue with the L hierarchy that the J hierarchy fixes is that in general $\mathrm{Def}(L_\alpha)$ is difficult to analyze since L_α satisfies very little of ZFC (e.g. it doesn't even satisfy the pairing axiom). In contrast, in Jensen's fine structure, we can define Skolem functions over the J_α structures in a nice way so that Σ_n definability over J_α is the same as Σ_1 definability over an associated structure.

The first major application of Jensen's fine structure was the proof that it is consistent that CH holds and there are no Suslin trees; the earlier result of Solovay and Tennenbaum that it is consistent that there are no Suslin trees constructed a model where CH failed. Jensen's proof relied on the following combinatorial principle $\Box_\kappa$: There is a sequence $\langle C_\alpha : \alpha \text{ is a limit ordinal } < \kappa^+\rangle$ such that for $\alpha < \kappa^+$:

- $C_\alpha \subseteq \alpha$ is club in α
- For β a limit point of C_α, $C_\alpha \cap \beta = C_\beta$, and
- For α such that $\mathrm{cf}(\alpha) < \kappa$, the order-type of C_α is less than κ.

[22]and building on earlier fine-structure-style analysis of Boolos and Putnam

Jensen showed using his new fine structure that $\Box_\kappa$ holds in L for each κ. Square is extremely useful for performing constructions of length κ^+ using approximations of cardinality less than κ.

21 Condensation in L and GCH

Our next goal is to prove that GCH is true in L. To do this, we'll suppose $x \subseteq \kappa$ and $x \in L$, and figure out the first level L_α where $x \in L_\alpha$. Our analysis will consider elementary substructures of some L_β such that $x \in L_\beta$. Then we'll use the fact that if M is a transitive set such that M satisfies a sufficiently large fragment of ZF and the sentence $V = L$, then M must actually be equal to L_α for some α. This is because the construction of L is absolute. This fact is called the condensation lemma:

Lemma 21.1 (The condensation lemma)**.** *There is a finite set S of axioms of ZF − powerset so that if M is a transitive set where $M \vDash S$ and $M \vDash V = L$, then $M = L_\lambda$ for some limit ordinal λ.*

Proof. Let the axioms of S be pairing, union, and those axioms of ZF used to prove that all the theorems leading up to the fact that for all α, L_α exists and the map $\alpha \mapsto L_\alpha$ is Δ_1 definable (and hence absolute). Suppose $M \vDash S$ and $M \vDash V = L$. Let λ be the least ordinal not in M. We must have that $\mathsf{ORD}^M = \lambda$ by absoluteness of being an ordinal. λ must be a limit ordinal since for each $\alpha \in M$, $\alpha + 1 = \alpha \cup \{\alpha\}$ is in M by the pairing and union axioms.

Since $M \vDash \forall x \exists \alpha \in \mathsf{ORD}(x \in L_\alpha)$, we have that $\forall x \in M \exists \alpha < \lambda (x \in L_\alpha^M)$. Since $L_\alpha^M = L_\alpha$ by the absoluteness of L_α we have $M \subseteq \bigcup_{\alpha \in M} L_\alpha = \bigcup_{\alpha<\lambda} L_\alpha = L_\lambda$.

Conversely, for each $\alpha < \lambda$, L_α^M exists in M (since S is strong enough to prove this), and $L_\alpha^M = L_\alpha$, so $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha \subseteq M$. □

A stronger form of condensation is true, but proving it is technical; one proof uses fine structure. There is a Π_2 sentence φ so that $M \vDash \varphi$ if and only if $M = L_\lambda$ for some limit ordinal λ. Hence if M is a transitive set and $M \prec_{\Sigma_1} L_\lambda$ for some limit λ, then $M = L_{\lambda'}$ for some $\lambda' \leq \lambda$.

In what follows, we'll heavily use the following version of the downward Löwenheim-Skolem theorem due to Tarski and Vaught:

Exercise 21.2. *Suppose M is an infinite structure in a language $\mathcal{L}$, $A \subseteq M$ is a subset of the universe of M. Then there is an elementary substructure $N \preceq M$ such that $A \subseteq N$, and N has cardinality at most $|A| + |\mathcal{L}| + \omega$.*

We're ready to calculate the levels of L when new subsets of κ appear.

Lemma 21.3. *Assume $V = L$. If κ is a cardinal and $x \subseteq \kappa$, then $x \in L_\lambda$ for some $\lambda < \kappa^+$.*

Proof. Let β be sufficiently large so that $x \in L_\beta$, and $L_\beta \vDash S + V = L$. Such a β exists by the reflection theorem.

By the downward Löwnheim Skolem theorem, there is an elementary substructure $N \prec L_\beta$ so that $\kappa \subseteq N$, $x \in N$, and $|N| = \kappa$. Let $\pi \colon N \to M$ be the Mostowski collapse of N to a transitive set model M.

By transfinite induction $\pi(\alpha) = \alpha$ for all $\alpha \leq \kappa$. Now $\alpha \in x \leftrightarrow \pi(\alpha) \in \pi(x) \leftrightarrow \alpha \in \pi(x)$, and so $\pi(x) = x$. Hence $\pi(x) = x$, and so $x \in M$.

Now $M \vDash S$ and $M \vDash V = L$ since N is an elementary substructure of L_β, and N and M are isomorphic. So we must have that $M = L_\lambda$ for some λ by the condensation lemma. Since $\kappa = |M| = |L_\lambda|$ and $|L_\alpha| = \alpha$ for all infinite α, we must have $|\lambda| = \kappa$, and hence $\lambda < \kappa^+$. □

Corollary 21.4. $L \vDash \mathsf{GCH}$.

Proof. If $V = L$, then $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$ by the above lemma. But $|L_{\kappa^+}| = \kappa^+$. □

Corollary 21.5. $\mathrm{Con}(\mathsf{ZF}) \to \mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \mathsf{GCH})$

In the above argument, we've used the reflection theorem to find an appropriate β to reflect from. It will be convenient to know that for regular cardinals κ, L_κ satisfies ZF − Powerset.

Lemma 21.6. *If κ is an uncountable regular cardinal, then $L_\kappa \vDash \mathsf{ZF}$ − Powerset*

Proof Sketch. Copy the proof that $L \vDash \mathsf{ZFC}$ replacing the use of the reflection theorem with Theorem 19.6 in the verification of the Replacement and Separation axiom. □

Recall that H_κ also satisfies ZF − Powerset. This isn't a coincidence.

Theorem 21.7. *If $V = L$, then for all infinite κ, $H_\kappa = L_\kappa$.*

Proof. $V_\omega = H_\omega = L_\omega$, so it suffices to prove this when κ is uncountable. Indeed, it suffices to prove this when κ is an uncountable successor cardinal, since at limit cardinals both H_κ and L_κ are the union of the previous levels.

First we show $L_{\kappa^+} \subseteq H_{\kappa^+}$. If $x \in L_{\kappa^+}$, then the transitive closure of x is in L_{κ^+} by Lemma 21.6 and the absoluteness of transitive closure. Hence $|\operatorname{TC}(x)| < \kappa^+$ since $|L_\alpha| = \alpha$ for every infinite ordinal α.

