Quantum algorithms for eigenvalue problems

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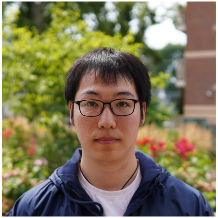
Joint work with



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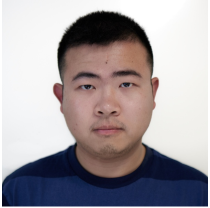

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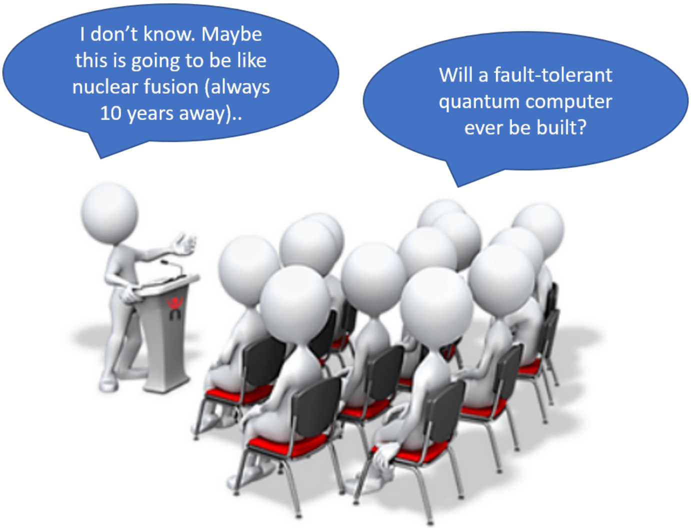
Outline

What is a quantum computer (mathematically)

- $|\psi\rangle \in \mathbb{C}^N \cong (\mathbb{C}^2)^{\otimes n}$, $N = 2^n$. n : number of qubits.
- Normalization condition $\langle\psi|\psi\rangle = \sum_{j=0}^{N-1} |\psi_j|^2 = 1$.
- Quantum gate: unitary matrix $U \in \mathbb{C}^{N\times N}$. For some U, application $U|\psi\rangle$ is efficient: cost is $\mathcal{O}(\mathrm{polylog}(N))$.
- Quantum algorithm: a series of large matrix-vector multiplications: $U_K \cdots U_1 |\psi\rangle$. Then measure some qubits and repeat M times for classical output.
- Quantum cost (roughly): $MK\mathrm{polylog}(N)$.
- Exponential quantum advantage (EQA): if $MK = \mathcal{O}(\mathrm{polylog}(N))$, and classical algorithm scales as $\mathcal{O}(\mathrm{poly}(N))$.

A fast growing industry

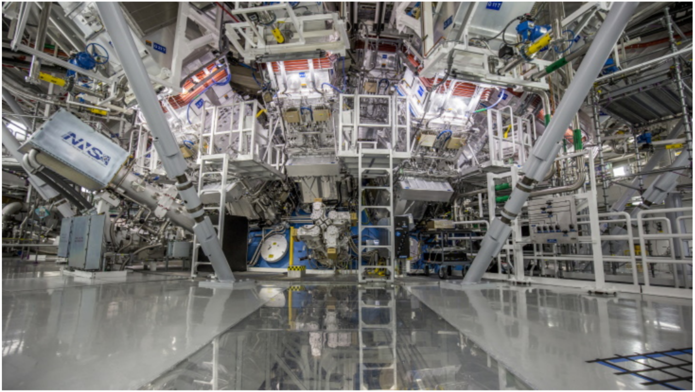

I don't know. Maybe this is going to be like nuclear fusion (always 10 years away)..
Will a fault-tolerant quantum computer ever be built?

US researchers achieve historic fusion ignition

13 December 2022

Share

The first ever controlled fusion experiment to produce more energy from fusion than the laser energy used to drive it was conducted at the National Ignition Facility (NIF) at the Lawrence Livermore National Laboratory (LLNL) on 5 December - a breakthrough that has been decades in the making.



The target chamber at NIF (Image: LLNL)

Quantum numerical linear algebra

- Solving numerical linear algebra problems on a quantum computer. Exciting progress in the past few years.
- Note "Quantum algorithms for scientific computation"[1]
- This talk is about eigenvalue problems:

$$H |\psi_0\rangle = \lambda_0 |\psi_0\rangle$$

  $H \in \mathbb{C}^{N\times N}$ Hermitian matrix (Hamiltonian).
  Find the smallest λ_0 and/or prepare $|\psi_0\rangle$
- One of the most important problems in quantum physics, quantum chemistry and materials science.

[1] arXiv:2201.08309. More frequently updated: https://math.berkeley.edu/~linlin/qasc/

Hamiltonian evolution input model

- Unitary matrix: $U_H = e^{-i\tau H}$ for some τ.
- e.g., $H = \sum_{i=1}^{n} Z_i$, $U_H = \prod_{i=1}^{n} e^{-i\tau Z_i}$ can be implemented with n single qubit gate rotations. Gate complexity is $n = \log_2 N$.
- Approximate implementation $\left\| U_H - e^{-i\tau H} \right\| \leq \epsilon$ via e.g., Trotter expansion is acceptable.
- Long time evolution $e^{-iTH} = U_H^d$.
  Runtime $T = d\tau$. Query depth d.
  Both measure query complexities

Assumptions in this talk

$$H\,|\psi_0\rangle = \lambda_0\,|\psi_0\rangle$$

- Hamiltonian evolution input model: $U_H = e^{-i\tau H}$ for some τ.
- A good[1] initial state $|\phi\rangle = U_I\,|0^n\rangle$, $p_0 = |\langle\phi|\psi_0\rangle|^2 = \Omega(1)$. $|0^n\rangle = |0\rangle^{\otimes n} = (1, 0, \dots, 0)^\top$.
- Ground-state energy estimation: estimate λ_0 to precision ϵ.
- Good initial state is a very strong assumption. But without it, the problem is theoretically intractable in the worst case[2] .
- Focus on methods with performance guarantee. Can be combined with e.g., VQE (prepare good initial state)

[1] Can be theoretically relaxed to $\gamma = \Omega(1/\mathrm{poly}(n))$.

