Minicourse: The algebraic structure of diffeomorphism groups

Abstract. This course introduces classical and new results on the algebraic structure of the identity component of the diffeomorphism group $\mathrm{Diff}_0(M)$ or homeomorphism group $\mathrm{Homeo}_0(M)$ of a compact manifold. These groups are algebraically simple (no nontrivial normal subgroups) – for deep topological reasons due to Epstein, Mather, Thurston... but nevertheless have a very rich algebraic structure. We'll see that:

a) The algebraic structure of $\mathrm{Diff}_0(M)$ determines M. If $\mathrm{Diff}_0(M)$ is isomorphic to $\mathrm{Diff}_0(N)$, then M and N are the same smooth manifold (Filipkiewicz for $\mathrm{Diff}_0(M)$, Whittaker for $\mathrm{Homeo}_0(M)$)

b) The algebraic structure of $\mathrm{Diff}_0(M)$ "captures the topology" of $\mathrm{Diff}_0(M)$
Any group homomorphism from $\mathrm{Diff}_0(M)$ to $\mathrm{Diff}_0(N)$ is necessarily continuous. Any homomorphism from $\mathrm{Homeo}_0(M)$ to any separable topological group is necessarily continuos (Hurtado, Mann)

We'll explore consequences of these theorems and related results, as well as other fascinating algebraic properties of diffeomorphism groups of manifolds (for instance, distorted elements, left-invariant orders, circular orders...). In the last lecture, we'll touch on recent work on the geometry and metric structure of diffeomorphism groupss.

Contents

1 Introduction, perfectness and simplicity of $\mathrm{Homeo}_0(M)$

1.1 Goals for the workshop

- My course: algebraic structure of homeomorphism and diffeomorphism groups and applications to (diverse!) problems in topology.
- Bena's course: topology (homotopy type, cohomology) of diffeomorphism groups, and applications to classifying bundles, flat bundles, and realization problems.

Basic motivation. For me, the most basic motivation comes from Klein's *Erlangen program.* Here is Klein's formulation (roughly)

> "Given a space X and transformation group acting (transitively) on X, investigate the properties of aspects of X invariant under G"

Klein's program gave rise to the modern idea of defining geometry as the study of (G, X) *structures.*

Definition 1.1. Let X be a topological space, and $G \subset \mathrm{Homeo}(X)$ a group. A manifold M has a (G, X) structure if it has charts to X with overlap maps in G. (Technically, one should say that overlap maps are *locally* in G: they are restrictions of elements of G to the sets on which they are defined.)

For example, hyperbolic 2-manifolds are those with a $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H}^2)$ structure; since the hyperbolic metric on $\mathbb{H}^2$ is invariant under $\mathrm{PSL}(2, \mathbb{R})$, these manifolds inherit a metric, and the isometries of M are exactly the automorphisms of the $(\mathrm{PSL}(2, \mathbb{R}), \mathbb{H}^2)$ structure. In Klein's perspective the *group* of automorphisms of a structure plays a central role – Klein himself applied this to projective geometry, translating geometric problems into algebraic statements.

In this formulation, an oriented topological manifold is a $(\mathrm{Homeo}_+(\mathbb{R}^n), \mathbb{R}^n)$ space, and a smooth structure on a manifold is a $(\mathrm{Diff}_+(\mathbb{R}^n), \mathbb{R}^n)$ structure. The automorphism groups of these structures are $\mathrm{Homeo}(M)$ and $\mathrm{Diff}(M)$ respectively. Just as a thorough knowledge of the algebraic (and Lie group) structure of $\mathrm{Isom}(\mathbb{H}^n)$ is essential to the hyperbolic geometer, we expect that understanding the structure of diffeomorphisms and homeomorphism groups should give us new tools in topology.

Exercise 1.2. In the language of (G, X) structures, what is a symplectic structure on a manifold? A contact structure? A Riemannian metric structure? A notion of volume? What are the automorphism groups of these structures?

1.2 Meet our friends for the week

Notation 1.3. M is always a connected manifold (smooth or topological, depending on context), usually assumed compact. We are interested in the following groups.

- $\mathrm{Homeo}(M)$ = group of homeomorphisms of M = continuous invertible maps $M \to M$ with continuous inverses.
- $\mathrm{Diff}^r(M)$ = group of C^r diffeomorphisms of M, i.e. r-times continuously differentiable maps with r-times continuously differentiable inverses. With this notation $\mathrm{Diff}^\infty(M)$ is the group of smooth diffeomorphisms (often abbreviated as $\mathrm{Diff}(M)$ – if no superscript appears, assume smooth), and $\mathrm{Diff}^0(M) = \mathrm{Homeo}(M)$.