Now we show $H_{\kappa^+} \subseteq L_{\kappa^+}$. Fix $x \in H_{\kappa^+}$. Since $V = L$, $\operatorname{TC}(\{x\})$ is contained in L_μ for some regular uncountable μ. By Löwenheim-Skolem, we can find an elementary substructure $M \prec L_\mu$ of cardinality κ such that $\operatorname{TC}(\{x\}) \in M$. Let $\pi \colon M \to N$ be the Mostowski collapse of M to a transitive set N. By condensation, $N = L_\lambda$ for some $\lambda < \kappa^+$. By transfinite induction, for all $y \in \operatorname{TC}(\{x\})$ we have $\pi(y) = y$, and so $\pi(x) = x$ is in $L_\lambda \subseteq L_{\kappa^+}$. □

Exercise 21.8. *Assume $V = L$. Show that for every countable ordinal α, there is some countable $\beta > \alpha$ such that $L_{\beta+1} \setminus L_\beta$ contains a subset of ω.*

Exercise 21.9. *Show that there is a Π_2 sentence φ so that $L_{\omega_1} \vDash \varphi$, but $L \vDash \neg\varphi$.*

22 $V = L$ implies $\diamondsuit$

In the early 1970s, Jensen proved that $V = L$ implies there is a Suslin tree. Jensen abstracted the $\diamondsuit$ principle as the key combinatorial tool used in his proof.

We'll recursively construct a $\diamondsuit$-sequence assuming $V = L$, making use of the wellordering $<_L$. We will prove this sequence is a $\diamondsuit$-sequence by showing there cannot be a $<_L$-least set X such that $X \cap \alpha \neq \alpha$ for a club C of α in ω_1. To see this, we'll analyze our construction in L_{ω_2}, taking a countable elementary submodel of L_{ω_2} containing X and C, and obtaining a contradiction to our definition of the $\diamondsuit$-sequence.

We'll begin by proving a quick lemma about countable elementary submodels of L_{ω_2}. Our use of ω_2 in this lemma (and in our proof of $\diamondsuit$) isn't so important; all that matters for this lemma is that $H_{\omega_1} \subseteq L_{\omega_2}$ assuming $V = L$, and L_{ω_2} satisfies a sufficiently large fragment of $\mathsf{ZF} + V = L$.

Lemma 22.1. *Assume $V = L$. Suppose M is a countable elementary submodel of L_{ω_2}. Then $\omega_1 \cap M = \alpha$ for some countable ordinal α.*

Proof. Suppose β is a countable ordinal such that $\beta \in M$. We need to show that if $\gamma < \beta$, then $\gamma \in M$. Then since M is countable, $M \cap \omega_1$ is a countable set of ordinals which is downwards closed, and hence it is a countable ordinal.

Since β is countable, there is a surjection from ω to β. Let f be the $<_L$-least surjection from ω to β. Then $f \in H_{\omega_1}$, and hence $f \in L_{\omega_2}$. Now since $<_L$ is a Δ_1-definable linear order, the fact that f is the $<_L$-least surjection from ω to β is a Π_1 fact, and hence it is true in L_{ω_2} by downwards absoluteness. Hence, f is definable in L_{ω_2}, and so it must be in the elementary submodel M. Now for each $n \in \omega$, $f(n)$ is also definable, and hence it must also be in M. But this implies every ordinal less than β is in M. $\square$

A similar proof gives the following:

Exercise 22.2. *Assume $V = L$, and suppose M is a countable elementary submodel of L_{ω_1}. Then M must be a transitive set and hence $M = L_\alpha$ for some countable ordinal α.*

Now we're ready to prove $V = L$ implies $\diamondsuit$.

Theorem 22.3 (Jensen)**.** *$V = L$ implies $\diamondsuit$.*

Proof. Assume $V = L$. We construct a sequence $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ where $A_\alpha \subseteq \alpha$ such that for all sets $X \subseteq \omega_1$, $\{\alpha \colon X \cap \alpha = A_\alpha\}$ is stationary. We will also construct a sequence $\langle C_\alpha \colon \alpha < \omega_1 \rangle$ where $C_\alpha \subseteq \alpha$ is club.

Let $A_0 = C_0 = \emptyset$ and at successor ordinals let $A_{\alpha+1} = A_\alpha$ and $C_{\alpha+1} = C_\alpha$. If α is a limit, let (A_α, C_α) be the $<_L$-least pair such that $A_\alpha \subseteq \alpha$, C_α is a club subset of α, and $A_\alpha \cap \beta \neq A_\beta$ for all $\beta \in C_\alpha$. If no such pair exists, let $A_\alpha = C_\alpha = \alpha$.

For a contradiction, assume that for some $X \subseteq \omega_1$, there is a club set C such that $X \cap \alpha \neq A_\alpha$ for all $\alpha \in C$. Let (X, C) be the $<_L$-least such pair.

Now $\langle A_\alpha \colon \alpha < \omega_1 \rangle$, $\langle C_\alpha \colon \alpha < \omega_1 \rangle$ and (X, C) are all hereditarily of cardinality $< \omega_2$, and so they are in H_{ω_2} and therefore in L_{ω_2}. By the Löwenheim-Skolem theorem, let M be a countable elementary submodel of L_{ω_2} which contains $\langle A_\alpha \colon \alpha < \omega_1 \rangle$, $\langle C_\alpha \colon \alpha < \omega_1 \rangle$ and (X, C). Let π be the Mostowski collapse of M. By condensation, $\pi(M) = L_\lambda$ for some countable ordinal λ.

By Lemma 22.1, $\omega_1 \cap M = \delta$ for some countable ordinal δ. By transfinite induction $\pi(\beta) = \beta$ for all $\beta < \delta$. Now $\pi(\omega_1)$ must be an ordinal $> \beta$ for all $\beta < \delta$, and hence $\pi(\omega_1) \geq \delta$. But then $\omega_1 \geq \pi^{-1}(\delta)$, and since $\pi^{-1} \in M$ we must have $\pi(\omega_1) = \delta$.

Similarly, $\pi(X) = X \cap \delta$, $\pi(C) = C \cap \delta$, $\pi(\langle A_\alpha \colon \alpha < \omega_1 \rangle) = \langle A_\alpha \colon \alpha < \delta \rangle$, and $\pi(\langle C_\alpha \colon \alpha < \delta \rangle) = \langle C_\alpha \colon \alpha < \delta \rangle$.

The statement that (C, X) is the $<_L$-least pair such that C is club in ω_1, and $X \cap \alpha \neq A_\alpha$ for all $\alpha \in C$ is Δ_1, hence it is true in L_{ω_2} by absoluteness. (Here we're using the fact that the ordering $<_L$ is Δ_1, and if $(C, X) \in L_\beta$, then that saying (C, X) is $<_L$-least only requires quantifying over all (C', X') in L_β).

Since M is an elementary submodel of L_{ω_2} and π is an isomorphism, L_λ models that $(C \cap \delta, X \cap \delta)$ is $<_L$-least such that $C \cap \delta$ is club in δ, and $X \cap \beta \neq A_\beta$ for all $\beta \in C \cap \delta$.

Hence by Δ_1 absoluteness, this statement is true in L, and hence by the definition of the sequences A_α and C_α, $A_\delta = X \cap \delta$, and $C_\delta = C \cap \delta$. But then $X \cap \delta = A_\delta$ by definition of $\langle A_\alpha \colon \alpha < \omega_1 \rangle$ and since $C \cap \delta$ is club in δ, $\delta \in C$. But then $X \cap \delta = A_\delta$ and $\delta \in C$ which is a contradiction to our choice of (X, C). $\square$

23 L and large cardinals

Our goal in this section is to prove two theorems about the relationship between large cardinals and L.

Recall a cardinal κ is **weakly inaccessible** if κ is a regular limit cardinal. Lets show first that weakly inaccessible cardinals are very large.

Lemma 23.1. *Suppose C is club in a weakly inaccessible κ. Then the set $C' = \{\alpha \in C \colon |C \cap \alpha| = \alpha\}$ is also club in κ.*

Proof. C' is closed: if $|C \cap \alpha_\xi| = \alpha_\xi$ for a sequence $\langle \alpha_\xi \colon \xi < \lambda \rangle$ and $\beta = \sup_{\xi<\lambda} \alpha_\xi$, then β is a cardinal since it is a sup of cardinals, and $|C \cap \beta|$, has cardinality greater than every $\alpha < \beta$, so $|C \cap \beta| = \beta$.