[2] The worst case is QMA-hard, which is a quantum analogue of NP hardness. In other words, the task can be difficult even with a perfect quantum computer.

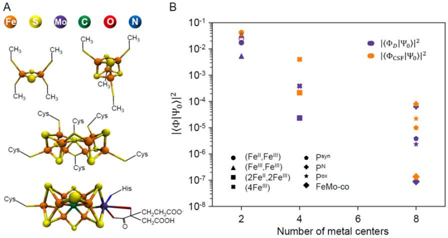
Exponential quantum advantage under debate

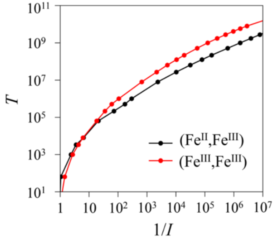
nature communications

Evaluating the evidence for exponential quantum advantage in ground-state quantum chemistry

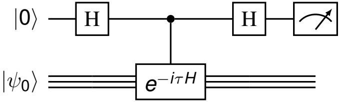
Seunghoon Lee, Joonho Lee, Huanchen Zhai, Yu Tong, Alexander M. Dalzell, Ashutosh Kumar, Phillip Helms, Johnnie Gray, Zhi-Hao Cui, Wenyuan Liu, Michael Kastoryano, Ryan Babbush, John Preskill, David R. Reichman, Earl T. Campbell, Edward F. Valeev, Lin Lin & Garnet Kin-Lic Chan

Nature Communications **14**, Article number: 1952 (2023) | Cite this article





Textbook algorithm 1: Hadamard test



$$\mathrm{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- Readout the success probability

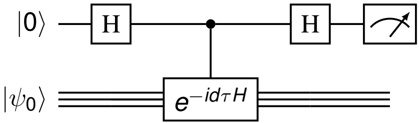
$$p(0) = \frac{1}{2}(1 + \mathrm{Re}\,\langle \psi_0 | e^{-i\tau H} | \psi_0 \rangle) = \frac{1}{2}(1 + \cos(\lambda_0 \tau)).$$

- Maximal runtime $T_{\mathrm{max}} = \tau$ can be arbitrarily small: very short circuit depth. At the expense of larger number of repetitions.

- Monte Carlo algorithm: To reach precision ϵ, the number of repetitions is $\mathcal{O}((\epsilon\tau)^{-2})$. Total runtime: $T_{\mathrm{total}} = \mathcal{O}(\epsilon^{-2}\tau^{-1})$

- Need to prepare exact eigenstate, i.e. $p_0 = |\langle \phi | \psi_0 \rangle|^2 = 1$.

Textbook algorithm 2: Kitaev's algorithm

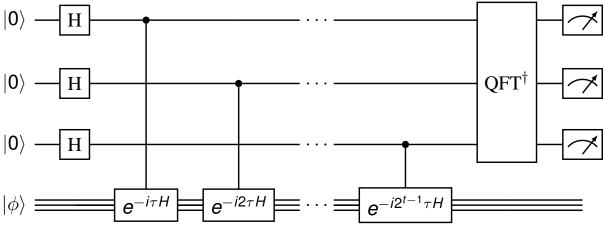


- $d = 1, 2, 4, \ldots, 2^{t-1} = \pi(\epsilon\tau)^{-1}$.
- Maximal runtime $T_{\max} = 2^{t-1}\tau = \pi/\epsilon$.
- Total runtime is $T_{\text{total}} = \widetilde{\mathcal{O}}(\epsilon^{-1})$. Heisenberg-limited scaling[1]
- Need to prepare exact eigenstate, i.e. $p_0 = |\langle\phi|\psi_0\rangle|^2 = 1$.
- Can be modified to accommodate inexact eigenstate[2]
  In this case, $T_{\max} = \mathcal{O}(\epsilon^{-1}p_0^{-1})$, $T_{\text{total}} = \widetilde{\mathcal{O}}(\epsilon^{-1}p_0^{-2})$.

[1] Best scaling allowed by quantum mechanics

[2] The first work is semi-classical QPE, or single ancilla QPE: (Griffiths, Niu, PRL 1996; Higgins et al, Nature 2007)

Textbook algorithm 3: Quantum phase estimation

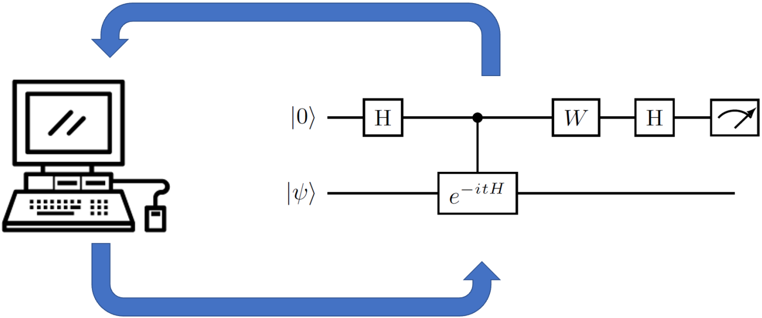


- Use many ancilla qubits.
- Exact eigenstate: $T_{\max} = 2^t\tau = 2\pi/\epsilon$, $T_{\text{total}} = \widetilde{\mathcal{O}}(\epsilon^{-1})$.
- Naturally accommodate inexact eigenstate $p_0 < 1$.
  In this case, $T_{\max} = \mathcal{O}(\epsilon^{-1}p_0^{-1})$, $T_{\text{total}} = \widetilde{\mathcal{O}}(\epsilon^{-1}p_0^{-2})$.