- $\mathrm{Diff}^r_+(M)$ = subgroup of orientation-preserving C^r diffeomorphisms.
- $\mathrm{Diff}^r_0(M)$ = connected component of $\mathrm{Diff}^r(M)$ containing the identity.

For non-compact manifolds, we usually focus on the subgroup of *compactly supported diffeomorphisms* –diffeomorphisms that are the identity outside of some compact subset. This group is denoted $\mathrm{Diff}^r_c(M)$, and $\mathrm{Diff}^r_0(M)$ denotes the subgroup of $\mathrm{Diff}^r_c(M)$ consisting of diffeomorphisms isotopic to the identity via a compactly supported isotopy. [1]

Topology and metric. The idea behind the topology on $\mathrm{Diff}^r(M)$ is to say that a diffeomorphism is close to the identity if it doesn't move any point too far in the manifold, and if its derivatives are "close to the identity" as well. For $\mathrm{Homeo}_0(M)$ this is easy to make precise by specifying a metric: Put a metric d on M, and define a distance on $\mathrm{Homeo}(M)$ by

$$\mathrm{dist}(f, g) := \sup_{x \in M} \{d(f(x), g(x))\}.$$

This definition feels natural, but we could have just as well defined

$$\mathrm{dist}_2(f, g) := \sup_{x \in M} \{d(f^{-1}(x), g^{-1}(x))\}$$

or even

$$\mathrm{dist}_3(f, g) := \mathrm{dist}(f, g) + \mathrm{dist}_2(f, g).$$

Exercise 1.4. Prove:

i) All of these distance functions are genuine metrics, and the topology they generate is independent of the choice of metric d on M.

ii) With the topology induced by dist, dist_1, and dist_2, right and left multiplication and inversion is continuous, hence $\mathrm{Homeo}(M)$ is a topological group. (In fact, all three distances induce the same topology. Even better, we'll see later that $\mathrm{Homeo}(M)$ has a *unique* complete separable topology – this is it.)

iii) Which metrics are invariant under left, resp. right, multiplication?
Remark: a theorem of Birkhoff-Kakutani states that every metrizable topological group admits a compatible left-invariant metric – perhaps you know this familiar fact about Lie groups. The existence of a metric invariant under *both left and right* multiplication is a nontrivial question. **Can you find such a metric on $\mathrm{Homeo}_0(M)$?

The topology on $\mathrm{Homeo}(M)$ induced by any of these metrics is separable – a countable sub-basis for the open sets consists of the sets

$$\{f \in \mathrm{Homeo}(M) : f(\overline{U}) \subset V\}$$

where U and V range over a countable basis for the topology of M. It also admits a *complete metric* (this is dist_3 above), making $\mathrm{Homeo}(M)$ a *Polish group.*

[1] There are many reasons to focus on $\mathrm{Diff}^r_c(M)$. From a topological perspective, the usual *compact-open topology* on $\mathrm{Diff}^r(M)$ doesn't "see" the behavior of diffeomorphisms at infinity. From an algebraic perspective, $\mathrm{Diff}^r_c(M)$ is a normal subgroup, so a good place to start the study of the algebraic structure of $\mathrm{Diff}^r(M)$.

The subset topology on $\mathrm{Diff}^r(M) \subset \mathrm{Homeo}(M)$ is not complete. For a better (finer) topology, one should take derivatives into account. Specifying how far a derivative is from the identity is most concretely done in coordinates, so to define the C^r *compact open* topology on $\mathrm{Diff}^r(M)$ we specify a sub-basis of open sets as follows. For $f \in M$, let $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ be coordinate charts, $K \subset U$ a compact set with $f(K) \subset V$ and let $\epsilon > 0$. Define the (ϵ, K, U, V)-neighborhood of f as

$$\{g \in \mathrm{Diff}^r(M) \mid \forall x \in \phi(K) \text{ and } 0 \leq k \leq r, \, \|D^k(\psi g \phi^{-1})(x) - D^k(\psi f \phi^{-1})(x)\| < \epsilon\}$$

(by convention, the 0^{th} derivative is just g). The topology on $\mathrm{Diff}^\infty(M)$ is induced by the inclusions $\mathrm{Diff}^\infty(M) \hookrightarrow \mathrm{Diff}^r(M)$ for all finite r; all of these spaces are also completely metrizable. We'll return to discussing metrics in Lecture 5.