C' is unbounded: given $\alpha \in \kappa$, let $\beta_0 \in C$ be such that $\beta_0 > \alpha$. Let $\beta_{n+1} \in C$ be least such that $|C \cap \beta_{n+1}| > \beta_n$. Such a β_{n+1} exists since κ is a limit cardinal (and $|C| = \kappa$ since κ is regular). Then $\beta = \sup \beta_n$ is a cardinal and $|C \cap \beta| > \beta_n$ for each n, so $|C \cap \beta| = \beta$. □

For example if κ is weakly inaccessible and we set $C = \kappa$, then $C' = \{\alpha \colon |\alpha| = \alpha\}$ is the set of cardinals in κ, and $C'' = \{\alpha \colon \omega_\alpha = \alpha\}$ is the set of fixed points of the $\aleph$ function, and these are both club in κ by the above lemma. So if κ is weakly inaccessible, it is far larger than the first fixed point of the $\aleph$ function.

We've shown that ZFC cannot prove there are strongly inaccessible cardinals; if κ is strongly inaccessible, then $V_\kappa \vDash \mathsf{ZFC}$. Similarly, we can show that ZFC cannot prove there are any weakly inaccessible cardinals, since if κ is weakly inaccessible, then $L_\kappa \vDash \mathsf{ZFC}$.

Theorem 23.2. *If κ is weakly inaccessible in V, then $L_\kappa \vDash \mathsf{ZFC}$. Hence, if ZFC is consistent, then ZFC cannot prove there are inaccessible cardinals.*

Proof. In Lemma 21.6 we already proved that $L_\kappa \vDash \mathsf{ZF} - \text{Powerset}$. L_κ satisfies Powerset since for each $x \in L_\kappa$, $x \in L_\mu$ for some cardinal $\mu < \kappa$ and so $\mathcal{P}(x)^L \in L_{\mu^+}$ by our work in Section 21. The statement $y = \mathcal{P}(x)$ is Π_1 so by downwards absoluteness, $\mathcal{P}(x)^L$ is the powerset of x in L_κ. $L_\kappa \vDash \mathsf{AC}$ is an exercise. □

Exercise 23.3. *Show that if κ is a regular cardinal, then $L_\kappa \vDash \mathsf{AC}$.*

Our second goal is to show that L is incompatible with the existence of measurable cardinals. Recall that a cardinal κ is **measurable** if there is a nonprincipal κ-complete ultrafilter on κ.

A use of measurable cardinals is they allow us to take an ultrapower of the universe V of set theory, analogously to how we have defined ultrapowers of structures whose universes are sets.

Definition 23.4. *Suppose U is an ultrafilter on a set I. Then we define the ultrapower $\prod_U V$ of V by U as follows. Consider the equivalence relation $\sim$ on all functions from I to M where $f \sim g$ if $\{i \in I \colon f(i) = g(i)\} \in U$. Now for each f, $\{g \colon g \sim f\}$ is a proper class in general, so using Scott's trick we* *define* $[f] = \{g\colon g \sim f \wedge (\forall h \sim f)(\mathrm{rank}(g) \leq \mathrm{rank}(h))\}$*, so* $[f]$ *is the set of representative of the* $\sim$*-class of* f *of minimal rank. Now we define* $\prod_U V$ *to be the structure in the language of set theory whose universe is the class of the set* $[f]$ *and where the* $\in$ *relation is given by* $[f] \in^{\prod_U V} [g]$ *if* $\{i\colon f(i) \in g(i)\} \in U$*. If the relation* $\in^{\prod_U V}$ *is wellfounded, we identify* $\prod_U V$ *with its Mostowski collapse.*

It is easy to check that the proof of Łos's theorem still works in this context, and so for every sentence φ, φ^V is true iff $\varphi^{\prod_U V}$ is true. Indeed,

Exercise 23.5. *Suppose* U *is an ultrafilter on an index set* I*. Show that the function* $j\colon V \to \prod_U V$ *defined by setting* $j(x)$ *to be the constant function* $i \mapsto x$ *is an elementary embedding of* V *to* $\prod_U V$*.*

If U is ω_1-complete, then $\prod_U V$ will always be wellfounded:

Lemma 23.6. *Suppose* U *is an ultrafilter on an index set* I*. Then if* M *is a transitive set or class model, and* U *is* ω_1*-complete, then the ultrapower* $\prod_U M$ *is wellfounded.*

Proof. Suppose $[f_0], [f_1], \dots$ was an infinite descending sequence. Then by Łos's theorem, for each n, $X_n = \{i \in I\colon f_{n+1}(i) \in f_n(i)\}$ is in the ultrafilter U. So $\bigcap_n X_n$ is in the ultrafilter. Pick $x \in \bigcap_{n\in\omega} X_n$. Then $f_0(x), f_1(x), \dots$ is an infinite descending sequence in V which is a contradiction. $\square$

In fact ω_1-completeness of U exactly characterizes when $\prod_U V$ is wellfounded.

Exercise 23.7. *Show that if* U *is an ultrafilter on* I *and* U *is not* ω_1*-complete, then the ultraproduct* $\prod_U V$ *is illfounded.*

Lemma 23.8. *Suppose* κ *is a measurable cardinal and* U *is a nonprincipal* κ*-complete ultrafilter on* κ*, and* j *is the elementary embedding from* V *into* $M = \prod_U V$ *given above. Then for every* $\alpha < \kappa$*,* $j(\alpha) = \alpha$*. However* $j(\kappa) > \kappa$*. Thus if there is a measurable cardinal, then there is a nontrivial (i.e. nonidentity) elementary embedding* j *from* V *into a transitive class model* M*.*

Proof. In this proof we will implicitly identify $\prod_U V$ with its Mostowski collapse. So for example, we will talk about whether some $f\colon \kappa \to V$ has the property that $[f]$ is an ordinal, we mean that the image of $[f]$ under the collapse is an ordinal. By Łos's theorem, $\prod_U V \vDash [f]$ is an ordinal iff $\{\alpha\colon f(\alpha)$ is an ordinal$\} \in U$, and $[f]$ is an ordinal iff $\prod_U V \vDash [f]$ is an ordinal by Δ_0 absoluteness.

First, we claim that $f(\beta) = \beta$ for all $\beta < \kappa$. This is by transfinite induction. $j(0) = 0$ since j is an elementary embedding, and 0 is definable in both V and M as the least ordinal. Now fix $\beta < \kappa$ and suppose that for all $\alpha < \beta$, $j(\alpha) = \alpha$. Then since $\alpha < \beta$ implies $j(\alpha) < j(\beta)$, and $j(\alpha) = \alpha$, we have $j(\beta) > \alpha$ for all $\alpha < \beta$, and so $j(\beta) \geq \beta$. We need to show $j(\beta) \leq \beta$. Suppose $f\colon \kappa \to V$ was such that $[f] < j(\beta)$. It suffices to show that $[f] = \alpha$ for some $\alpha < \beta$. Now since $j(\beta)$ is the constant function β, by definition of the ϵ relation in $\prod_U M$, we have that $X = \{\alpha\colon f(\alpha) < \beta\} \in U$. Now for each $\gamma < \beta$, let $X_\gamma = \{\alpha\colon f(\alpha) = \gamma\}$. Now $X = \bigcup_{\gamma<\beta} X_\gamma$, so since U is κ-complete, some X_γ must be in U. For this $X_\gamma \in U$, we have $\{\alpha\colon f(\alpha) = \gamma\} \in U$, and so $[f] = j(\gamma) = \gamma < \beta$.