Outline

Classical post-processing of Hadamard test circuit

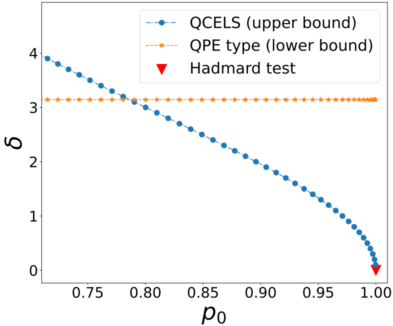
Classical postprocessing



Evolution time, number of measurements

Simple and surprisingly powerful.

Theoretical comparison of preconstants with large p_0



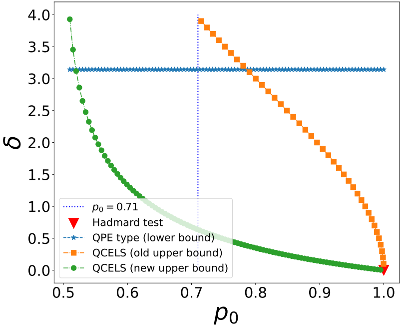
$$T_{\max} = \frac{\delta}{\epsilon}$$

QCELS: quantum complex exponential least squares

(Ding-**Lin**, PRX Quantum 2023)

Further improvement of preconstant

$$\delta = \Theta(\sqrt{1-p_0}) \to \delta = \Theta(1-p_0), \quad T_{\max} = \frac{\delta}{\epsilon}.$$



Earlier bound (Ding-**Lin**, , PRX Quantum 2023); New bound: (Ding-**Lin**, 2303.05714)

See also (Ni-Li-Ying, 2302.0245) for proving robust phase estimation (RPE) satisfies $\delta = \Theta(1 - p_0)$.

Large p_0

Theorem (Ding-Lin, 2211.11973)

If $p_0 > 0.71$, choose

$$\delta = \Theta(\sqrt{1-p_0}).$$

There exists an algorithm that uses 1 ancilla qubit to estimate λ_0 to precision ϵ with

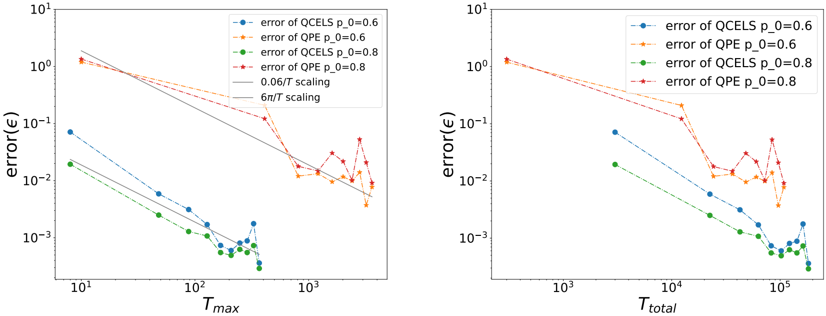
$$T_{\max} = \frac{\delta}{\epsilon}, \quad T_{\text{total}} = \widetilde{\Theta}\left(\delta^{-(1+o(1))}\epsilon^{-1}\right).$$

- Distinct feature: the preconstant δ can be arbitrarily small as $p_0 \to 1$.

(Ding-**Lin**, PRX Quantum 2023)

Numerical results for large p_0

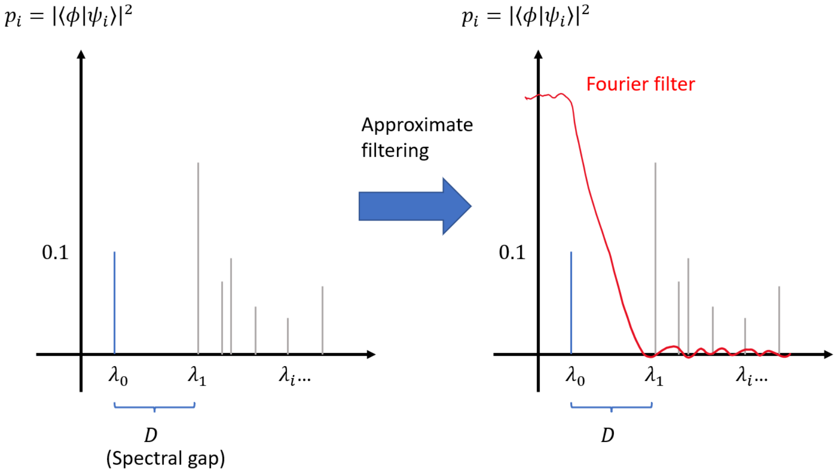
Transverse field Ising model (TFIM)[1]



- Numerical performance is much better than theoretical prediction, and the bound 0.71 can be pushed downward.
- Two order of magnitude reduction of maximal runtime!