An infinite dimensional Lie group. The groups $\mathrm{Diff}^r(M)$ have a natural *smooth* structure (as a *Banach manifold*, or *Frechet manifold* for the case of $\mathrm{Diff}^\infty(M)$); the tangent space at the identity can be identified with the space of C^r vector fields on M. A local chart from a neighborhood of the identity in $\mathrm{Diff}^r(M)$ to the space of vector fields is given by mapping g to the vector field

$$X(p) := \exp_p^{-1} g(p)$$

where $\exp_p$ is the Riemannian exponential map on a neighborhood of the identity in $T_p(M)$. We say that $\mathrm{Diff}^r(M)$ is *locally modeled on the space of C^r vector fields.*

This smooth structure, together with the Lie algebra structure on $\mathrm{Diff}^r(M)$ makes $\mathrm{Diff}^r(M)$ an *infinite dimensional Lie group*, with Lie algebra the algebra of C^r vector fields.

> *Warning!* although there is a "smooth chart" from M a neighborhood of the 0 vector field, this is *not* given by the usual Lie algebra exponential map. The Lie algebra exponential in this case assigns to a vector field the time 1 map of the flow generated by this vector field, but there exist diffeomorphisms, even of S^1, that are arbitrarily close to the identity and not the time 1 map of any flow.

See Chapter 1 of [2] for a very short introduction and [36], especially Chapters II.2 and II.3, for more development and context.

1.3 Perfectness and simplicity

A recurring theme in this course will be that $\mathrm{Diff}_0(M)$ and $\mathrm{Homeo}_0(M)$ share many common traits (perfectness, simplicity, group determines the manifold, etc.), but the tools needed to prove these in the smooth and topological categories are often completely different. As a first example, we'll prove a fundamental tool called the *Fragmentation property*, first for $\mathrm{Homeo}_0(M)$ using Kirby's torus trick, then for $\mathrm{Diff}_0(M)$, which will be comparatively elementary.

Fragmentation is a particular way of decomposing a homeomorphism into "smaller" ones, precisely, ones with smaller support.

Definition 1.5. The *support* of a homeomorphism f, denoted $\mathrm{supp}(f)$, is the closure of the set $\{x \in M : f(x) \neq x\}$

Definition 1.6 (Fragmentation)**.** A group $G \subset \mathrm{Homeo}_0(M)$ has the *fragmentation property* if, given any finite open cover $\mathcal{U}$ of M and any element $g \in G$, there is a decomposition $g = g_1 \circ g_2 .... \circ g_n$ with each g_i supported in some set in $\mathcal{U}$.

Theorem 1.7 (Kirby [34], Edwards–Kirby [17])**.** $\mathrm{Homeo}_0(M)$ has the fragmentation property.

Proof sketch. The proof is a straightforward application of Kirby's famous "torus trick"

1. *Reduction to neighborhood of id.* Since a topological group is generated by a neighborhood of the identity, it suffices to prove that homeomorphisms close to the identity can be fragmented.

2. *The torus trick.* If $f \in \mathrm{Homeo}(\mathbb{R}^n)$ is sufficiently close to the identity, and $K \subset \mathbb{R}^n$ a compact set, then there is an isotopy f_t, with $f_0 = f$ and $K \subset \mathrm{fix}(f_1)$.

3. *First application of torus trick.* If M is a manifold, $U \subset M$ and open set and $K \subset U$ compact, then any $f \in \mathrm{Homeo}(M)$ sufficiently close to the identity can be isotoped to a homeomorphism f_1 fixing K, and the isotopy can be taken to be constant (agreeing with f) outside of U.

4. *Control of norm.* We can control the norm of f_1 from step 2. Given ϵ, there exists δ so that $\|f\| < \delta \Rightarrow \|f_1\| < \epsilon$. [proof omitted]

5. *Fragmentation.* Let $\mathcal{U}$ be an open cover of M. For simplicity, I'll assume $\mathcal{U} = \{U_1, U_2\}$, if there are more sets you make an inductive argument. Take $V_1 \subset U_1$ so that $\{V_1, U_2\}$ is still a cover. Let $f \in \mathrm{Homeo}(M)$. If f is close enough to the identity, we can isotope f to f_1, where f_1 fixes V_1 and agrees with f outside of U_1. Then $f_2 := f \circ f_1^{-1}$ agrees with f on V_1 and is supported on U_1. Since f_1 fixes V_1, $\mathrm{supp}(f) \subset U_2$, so $f = f_2 f_1$ is our desired fragmentation.

□

Remark 1.8. The torus trick uses topological Schönflies (which itself is not too hard), and otherwise is quite elementary. A more refined version, using Gauld's "cannonical Schönflies" theorem shows that there is a *cannonical* means of fragmenting a homeomorphism close to the identity, and was used by Edwards–Kirby to prove that $\mathrm{Homeo}_0(M)$ is locally contractible. Local contractibility of $\mathrm{Homeo}_0(M)$ was proved earlier by Cernavskii, using difficult surgery theory (handle-straightening). For a thorough introduction to the topic (published in 1973, when it was a very hot topic) see [60].