Now we show that $j(\kappa) > \kappa$. Consider the function $d\colon \kappa \to \kappa$ where $d(\alpha) = \alpha$. Then $[d]$ is an ordinal, and $[d] > \alpha$ for each $\alpha < \kappa$, since d is eventually greater than α, and a set of size less than κ cannot be in U. So $[d] \geq \kappa$. However, $[d] < j(\kappa)$, since $d(\alpha) < \kappa$ for each $\alpha \in \kappa$. Hence, $j(\kappa) > \kappa$. □

Exercise 23.9. *Suppose that U is a normal κ-additive ultrafilter on a measurable cardinal κ. (Normal in the sense of Section 12, and hence by Exercise 14.17 it extends the club filter). Then show that for the function d defined in the proof of Lemma 23.8, $[d] = \kappa$. Further, U is normal if and only if $[d] = \kappa$.*

The converse of the above lemma is true.

Exercise 23.10. *If there is a nontrivial elementary embedding $j\colon V \to M$ where M is a transitive class model, then there is a measurable cardinal. [Hint: First prove that if $j(\alpha) = \alpha$ for all α, then j is trivial. To show this, argue that if $\operatorname{rank}(j(x)) = \operatorname{rank}(x)$ for all $x \in V$, then by induction on rank, we'd have $j(x) = x$ for all x. Now let κ be the least number such that $j(\kappa) \neq \kappa$. For each subset X of κ, put $X \in U$ iff $\kappa \in j(X)$. Show that U is a κ-complete ultrafilter.]*

Now we see measurable cardinals cannot exist in L.

Theorem 23.11 (Scott)**.** *If there is a measurable cardinal, then $V \neq L$.*

Proof. Suppose there is a measurable cardinal. Let κ be the least measurable cardinal, and let U be a nonprincipal κ-complete ultrafilter on κ. Let $M = \prod_U V$, and let $j\colon V \to M$ be the corresponding elementary embedding.

If $V = L$, then the only transitive class model containing all the ordinals is L, so $V = M = L$. But $V \vDash \kappa$ is the least measurable cardinal, and since j is an elementary embedding $M \vDash j(\kappa)$ is the least measurable cardinal, and by Lemma 23.8, $j(\kappa) > \kappa$. So we cannot have $M = V$. Contradiction! □

23.1 Finding right universe of set theory*

L is an incredibly important object of study for set theorists. It reveals a huge amount about ZFC. It has an intricate structure theory with myriad uses.

But if your goal is to find the "right" platonic realm of sets, L is a terrible universe in which to live. We understand L well enough to answer most all questions of set theory inside it. But the answers are almost always myopic. L is like the tiny town where I grew up; people refused to acknowledge anything outside the town borders and nothing interesting ever happened.

The standout deficiency of L is that $V = L$ implies there are no measurable cardinals. Large cardinals are an incredibly important part of set theory and are intimately tied to most all its branches. From extensive work, we have deep intuitions about how they behave, we have detailed fine-structural descriptions of models containing them (from the inner model theory of large cardinals), and myriad connections and uses of them which touch many other areas of mathematics. The consistency of these large cardinal axioms (as opposed to the set-theoretic statement that they exist) are also Π^0_1 statements which make predictions about reality (we'll never find a proof of their inconsistency), which have held true for more than a century. Since set theorists strongly believe these large cardinal axioms are consistent for so many reasons, living in a universe where they don't exist feels very wrong.

An important focus of modern set theory is the inner model program: finding L-like models which contain large cardinals, and help us understand their fine structure, consistency strength, and what can be forced using them.

A research programme of Hugh Woodin is to build an inner model called Ultimate L. It would help us answer the technical questions about forcing and large cardinals alluded to above. But in addition, Woodin's vision is for there to be a convincing philosophical argument that this is the "right" universe of sets, and that mathematicians should add the axiom $V = \text{Ultimate } L$ to ZFC.

24 The basics of forcing

Forcing was introduced by Cohen in order to prove that CH is independent of ZFC; if there is a model of ZFC, then there is a model of $\mathsf{ZFC} + \neg\mathsf{CH}$. Forcing is a way of taking a countable transitive model M of ZFC, and constructing from it a "generic extension" $M[G]$ which will be another countable transitive model of ZFC which contains the same ordinals. The model $M[G]$ will contain M as a submodel, M will contain a set G where $G \in V$, but $G \notin M$, and $M[G]$ will contain all the other sets we can reasonably define from M and G.

By way of analogy, we can think of taking a field F, and forming an algebraic extension of it. Our original field F must be missing roots of some polynomials which exist in the extension, and we can understand possible algebraic field extensions of F by studying polynomials in F. Similarly, any transitive set model M of ZFC is missing some sets (since M is not a proper class) and we can add some of these missing sets and then understand the model $M[G]$ by analyzing it from the perspective of M.

We can't adjoin just any set G to a countable transitive model M and get a model of ZFC. Indeed, it is consistent that there are a transitive set model M of ZFC and a set X so that there is no transitive set model N of ZFC so $M \subseteq N$ and $X \in N$. In forcing, we require G to have good "approximations" inside M, and we also require that G is sufficiently "generic". To make a forcing extension of a countable transitive model M, we need to choose a partial order $(\mathbb{P}, \leq)$ to "guide" our construction. We will always assume our partial order has a maximal element which we denote $1_{\mathbb{P}}$. The set G will be a subset of $\mathbb{P}$ that always includes $1_{\mathbb{P}}$.

To define $M[G]$, we'll use the concept of $\mathbb{P}$-names. Every element of $M[G]$ will have a "name" in M.

Definition 24.1. *Fix a partial order $(\mathbb{P}, \leq)$. By recursion, say that a set τ is a $\mathbb{P}$-name if every element of τ is an ordered pair (σ, p) where σ is a $\mathbb{P}$-name, and $p \in \mathbb{P}$.*

Alternatively, we could define a hierarchy of $\mathbb{P}$-names, where $V_0^{\mathbb{P}} = \emptyset$, $V_{\alpha+1}^{\mathbb{P}} = V_\alpha^{\mathbb{P}} \times \mathbb{P}$, and $V_\lambda^{\mathbb{P}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{P}}$, and $V^{\mathbb{P}} = \bigcup_\alpha V_\alpha^{\mathbb{P}}$ is the class of $\mathbb{P}$-names. Note that being a $\mathbb{P}$-name is absolute. So if M is a transitive model which contains $\mathbb{P}$, then the set of $\tau \in M$ such that $M \models$ "τ is a $\mathbb{P}$-name" is equal to $V^{\mathbb{P}} \cap M$.

Now we describe how to interpret these names:

Definition 24.2. *Suppose $\mathbb{P}$ is a partial order and G is a subset of $\mathbb{P}$, and τ is a $\mathbb{P}$-name. Then the* ***value of*** τ ***with respect to*** G*, denoted $\tau[G]$, is defined by recursion as $\tau[G] = \{\sigma[G] \colon (\sigma, p) \in \tau \wedge p \in G\}$. Finally, let $M[G] = \{\tau[G] \colon \tau$ is a $\mathbb{P}$-name and $\tau \in M\}$.*

It is clear that if M is countable, then $M[G]$ is countable; it is the image of M under some function. We prove some basic facts about $M[G]$.