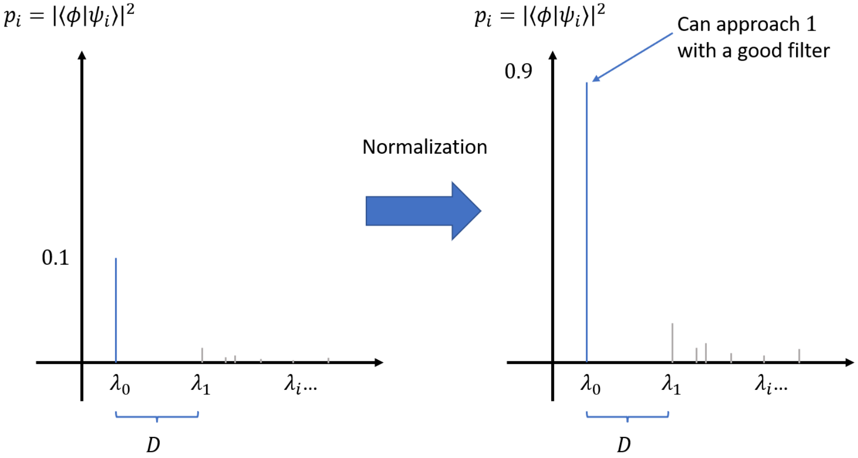
[1] QCELS refers to the multi-level version of quantum complex exponential least squares

Small p_0: convert to large p_0 with a spectral gap



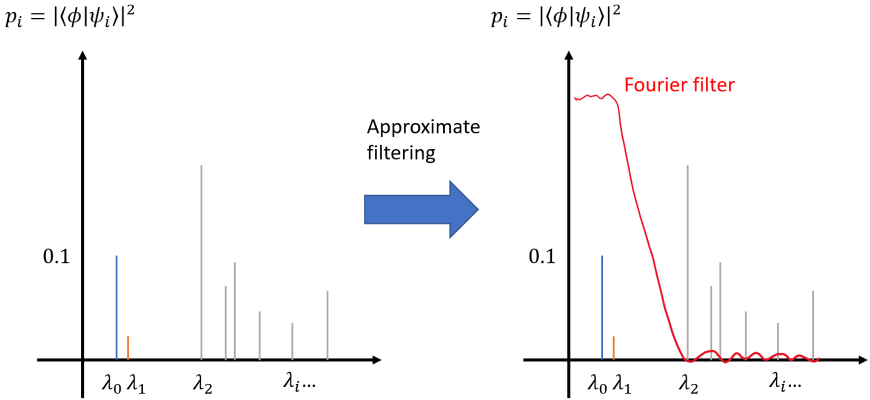
(**Lin**-Tong, PRX Quantum 2022): Effective approximate Fourier filtering algorithm with the Hadamard test circuit

Small p_0: convert to large p_0 with a spectral gap



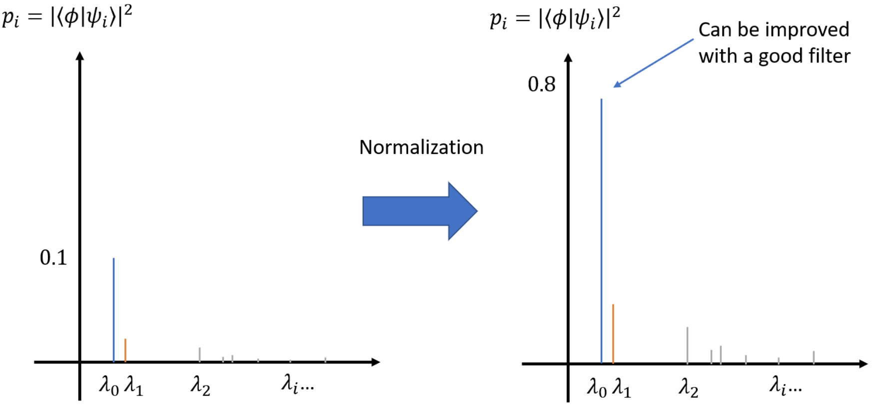
Apply the algorithm for the case of large p_0!

Small p_0: what if the spectral gap is small?



(**Lin**-Tong, PRX Quantum 2022): Effective approximate Fourier filtering algorithm with the Hadamard test circuit

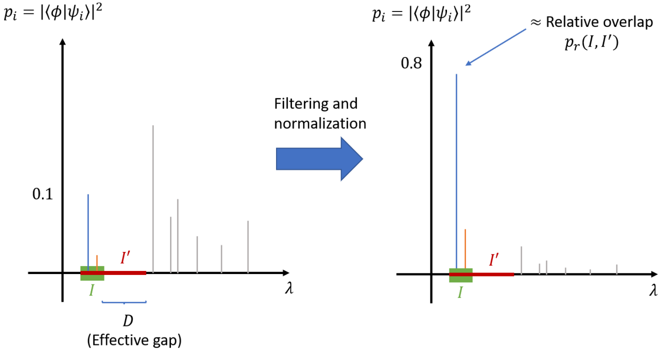
Small p_0: what if the spectral gap is small?



Apply the algorithm for the case of large p_0!

Relative overlap

$$p_r(I, I') = \frac{|\langle\psi|\psi_0\rangle|^2 \, \mathbb{1}_{\lambda_0 \in I}}{\sum_{\lambda_k \in I'} |\langle\psi|\psi_k\rangle|^2}.$$



The concept of relative overlap is applicable to certain small gapped quantum systems, and is aware of the information of the initial state.