Lemma 24.3. *Suppose M is a transitive model, $(\mathbb{P}, \leq)$ is a partial order in M, and G is a subset of $\mathbb{P}$. Then*

1. *$M[G]$ is transitive,*

2. *$M \subseteq M[G]$,*

3. *$G \in M[G]$, and*

4. *$\mathsf{ORD} \cap M = \mathsf{ORD} \cap M[G]$.*

Proof. (1) To see that $M[G]$ is transitive, suppose $x \in M[G]$ and $y \in x$. Then $x = \tau[G]$ for some $\mathbb{P}$-name $\tau \in M$, so by definition of $\tau[G]$, $y = \sigma[G]$ for some $\mathbb{P}$-name σ. Hence, $y \in M[G]$.

(2) We define a map $x \mapsto \check{x}$ by recursion so that for each $x \in M$, $\check{x}[G] = x$. Let

$$\check{x} = \{(\check{y}, 1_{\mathbb{P}}) \colon y \in x\}.$$

So for example, $\check{\emptyset} = \emptyset$ always takes the value $\emptyset$ no matter what G is. Then by induction on the rank of names,

$$\check{x}[G] = \{\check{y}[G] \colon (\check{y}, 1_{\mathbb{P}}) \in \check{x}\}$$

which is equal to $\{\check{y}[G] \colon y \in x\} = \{y \colon y \in x\} = x$ where the penultimate equality is by our induction hypothesis.

(3) Consider the name $\tau = \{(\check{p}, p) \colon p \in P\}$. Then $\tau[G] = \{\check{p}[G] \colon p \in G\} = \{p \colon p \in G\} = G$.

(4) By induction, it is easy to check $\operatorname{rank}(\tau[G]) \leq \operatorname{rank}(\tau)$. So if τ is a $\mathbb{P}$-name such that $\tau[G] = \alpha$ is an ordinal, then $\operatorname{rank}(\tau[G]) \leq \operatorname{rank}(\tau) \in M$. Hence, M contains an ordinal greater than or equal to α, thus M contains α since M is transitive. So $M[G] \cap \mathsf{ORD} \subseteq M$. We've already proved $M \subseteq M[G]$. Thus, $M \cap \mathsf{ORD} = M[G] \cap \mathsf{ORD}$. $\square$

We get a few other axioms of ZFC:

Lemma 24.4. *Suppose M is a transitive model of ZFC, $\mathbb{P}$ is a partial order in M, and $G \subseteq \mathbb{P}$ is nonempty. Then $M[G]$ satisfies the axioms of infinity, and pairing.*

Proof. We have $\omega \in M[G]$ since M satisfies ZFC, so $\omega \in M$ and $M \subseteq M[G]$. This witnesses the axiom of infinity in $M[G]$.

Pairing is true since if $x, y \in M[G]$, where $x = \tau[G]$ and $y = \sigma[G]$, then $\{x, y\}$ is the evaluation of the name $\{(\tau, 1), (\sigma, 1)\}$. $\square$

Now if $G \subseteq \mathbb{P}$ is an arbitrary set, then we cannot prove that $M[G]$ satisfies ZFC. For example, consider the partial order $\mathbb{P}$ of finite partial functions from $\omega \times \omega$ to 2 ordered under reverse inclusion. Fix a countable transitive model M, and let α be a countable ordinal not in M. Then there is an ordering of ω isomorphic to α, and the finite subsets of the characteristic function of this ordering relation form a filter in $\mathbb{P}$. However, if $M[G]$ satisfies ZFC, then the Mostowski collapse of $\bigcup G$ would be in $M[G]$, but this would be α which is an ordinal not in M contradicting the above lemma that $\mathsf{ORD} \cap M = \mathsf{ORD} \cap M[G]$.

(Indeed, it is consistent that there is no model of ZFC which contains both M and α).

The problem in the previous paragraph is that the G we chose contains too much information. In order to ensure that $M[G]$ is a model of ZFC, we will require that G is M-generic in the following sense:

Definition 24.5. *Suppose $(\mathbb{P}, \leq)$ is a forcing partial order. If $p, q \in \mathbb{P}$, and $p \leq q$, then we say that p **refines** q or p **extends** q. If $p, q \in \mathbb{P}$ we say p and q are **compatible** if there exists r such that r extends p and r extends q. Otherwise, we say p and q are **incompatible**. Finally, we say that a subset F of $\mathbb{P}$ is a **filter** if all $p, q \in F$ are compatible, and $p \in F$ and $r \geq p$ implies $r \in F$. Note that every nonempty filter contains $1_{\mathbb{P}}$. If $(\mathbb{P}, \leq)$ is a partial order, and $X \subseteq \mathbb{P}$ then we say X is **dense** if for every $p \in \mathbb{P}$ there is some $q \in X$ such that $q \leq p$. If M is a transitive model that contains $\mathbb{P}$, a filter $G \subseteq \mathbb{P}$ is M**-generic** if for every dense set $D \subseteq \mathbb{P}$, we have $G \cap D \neq \emptyset$.*

M-genericity is a sort of Murphy's law: anything that can happen will happen. We'll use this type of genericity to show we can compute what's true about $M[G]$ internally in M, and show that $M[G]$ satisfies the axioms of ZFC.

It is important that M is countable in order to guarantee that M-generic filters exist.

Lemma 24.6. *Suppose M is a countable transitive model of ZFC, and $\mathbb{P}$ is a forcing poset such that $\mathbb{P} \in M$, and $p \in P$. Then there exists an M-generic filter G such that $p \in G$.*

Proof. Let $\langle D_n \colon n \in \omega \rangle$ enumerate the dense subsets of $\mathbb{P}$ that are in M. Since M is countable there are only countably many such D_n. Now inductively define a sequence $\langle p_n \colon n \in \omega \rangle$ where $p_0 = p$, and $p_{n+1} \leq p_n$ is such that $p_{n+1} \in D_n$. Such a p_{n+1} exists since D_{n+1} is dense. Finally, let G be $\{q \colon \exists n q \geq p_n\}$. Then G is an M-generic filter. $\square$

The above lemma fails in general if M is not countable. For example, suppose $\mathbb{P} \in M$ is **non-atomic**, meaning for every $p \in \mathbb{P}$ there exist incompatible q, r such that $q \leq p$ and $r \leq p$. Then for every filter $F \subseteq \mathbb{P}$, $D_F = \{p \colon p \notin F\}$ is a dense set. However, there cannot be any filter F meeting all of these dense sets, since F doesn't meet D_F. So if $\mathcal{P}(\mathbb{P})^M = \mathcal{P}(\mathbb{P})^V$, then there do not exist any M-generic filters.

In general, you should think of the elements of $\mathbb{P}$ as being "approximations" to the generic filter G we want to add to M. In this way, from the forcing poset $\mathbb{P}$ that we force with we can readily get an understanding of the generic G that we will add. For example,

- $\mathbb{P}_{\text{Cohen}}$ is the partial order of all elements of $2^{<\omega}$; functions from n to 2 for some natural number n, ordered by reverse inclusion. Here the empty function is the maximal element of $\mathbb{P}_{\text{Cohen}}$. We think of such a finite partial function as an "approximation" to a total function from $\omega \to \{0, 1\}$. And indeed, Cohen forcing adds an infinite binary sequence to a countable model M of ZFC. (Since M is countable, it is missing uncountably many such sequences).

- The random poset $\mathbb{P}_{\text{random}}$ consists of all closed subsets of $[0,1]$ of positive Lebesgue measure, ordered by inclusion. We think of these sets as being approximations to their intersection which will be a single "random" real number.

- The Lévy collapse $\mathbb{P}_{\mathrm{Col}(\kappa,\omega)}$ of κ to ω is the partial order of finite partial functions from ω to κ, ordered by extension. We think of an element of this partial order as an approximation to a single function from ω to κ. A generic for this poset will give a total function from ω to κ. For each $\alpha \in \kappa$, the set of finite partial functions p so that $\alpha \in \mathrm{ran}(p)$ is a dense set. Hence, a generic such function will be onto. Of course if M is a countable transitive model, and $\kappa \in M$, then κ is really countable, and hence in the real universe there will be some function from ω onto κ.

- The poset for "shooting a club through a stationary set $S \subseteq \omega_1$" is the set of all closed countable sequences in S ordered under reverse inclusion. We think of elements of this poset as approximating a closed subset of ω_1.

25 Forcing = Truth

For this section, fix a transitive model M of ZFC and a forcing poset $\mathbb{P} \in M$. Our next goal is to introduce the forcing relation which gives us a way of understanding what will be true in the generic extension $M[G]$. We'll want to understand when $M[G] \models \varphi(x_1, \ldots, x_n)$ for a formula φ. Each $x_i \in M[G]$ comes from some $\mathbb{P}$-name τ_i in M, so we can rewrite this as $M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G])$. We will show there is a relation definable inside M called the forcing relation and and noted $\Vdash^M$ so that

$$M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G]) \text{ iff } (\exists p \in G) p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$$

The generic G is not an element of M, but each element of $\mathbb{P}$ is, and so we can understand part of what can possibly be true in $M[G]$ using the forcing relation inside M. The forcing relation $\Vdash^M$ depends on the poset $\mathbb{P}$ we choose and if we want to emphasize the particular poset we use we use the notation $\Vdash^M_{\mathbb{P}}$.

We caution that we think of $\varphi(\tau_1, \ldots, \tau_n)$ as a formal expression, and we do not interpret it using the usual relations $\in$ and $=$. Technically it is part of what is called the **forcing language**, and we think of this expression as something that will fully gain meaning once we obtain a generic G and interpret each of the names $\tau_1, \ldots, \tau_n$. Furthermore, from more and more partial information about G we will be able to discover more and more of what is true about these types of sentences.

For example, let $\tau_1 = \emptyset$ (which is the canonical name for $\emptyset$, and $\tau_2 = \{(\emptyset, p)\}$. Then if p is an element of the generic G, then $\tau_2[G] = \{\emptyset\}$, and so $\tau_1[G] \in \tau_2[G]$. In this way, p being in the generic G "forces" that $\tau_1[G] \in \tau_2[G]$. If however, p is not an element of the generic G, then $\tau_2[G] = \emptyset$ and so $\tau_1[G] \in \tau_2[G]$. Hence, if $q \in G$ and q is incompatible with p (so p cannot be in G), then q "forces" $\tau_1[G] \notin \tau_2[G]$. Note that the set $X = \{q \colon q \leq p \text{ or } q \text{ is incompatible with } p\}$ is dense, and so if G is M-generic, then it meets this set, and so we must either force whether or not $\tau_1[G] \in \tau_2[G]$ in the above way.

We begin with a useful characterization of M-genericity:

Definition 25.1. *If $(\mathbb{P}, \leq)$ is a partial order, and $X \subseteq \mathbb{P}$ then we say X is **dense below** p if for all $q \leq p$, there is some $r \leq q$ such that $r \in X$.*

We more often deal with sets that are dense below a condition p in our definitions (e.g. of the forcing relation). By the following lemma, this type of density can also characterize when G is M-generic.

Lemma 25.2. *A filter $G \subseteq \mathbb{P}$ is M-generic if and only if for every set $X \in M$ with $X \subseteq \mathbb{P}$, if $p \in G$ and X is dense below p, then G meets X.*

Proof. The reverse implication follows since if X is dense, it is dense below $1_{\mathbb{P}}$, and every generic filter contains $1_{\mathbb{P}}$..

The forward implication is since if X is dense below p, then $X \cup \{q \in \mathbb{P} \colon q \text{ is incompatible with } p\}$ is dense and it is Δ_1 definable and hence in M, so G must meet X. □

Note also that if X is dense below p and $q \leq p$, then X is dense below q.

Definition 25.3 (The forcing relation for atomic formulas). *The forcing relation $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ is defined between elements $p \in \mathbb{P}$, formulas φ, and n-tuples of names as follows. For atomic formulas, we define $p \Vdash^M \tau \in \sigma$ and $p \Vdash^M \tau = \sigma$ by simultaneous induction on the rank of σ and τ.*

- *$p \Vdash^M \tau \in \sigma$ iff $\{p' \in \mathbb{P} \colon (\exists q \geq p')(\exists \sigma' \in \sigma)(\sigma', q) \in \sigma \wedge p' \Vdash^M \tau = \sigma'\}$ is dense below p. Note that the set*
- *$p \Vdash^M \tau = \sigma$ iff for all $(\tau', q) \in \tau$, $\{p' \in \mathbb{P} \colon q \geq p' \to p' \Vdash^M \tau' \in \sigma\}$ is dense below p and for all $(\sigma', q) \in \sigma$, $\{p' \in \mathbb{P} \colon q \geq p' \to p' \Vdash^M \sigma' \in \tau\}$ is dense below p.*

Note that the definition of the forcing relation $\Vdash^M$ for atomic formulas is Δ_1, definable in M, and absolute between M and V. Another important technical fact that follows from this definition is that if $p \leq q$ and $q \Vdash^M \varphi$ for some atomic formula φ, then $p \Vdash^M \varphi$.

Lemma 25.4 (Forcing = Truth for atomic formulas). *Suppose M is a transitive set model of* ZFC *and G is M-generic. Then*

$$M[G] \vDash \tau[G] = \sigma[G] \text{ iff } (\exists p \in G) p \Vdash^M \tau = \sigma.$$

and

$$M[G] \vDash \tau[G] \in \sigma[G] \text{ iff } (\exists p \in G) p \Vdash^M \tau \in \sigma.$$

Proof. We prove these simultaneously by induction on the rank of σ and τ. Suppose G is M-generic.

If $p \Vdash^M \tau \in \sigma$ and $p \in G$, then the set $\{p' \in \mathbb{P} \colon (\exists q \geq p')(\exists \sigma' \in \sigma)(\sigma', q) \in \sigma \wedge p' \Vdash^M \tau = \sigma'\}$ is dense below p, and hence there is some p' in the set such that $p' \in G$, since G is M-generic. Let (σ', q') witness the membership of p' in this set. Then $p' \Vdash^M \tau = \sigma'$, so by our induction hypothesis, $M[G] \vDash \tau[G] = \sigma'[G]$, and since G is a filter, $q \in G$, and so $\sigma'[G] \in \sigma[G]$, so $M[G] \vDash \tau[G] \in \sigma[G]$.

For the converse, suppose $M[G] \vDash \tau[G] \in \sigma[G]$. Then $M[G] \vDash \tau[G] = \sigma'[G]$ for some $\sigma'[G] \in \sigma[G]$. Hence, by our induction hypothesis, there is some p such that $p \Vdash^M \tau = \sigma'$. But then by definition of the forcing relation, $p \Vdash^M \tau \in \sigma$.

Now suppose $p \Vdash^M \tau = \sigma$ and $p \in G$. By symmetry it suffices to show that $M[G] \vDash \tau[G] \subseteq \sigma[G]$. Suppose $(\tau', q) \in \tau$ and $q \in G$, so $\tau'[G] \in \tau[G]$. Let r extend both p and q. Then $\{p' \in \mathbb{P} \colon p' \Vdash^M \tau' \in \sigma\}$ is dense below r, and so G meets this set at some p' where $p' \Vdash^M \tau' \in \sigma$, and hence $M[G] \vDash \tau'[G] \in \sigma[G]$.