Small p_0

Theorem (Ding-Lin, 2211.11973)

Given relative overlap $p_r(I, I') \geq 0.71$, $D = \min_{x_1 \notin I', x_2 \in I} |x_1 - x_2|$, *choose*

$$\delta = \Theta(\sqrt{1 - p_r(I, I')}).$$

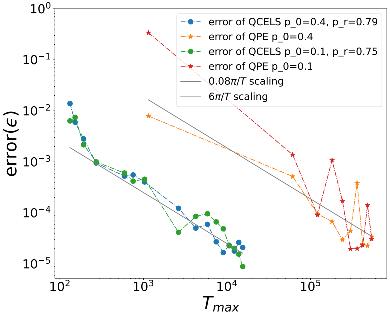
There exists an algorithm that uses 1 ancilla qubit to estimate λ_0 *to precision* ϵ *with*

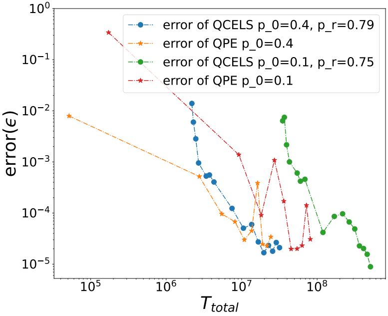
$$T_{\max} = \widetilde{\Theta}(D^{-1}) + \delta/\epsilon, \quad T_{\text{total}} = \widetilde{\Theta}\left(p_0^{-2}\delta^{-(2+o(1))}\left(D^{-1} + \delta/\epsilon\right)\right).$$

- Distinct feature: can use the information of relative overlap (all previous algorithms are agnostic to it)
- Reduce circuit depth when $D \gg \epsilon$ and $p_r(I, I')$ is large.

Numerical results for small p_0

Hubbard model





Two order of magnitude reduction of maximal runtime!

Criterion for comparing quantum algorithms

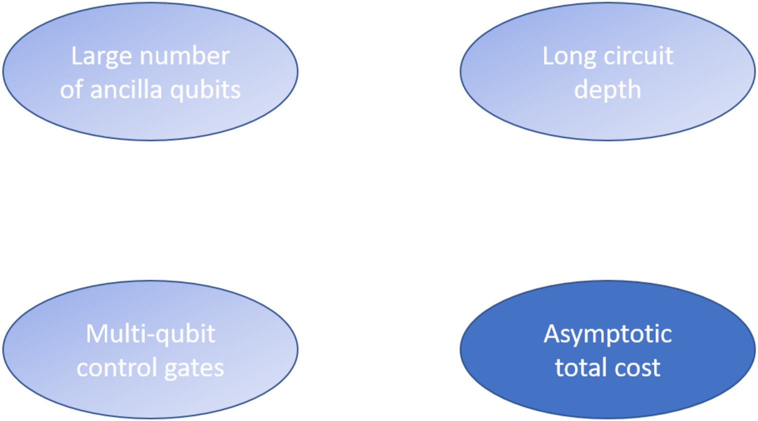
Number of ancilla qubits

Circuit depth

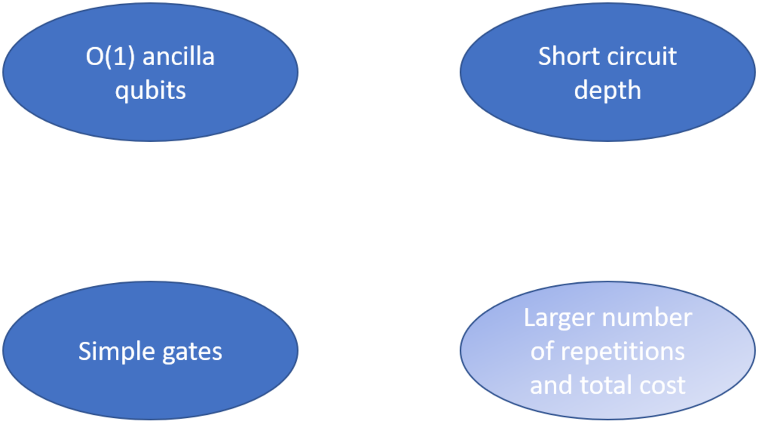
Gate set

Total cost

Full fault-tolerant quantum computer



Early fault-tolerant quantum computer



Eventually, lead to a small non-Clifford (Toffoli/T) gate count.

Progresses for ground-state energy estimation

	Maximal runtime	Total runtime	# ancilla qubits	Need MQC?	Input model
QPE (high confidence)	$\widetilde{\mathcal{O}}(\epsilon^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-2})$	$\mathcal{O}(\text{polylog}(\gamma^{-1}\epsilon^{-1}))$	High	HE
QPE (1 ancilla)	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-2})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-4})$	$\mathcal{O}(1)$	No	HE
Som19 (short depth)	$\widetilde{\mathcal{O}}(\epsilon^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-4}\gamma^{-4})$	$\mathcal{O}(1)$	No	HE
GTC19	$\widetilde{\mathcal{O}}(\epsilon^{-3/2}\gamma^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-3/2}\gamma^{-1})$	$\mathcal{O}(\log(\epsilon^{-1}))$	High	HE
LT20*	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-1})$	$m+\mathcal{O}(\log(\epsilon^{-1}))$	High	BE
LT22 (short depth)	$\widetilde{\mathcal{O}}(\epsilon^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-4})$	$\mathcal{O}(1)$	No	HE
DLT22 (short depth)	$\widetilde{\mathcal{O}}(\epsilon^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-2})$	$\mathcal{O}(1)$	No	HE
DLT22*	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-1})$	$\widetilde{\mathcal{O}}(\epsilon^{-1}\gamma^{-1})$	$\mathcal{O}(1)$	Low	HE
DL22 (even shorter depth)$^\diamond$	$\widetilde{\mathcal{O}}(D^{-1})+\frac{\delta}{\epsilon}$	$\widetilde{\mathcal{O}}((D^{-1}+\delta/\epsilon)\,\gamma^{-4})$	$\mathcal{O}(1)$	No	HE

Initial guess $p_0 = |\langle\phi|\psi_0\rangle|^2 = \gamma^2$.
MQC: Multi-qubit control. HE: Hamiltonian evolution. BE: Block encoding
$\star$ Achieves near optimal complexity w.r.t. γ, ϵ.
$\diamond$ Significantly reduced preconstant in depth with large overlap / relative overlap.