Now suppose there is no $p \in G$ such that $p \Vdash^M \tau = \sigma$. The set of p' such that $\exists(\tau', q) \in \tau$ such that $q \geq p'$ and $(\forall r \leq p) r \nVdash^M \tau' \in \sigma$ or $\exists(\sigma', q) \in \sigma$ such that $q \geq p'$ and $(\forall r \leq p) r \nVdash^M \sigma' \in \tau$ must be dense below p, so G must meet this set. Without loss of generality, assume there is some (τ', q) and $p' \in G$ such that $q \geq p'$ and $(\forall r \leq p) r \nVdash^M \tau' \in \sigma$. Then by our induction hypothesis, $M[G] \vDash \tau'[G] \notin \sigma[G]$, and $M[G] \vDash \tau'[G] \in \tau[G]$, so $M[G] \vDash \tau[G] \neq \sigma[G]$. $\square$

Definition 25.5 (The forcing relation). *We define the forcing relation $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ on all formulas by induction as follows:*

- *$p \Vdash^M \varphi(\tau_1, \ldots, \tau_n) \wedge \psi(\sigma_1, \ldots, \sigma_n)$ iff $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ and $p \Vdash^M \psi(\sigma_1, \ldots, \sigma_n)$.*
- *$p \Vdash^M \neg\varphi(\tau_1, \ldots, \tau_n)$ if there is no $q \leq p$ such that $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$.*
- *$p \Vdash^M \exists x\varphi(x, \tau_2, \ldots, \tau_n)$ if the set of $p' \leq p$ such that $(\exists \tau_1 \in M)p' \Vdash^M \varphi(\tau_1, \tau_2, \ldots, \tau_n)$ is dense below p.*

Note that this is a definition scheme, and for each formula φ, the forcing relation is definable, but there is no single definition of the forcing relation for all formulas; the class of $\mathbb{P}$-names is a proper class, and so the complexity of the definition of the forcing relation increases in the Levy hierarchy as the complexity of φ increases.

The only place where we use have used M in the definition of the forcing relation is when we are quantifying over names in M in the clause of the forcing relation for an existential quantifier. In this sense, $\Vdash^M$ really is the realizvization of the definition of the full forcing relation $\Vdash$, where we restrict all the quantfiers to be over M. Often we think of working inside the model M and in this case when we just write $\Vdash$ instead of writing $\Vdash^M$.

Lemma 25.6 (Forcing = Truth)**.** *Suppose M is a transitive set model of ZFC, G is M-generic, and φ is a formula. Then*

$$M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G]) \text{ iff } (\exists p \in G)p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$$

Proof. We prove this by induction on formula complexity. By Lemma 25.4, the lemma is true for atomic formulas.

For conjunction,

$$\begin{aligned}
&M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G]) \wedge \psi(\sigma_1[G], \ldots, \sigma_n[G]) \\
&\leftrightarrow M \models \varphi(\tau_1[G], \ldots, \tau_n[G]) \text{ and } M \models \psi(\sigma_1[G], \ldots, \sigma_n[G]) \\
&\leftrightarrow (\exists p_1 \in G)p_1 \Vdash^M \varphi(\tau_1, \ldots, \tau_n) \text{ and } (\exists p_2 \in G)p_2 \Vdash^M \psi(\sigma_1, \ldots, \sigma_n) \\
&\leftrightarrow (\exists p \in G)p \Vdash^M \varphi(\tau_1, \ldots, \tau_n) \wedge \psi(\sigma_1, \ldots, \sigma_n).
\end{aligned}$$

The last equivalence here is since given any $p_1, p_2 \in G$, there is some common refinement $p \leq p_1$ and $p \leq p_2$, since G is a filter.

For negation, $M[G] \models \neg\varphi(\tau_1[G], \ldots, \tau_n[G])$ iff it's not the case that $M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G])$ iff there is no $p \in G$ such that $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$. We need to show this is equivalent to $p \Vdash^M \neg\varphi(\tau_1, \ldots, \tau_n)$.

If there is some $p \in G$ so that no $q \leq p$ has $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$, then clearly we cannot have $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ for any $q \in G$. For the reverse direction suppose there is no $p \in G$ such that $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$. Consider the set X of q such that $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ or $q \Vdash^M \neg\varphi(\tau_1, \ldots, \tau_n)$. The set X is dense by the definition of the forcing relation of a negation, and so G meets X, hence there must be some $q \in G$ such that $q \Vdash^M \neg\varphi(\tau_1, \ldots \tau_n)$, otherwise we would contradict our induction hypothesis that $M[G] \models \varphi(\tau_1[G], \ldots, \tau_n[G])$ iff $(\exists p \in G)p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$.

We leave the existential case as an exercise. $\square$

Lemma 25.7. *Suppose M is a transitive model and G is M-generic. Then $M[G] \vDash \mathsf{ZFC}$.*

Proof. We begin by showing that separation holds. Suppose $\varphi(x, w_1, \ldots, w_n)$ is a formula, and $\tau_1, \ldots, \tau_n$ and σ are names. We want to show $M[G] \vDash \exists y \forall z(z \in y \leftrightarrow z \in \sigma[G] \wedge \varphi(z, w_1, \ldots, w_n)$. We construct a name ρ which well be a witness for the y in this existential statement. Consider the name $\nu = \{(\sigma', p) \in \sigma \colon p \Vdash^M \varphi(\sigma', \tau_1, \ldots, \tau_n)\}$. Then if G is M-generic, then by the forcing = truth lemma, for each $(\sigma', p) \in \sigma$, $\sigma'[G] \in \nu[G]$ iff $M[G] \vDash \varphi(\sigma'[G], \tau_1[G], \ldots, \tau_n[G])$.

We leave the remaining axioms as exercises. □

Exercise 25.8. *Suppose M is a model of* ZFC*, and G is M-generic. Then show that $M[G]$ satisfies replacement, and the axiom of choice.*

Exercise 25.9. *Suppose φ is a formula, $\tau_1, \ldots, \tau_n$ are $\mathbb{P}$-names,*

1. *If $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ and $q \leq p$, then $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$.*
2. *If the set of $q \leq p$ such that $q \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ is dense below p. Then $p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$.*

Exercise 25.10. *Suppose M is a countable transitive model of* ZFC*, and $\mathbb{P} \in M$ is a forcing poset.*

1. *$p \Vdash^M \varphi(\tau_1, \ldots, \tau_n)$ if and only if for every M-generic G such that $p \in G$, we have $M[G] \vDash \varphi(\tau_1[G], \ldots, \tau_n[G])$.*
2. *The forcing relation in M is deductively closed. If $p \Vdash^M \varphi$, and from φ we can prove ψ, then $p \Vdash^M \psi$.*

26 The consistency of $\neg$CH

We'll prove in this section that if there's a countable transitive model of ZFC, then there is a countable transitive model of $\mathsf{ZFC} + \neg CH$.

Definition 26.1. *Suppose $\mathbb{P}$ is a poset. We say that $\mathbb{P}$ is ccc (has the countable chain condition) if every antichain (a set of pairwise incompatible elements) in $\mathbb{P}$ is countable.*

This should be called the countable antichain condition, but there is a very long history of calling it the "countable chain condition" so we perpetuate this badly chosen name.

Theorem 26.2 (ccc forcing preserves cardinals)**.** *Suppose M is a countable transitive model of* ZFC*, $\mathbb{P} \in M$ is a forcing poset, G is an M-generic $\mathbb{P}$-filter, and $M \models$ "$\mathbb{P}$ is a ccc poset". Then for every ordinal $\kappa \in M$, $M[G] \models$ "κ is a cardinal" iff $M \models$ "κ is a cardinal".*

Proof. Since $M \subseteq M[G]$ are both transitive, if $M[G] \models$ "κ is a cardinal", then $M \models$ "κ is a cardinal" by downwards absoluteness since being a cardinal is Π_1.