Som19: (Somma New J. Phys., 2019; slightly improved by LT22); GTC19: (Ge-Tura-Cirac, J. Math. Phys. 2019)
(**Lin**-Tong, Quantum 2020); (**Lin**-Tong, PRX Quantum 2022); (Dong-**Lin**-Tong, PRX Quantum 2022);
(Ding-**Lin**, PRX Quantum 2023)

Exponential improvement of dependence on precision for gapped system

Corollary (Ding-Lin, 2211.11973)

If $\epsilon \ll D$, there exists an algorithm that uses 1 ancilla qubit to estimate λ_0 to precision ϵ with high probability using

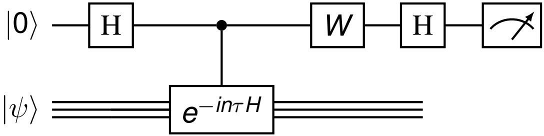
$$T_{\max} = \widetilde{\Theta}(D^{-1}), \quad T_{\text{total}} = \widetilde{\Theta}\left(D/(p_0^2\epsilon^2)\right).$$

- $T_{\max}$ is independent of ϵ, though this does not satisfy Heisenberg-limited scaling.

This matches the result in (Wang et al, 2209.06811).

Outline

Classical post-processing of Hadamard test circuit



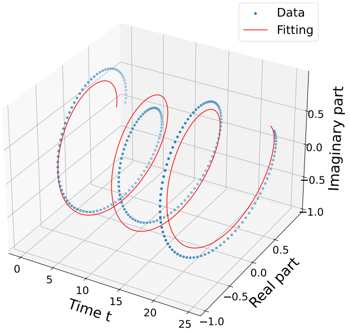
- $W = I$, $\mathbb{E}(X_n) = \mathrm{Re}\left(\langle\psi| \exp(-in\tau H) |\psi\rangle\right)$.
- $W = S^\dagger$, $\mathbb{E}(Y_n) = \mathrm{Im}\left(\langle\psi| \exp(-in\tau H) |\psi\rangle\right)$.
- Sample N_s times

$$Z_n = \frac{1}{N_s} \sum_{k=1}^{N_s} \left(X_{k,n} + iY_{k,n}\right) \approx \langle\psi| \exp(-in\tau H) |\psi\rangle .$$

- Post-processing of time series

$$\mathcal{D}_H = \{(n\tau, Z_n)\}_{n=0}^{N-1} .$$

Quantum complex exponential least squares



- Minimize mean square error (MSE)

$$(r^*, \theta^*) = \underset{r\in\mathbb{C},\theta\in\mathbb{R}}{\arg\min}\, L(r,\theta), \quad L(r,\theta) = \frac{1}{N}\sum_{n=0}^{N-1} \left|Z_n - r\exp(-i\theta n\tau)\right|^2 .$$

- Fitting can be inexact when $p_0 < 1$, but can still accurately estimate λ_0 (*not* an obvious fact).

Solve the optimization problem

- Fix θ, optimize r

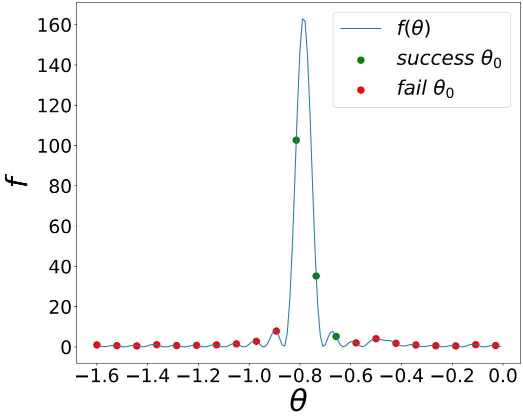
$$\min_{r\in\mathbb{C}} L(r,\theta) = \frac{1}{N}\sum_{n=0}^{N-1} |Z_n|^2 - \frac{1}{N}\left|\sum_{n=0}^{N-1} Z_n e^{i\theta n\tau}\right|^2.$$

- Only optimize w.r.t. θ:

$$\theta^* = \arg\max_{\theta\in\mathbb{R}} f(\theta), \quad f(\theta) = \frac{1}{N}\left|\sum_{n=0}^{N-1} Z_n e^{i\theta n\tau}\right|^2.$$

- Energy landscape is rugged but can be handled classically (there is only one scalar variable θ).

Optimization landscape



Convergence

Theorem (Basic version QCELS)

Given $p_0 > 0.71$, *we can choose*

$$\delta = \Theta(\sqrt{1-p_0}), \quad T_{\max} = \frac{\delta}{\epsilon}, \quad NN_s = \widetilde{\Omega}(\delta^{-(2+o(1))}).$$

Let θ^* *be the optimizer. Then with high probability*

$$|(\theta^* - \lambda_0) \bmod [-\pi/\tau, \pi/\tau)| < \epsilon.$$

- Short maximal runtime (circuit depth).
- Does not achieve Heisenberg-limited scaling.
  $T_{\max} = N\tau \quad \Rightarrow \quad N = \mathcal{O}(\epsilon^{-1})$ if τ is small

$$T_{\text{total}} = \tau N_s N(N-1)/2 = \widetilde{\mathcal{O}}(\epsilon^{-2})$$

Multi-level QCELS

- The result $T_{\max} = \frac{\delta}{\epsilon}, NN_s = \widetilde{\Omega}(\delta^{-(2+o(1))})$ is independent of τ

$$|(\theta^* - \lambda_0) \bmod [-\pi/\tau, \pi/\tau)| < \epsilon.$$

- Choose $\tau_{j+1} = 2\tau_j$ to refine the search interval

Algorithm

For $j = 1, \ldots, J$

Generate data set $\mathcal{D}_{H,j} = \left\{ \left(n\tau_j, Z_{n,j} \right) \right\}_{n=0}^{N-1}$.