Suppose κ is an infinite cardinal, $M \models$ "κ is a cardinal", and for a contradiction, suppose that there is some $f \in M[G]$ and $\alpha < \kappa$ so $M[G] \models$ "f is a surjection from α to κ". Let $\dot{f}$ be a $\mathbb{P}$-name for f, so $\dot{f}[G] = f$, and let p_0 be such that $p_0 \Vdash^M$ "$\dot{f}$ is a surjection from $\check{\alpha}$ to $\check{\kappa}$". Such a p_0 exists since forcing = truth.

For each $\beta < \alpha$, there is some $\gamma \in \kappa$ such that $M[G] \models f(\beta) = \gamma$, and hence some $p \leq p_0$ such that $p \Vdash^M \dot{f}(\check{\beta}) = \check{\gamma}$.

From now on, work inside M. For each $\beta < \alpha$, let X_β be the set of $\gamma < \kappa$ such that there is some $p \leq p_0$ such that $p \Vdash \dot{f}(\check{\beta}) = \check{\gamma}$. Let $\check{X}_\beta = \{(\check{\gamma}, 1_{\mathbb{P}}) \colon \gamma \in X_\beta\}$ be the set of canonical names $\check{\gamma}$ for the elements of X_β.

Claim: $p_0 \Vdash \dot{f}(\check{\beta}) \in \check{X}_\beta$. If this was not true, then by Exercise 25.9 there would be some $q \leq p_0$ such that no $q' \leq q$ has $q' \Vdash \dot{f}(\check{\beta}) \in \check{X}_\beta$. Take a generic G' extending q (which exists by Lemma 24.6). By our choice of q, there can be no $p \in G'$ such that $p \Vdash \dot{f}(\check{\beta}) \in X_\beta$, so by forcing = truth, $M[G'] \models f(\beta) \notin X_\beta$. Since $p_0 \in G'$, $M[G'] \models$ f is a surjection from α to κ, hence $M[G'] \models f(\beta) = \gamma'$ for some $\gamma \in \kappa$. So by forcing = truth, there must be some p' such that $p' \Vdash f(\beta) = \gamma$. Since p' and p_0 are compatible, there is some p extending both of them, so $p \Vdash f(\beta) = \gamma$. But then $\gamma \in X_\beta$ by definition, which is a contradiction). This proves the claim.

A similar proof shows $p_0 \Vdash \operatorname{ran}(\dot{f}) \subseteq \bigcup_{\beta<\alpha} \check{X}_\beta$.

Claim: $M \models X_\gamma$ is countable. Still working inside M, for each $\gamma \in X_\beta$, by the axiom of choice, we can pick some p_γ such that $p_\gamma \Vdash \dot{f}(\check{\beta}) = \check{\gamma}$. Then for any two distinct $\gamma, \gamma' \in X_\beta$, we must have that p_γ, p'_γ are incompatible, since p_0 forces f to be a function, and hence we can't have two compatible conditions forcing two different values of $f(\beta)$. Thus, since $\mathbb{P}$ has the ccc in M, we have that $M \models X_\gamma$ is countable. But in M, $\bigcup_{\beta<\alpha} X_\beta$ therefore has cardinality at most $\omega \cdot |\alpha| < \kappa$, and hence $\bigcup_{\beta<\alpha} X_\beta$ is a proper subset of κ. This contradicts that p_0 forces that $\operatorname{ran}(f) = \kappa$. $\square$

Lemma 26.3. *Let $\mathbb{P}$ be the set of finite partial functions from $\omega \times \omega_2 \to 2$ ordered under reverse inclusion, so $p \leq_{\mathbb{P}} q$ if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and for all x in $\mathrm{dom}(q)$, $q(x) = p(x)$. $\mathbb{P}$ has the ccc.*

Proof. Suppose A is an uncountable set of conditions in $\mathbb{P}$. We claim these conditions cannot all be incompatible.

Let $D = \{\mathrm{dom}(p) \colon p \in A\}$ be the set of domains of the conditions in A. Since for each possible finite $d \subseteq \omega \times \omega_2$, there are only finitely many functions from d to 2, we must have that D is also uncountable. By the Δ-system lemma, there is a uncountable subset $D' \subseteq D$ which is a Δ-system with root r. Since there are only finitely many functions from r to 2, there must be some $q \colon r \to 2$ such that there are uncountably many $p \in A$ such that $p \upharpoonright r = q$. But any two such p must be compatible in $\mathbb{P}$, since their union is a common extension of them both. $\square$

Theorem 26.4. *Suppose there is a countable transitive model M of ZFC. Then there is a countable transitive model of $\mathsf{ZFC} + \neg\mathsf{CH}$.*

Proof. Working inside M, let $\mathbb{P}$ be the partial order of finite partial functions from $\omega \times \omega_2 \to 2$ ordered under reverse inclusion. Let G be an M-generic filter for $\mathbb{P}$. By the previous two lemmas, $M[G]$ has the same cardinals as M, so $\omega_1^{M[G]} = \omega_1^M$ and $\omega_2^{M[G]} = \omega_2^M$. We claim that $M[G] \vDash$ "there is an injection from ω_2 to $\mathcal{P}(\omega)$, and hence $|\mathcal{P}(\omega)|^{M[G]} > \omega_1$, so $M[G] \vDash \neg\mathsf{CH}$.

Working inside M, for each $(n, \alpha) \in \omega \times \omega_2$, the $D_{(n,\alpha)} = \{p \in \mathbb{P} \colon (n, \alpha) \in \mathrm{dom}(p)\}$ is is dense in $\mathbb{P}$. To see this, given any $q \in \mathbb{P}$, let $p \leq q$ be defined by $p = q$ if $(n, \alpha) \in \mathrm{dom}(q)$, and $p = q \cup ((n, \alpha), 0)$ otherwise. Now G meets each of these dense sets $D_{(n,\alpha)}$. So in $M[G]$, if we let $g = \bigcup G$, then g is a function from $\omega \times \omega_2 \to 2$.

In $M[G]$, we claim that the function f defined by $f(\alpha) = \{n \colon g(n, \alpha) = 1\}$ is an injection from $\omega_2^{M[G]}$ to $\mathcal{P}(\omega)$. Working in M, suppose $\alpha, \beta < \omega_2$. Then the set of $p \in \mathbb{P}$ such that there exists some n such that $(n, \alpha) \in \mathrm{dom}(p) \wedge (n, \beta) \in \mathrm{dom}(p) \wedge p(n, \alpha) \neq p(n, \beta)$ is dense by a similar argument to the above. Hence, G meets this dense set, and so $M[G] \vDash f(\alpha) \neq f(\beta)$ for all $\alpha, \beta < \omega_2^M = \omega_2^{M[G]}$. $\square$

Exercise 26.5. *Assume M in the above theorem has $M \vDash \mathsf{CH}$. Show that $M[G] \vDash 2^\omega = \omega_2$. Show that if we replace ω_2 with any regular cardinal $\kappa \in M$ of uncountable cofinality in M, then $M[G] \vDash 2^\omega = \kappa$.*

Now as we proved in an exercise, $\mathrm{Con}(\mathsf{ZFC})$ does not prove that there is a countable transitive model of ZFC. However, by the reflection theorem and Lowenheim-Skolem, $\mathrm{Con}(\mathsf{ZFC})$ does prove that for any finite set S of axioms of ZFC, there is a countable transitive model of S. Now let S be an arbitrarily large finite set that includes all the finitely many axioms of ZFC that we have used in these notes to prove the above theorem. Then the above theorem shows there is a model of $S + \neg\mathsf{CH}$, for arbitrarily large finite $S \subseteq \mathsf{ZFC}$. By the compactness theorem, this implies $\mathrm{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$.

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