Solve

$$(r_j^*, \theta_j^*) \leftarrow \underset{r \in \mathbb{C}, \theta \in [-\lambda_{\min}, \lambda_{\max}]}{\arg\min} L(r, \theta),$$

Shrink search interval

$$\lambda_{\min} \leftarrow \theta_j^* - \frac{\pi}{2\tau_j}, \quad \lambda_{\max} \leftarrow \theta_j^* + \frac{\pi}{2\tau_j}$$

Convergence

Theorem (Multi-level QCELS)

If $p_0 > 0.71$, choose

$$\delta = \Theta(\sqrt{1 - p_0}),$$

and

$$T_{\max} = \frac{\delta}{\epsilon}, \quad T_{\mathrm{total}} = \widetilde{\Theta}\left(\delta^{-(1+o(1))}\epsilon^{-1}\right).$$

Let θ^* be the output of multi-level QCELS. Then with high probability

$$|(\theta^* - \lambda_0) \bmod [-\pi, \pi)| < \epsilon.$$

- Short maximal runtime (circuit depth).
- Achieve Heisenberg-limited scaling.

Outline

Estimating multiple eigenvalues

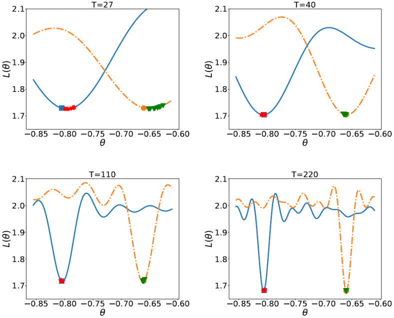
$$H|\psi_i\rangle = \lambda_i |\psi_i\rangle, \quad i = 1, 2, \ldots, M$$

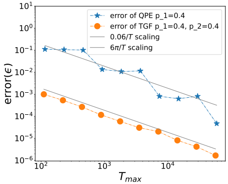
- Dominant modes $\lambda_m, m \in \mathcal{D} \subset \{1, 2, \cdots, M\}, |\mathcal{D}| = K$.
- Overlap $p_m = |\langle\psi_m|\psi\rangle|^2$. Residual overlap $R^K = \sum_{m' \in \mathcal{D}^c} p_{m'}$.
- Assume $p_{\min}^K = \min_{m \in \mathcal{D}} p_m = \Omega(R^K)$.
- Heisenberg-limited scaling: Total cost $\widetilde{\mathcal{O}}(\epsilon^{-1})$.
- Short-depth: $T_{\max} = \delta/\epsilon, \delta = \widetilde{\Theta}(R^K/p_{\min}^K)$.

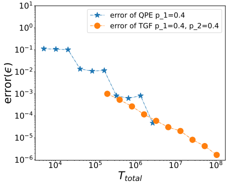
Estimating multiple eigenvalues with short-depth quantum circuit

$$\left(\{r_k^*\}_{k=1}^K, \{\theta_k^*\}_{k=1}^K\right) = \underset{r_k\in\mathbb{C},\theta_k\in\mathbb{R}}{\arg\min}\; L_K\left(\{r_k\}_{k=1}^K, \{\theta_k\}_{k=1}^K\right) .$$

$$L_K\left(\{r_k\}_{k=1}^K, \{\theta_k\}_{k=1}^K\right) = \frac{1}{N}\sum_{n=1}^{N}\left|Z_n - \sum_{k=1}^{K} r_k \exp(-i\theta_k t_n)\right|^2$$







Transverse field Ising model (TFIM)

(Ding-**Lin**, 2303.05714)

Complexity

Theorem (Ding-**Lin**, 2303.05714)

If $p_{\min}^K > 3R^K$, choose

$$\delta = \widetilde{\Theta}(R^K / p_{\min}^K).$$

There exists an algorithm that uses 1 ancilla qubit to estimate dominant $\{\lambda_m\}_{m\in\mathcal{D}}$ to precision ϵ with high probability using

$$T_{\max} = \frac{\delta}{\epsilon}, \quad T_{\text{total}} = \widetilde{\Theta}\left(\frac{1}{\left(p_{\min}^K\right)^4 \delta^{1+o(1)}\epsilon}\right).$$

- Direct generalization of QCELS
- Current drawback: classical optimization cost can be $\exp(cK)$ in the worst case (this has not effect on the quantum cost)

Classical signal processing approaches for gapped and gapless systems (Li-Ni-Ying, 2303.08099)

Ground-state energy estimation with global depolarized noise

- Global depolarized noise channel

$$\rho \mapsto e^{-\alpha\tau}\rho + \frac{1-e^{-\alpha\tau}}{M} I,$$

- Not possible to run to $T_{\max} \gg \alpha^{-1} \Rightarrow$ No Heisenberg-limited scaling.

- New result: For gapped system $\Delta_\lambda > 0$, choose

$$T_{\max} = \Theta\left(\frac{1}{\Delta_\lambda} \log\left(\frac{1}{\epsilon}\right)\right), \quad N = \Theta(\mathrm{poly}(\epsilon^{-1})),$$

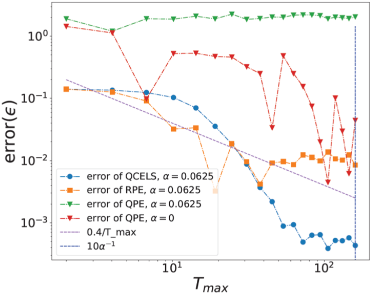
can still approximate ground-state energy to arbitrary precision ϵ.

(Ding-Dong-Tong-**Lin**, in preparation)

QCELS with global depolarized noise channel

$$L_\beta(r,\theta) = \frac{1}{N_t}\sum_{n=1}^{N_t} \left|\exp(\beta|t_n|)Z_n - r\exp(-i\theta t_n)\right|^2$$

$$(r^*,\theta^*) = \mathrm{argmin}_{r\in\mathbb{R},\theta\in[-\pi,\pi]} L_\beta(r,\theta)$$



(Ding-Dong-Tong-**Lin**, in preparation)

Conclusion

- Early fault-tolerant quantum algorithm:
  Small number of ancilla qubits, simple gates, short circuit depth

- Recommend QCELS for short-depth simulation, and in general signal processing based methods.

- Compare with quantum subspace methods / matrix pencil methods; More general noise channel Applications and initialize with VQE / DMRG; Excited state properties and Green's functions.

- Not discussed:
  - Randomized implementation of Fourier filtering and binary search based ground-state energy estimation[1];
  - Quantum eigenvalue transformation of unitary matrices and preparation of ground state[2]

[1] (**Lin**-Tong, PRX Quantum 3, 010318, 2022)

[2] (Dong-**Lin**-Tong, PRX Quantum 3, 040305, 2022)

Acknowledgment

Thank you for your attention!

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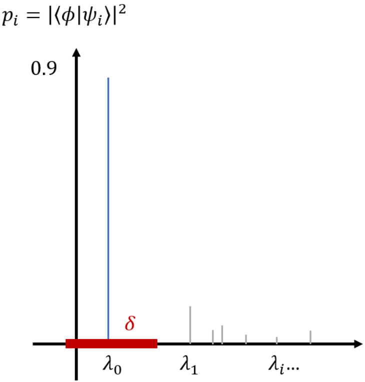
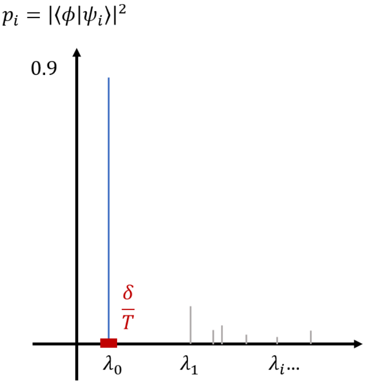
Outline

Phase cancellation in long time simulation



Uncertainty in short time simulation



Uncertainty in long time simulation

Intuitive analysis of basic version of QCELS

- Recall

$$\theta^* = \arg\max_{\theta\in\mathbb{R}} f(\theta), \quad f(\theta) = \frac{1}{N}\left|\sum_{n=0}^{N-1} Z_n e^{i\theta n\tau}\right|^2.$$

- Need to bound

$$R_0 = \left|(\lambda_0 - \theta^*)\tau \bmod [-\pi, \pi)\right|.$$

- Lower bound $f(\lambda_0)$

$$(2p_0 - 1)\, N \leq \sqrt{f(\lambda_0)}$$

- Upper bound $f(\theta^*)$

$$\sqrt{f(\theta^*)} \leq \left|\frac{\sin(NR_0/2)}{\sin(R_0/2)}\right| + (1 - p_0)N.$$

Intuitive analysis of basic version of QCELS

- Optimality $\sqrt{f(\theta^*)} \geq \sqrt{f(\lambda_0)}$ gives

$$\left|\frac{\sin(NR_0/2)}{\sin(R_0/2)}\right| \geq (3p_0 - 2)N \equiv \frac{\sin(N(\delta/2N))}{\sin(\delta/2N)} \approx N\left(1 - \frac{\delta^2}{24}\right)$$

- $\delta^2 \approx 72(1 - p_0) \Rightarrow \delta \to 0$ as $p_0 \to 1$.
- $\frac{\sin(Nx)}{\sin(x)}$ is decreasing on $[0, \pi/(2N)] \Rightarrow R_0 \leq \frac{\delta}{N}$ or

$$|(\lambda_0 - \theta^*) \mathrm{mod}\, [-\pi/\tau, \pi/\tau)| < \frac{\delta}{T_{max}} = \epsilon$$

- This gives $T_{max} = \delta/\epsilon$: short runtime!

A more careful analysis

- Everything is noisy. Take into account failure probability.

- Need to bound Monte Carlo error

$$E_n = Z_n - \langle\psi|\exp(-in\tau H)|\psi\rangle\,, \quad \overline{E}_\theta = \frac{1}{N}\sum_{n=0}^{N-1} E_n \exp(i\theta n\tau)$$

- Lipschitz continuity and Hoeffding's inequality

$$\mathbb{P}\left(\sup_{\theta\in[\lambda_0-\frac{\rho}{T},\lambda_0+\frac{\rho}{T}]}\left|\overline{E}_\theta - \overline{E}_{\lambda_0}\right| \geq \left(4\sqrt{2}\log^{1/2}\left(\frac{8\sqrt{N_s N}}{\eta}\right)+1\right)\frac{\rho}{\sqrt{N_s N}}\right) \leq \eta$$

- A somewhat elaborate iterative refinement procedure to improve the δ dependence of NN_s.

- Finally, multi-level QCELS just applies the analysis to each level.