Fragmentation has many other important applications. We'll use it now to show an algebraic property of $\mathrm{Homeo}_0(M)$. Recall that a group G is *perfect* if it is equal to its commutator subgroup, equivalently, if $H_1(G;\mathbb{Z}) = 0$ (in group homology).

Corollary 1.9. $\mathrm{Homeo}_0(M)$ is perfect.

Proof. ("folklore," perhaps due to Anderson [1]) Since $\mathrm{Homeo}_0(M)$ is generated by any neighborhood of the identity, it suffices to prove that any homeomorphism close to the identity can be written as a product of commutators. By fragmentation, it will suffice to prove that any $f \in \mathrm{Homeo}_0(M)$ supported in a small ball B can be written as a commutator. So Let B be an open ball in M, and suppose $\mathrm{supp}(f) \subset B$. Choose $b \in \mathrm{Homeo}_0(M)$ such that for any $m \neq n$, $b^n(B) \cap b^m(B) = \emptyset$ (one can easily construct such explicitly using a chart in $\mathbb{R}^n$. Define a by

$$a(x) = \begin{cases} b^n f b^{-n}(x) & \text{if } x \in b^n(B) \text{ for some } n \\ x & \text{otherwise} \end{cases}$$

Then it is easily verified that $[a, b] = f$. □

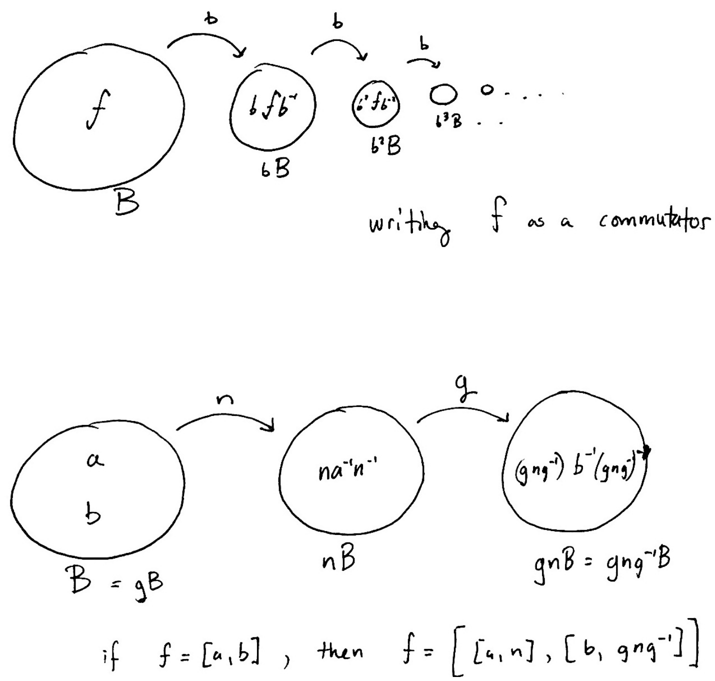

Figure 1: Schematics for the proofs of Corollaries 1.9 and 1.10

Corollary 1.10. $\mathrm{Homeo}_0(M)$ is algebraically simple.

Proof. . Let $n \neq id$ be an element of $\mathrm{Homeo}_0(M)$. We want to show that every element of $\mathrm{Homeo}_0(M)$ is in the normal closure of n. Let $B \subset M$ be a small ball such that $nB \cap B = \emptyset$. First we show that if $\mathrm{supp}(f) \subset B$, then f is in the normal closure of n.

Let $g \in \mathrm{Homeo}_0(M)$ be such that $g|_B = \mathrm{id}$ and $g(nB) \cap nB = \emptyset$. The proof of Corollary 1.9 in fact shows that $f = [a, b]$ for some a and b supported on B. (Since $\mathrm{supp}(f)$ is closed, it is contained in some smaller sub-ball $B' \subset B$, run the proof of the corollary with B' in place of B, and take b to be supported on B.) Now verify (exercise!) that

$$f = \big[[a, n], [b, gng^{-1}]\big]$$

which is what we wanted to show.

If B' is another small embedded ball, then there exists $h \in \mathrm{Homeo}_0(M)$ such that $h(B') \subset B$. If $\mathrm{supp}(f) \subset B'$, then $\mathrm{supp}(hfh^{-1}) \subset B$, so hfh^{-1} (and hence h) lies in the normal closure of n. Finally, fragmentation tells us that $\mathrm{Homeo}_0(M)$ is generated by elements supported on small balls, hence lies in the normal closure of n. $\square$

Exercise 1.11. Fill in the details of the proof above. Does this also show that $\mathrm{Diff}_0(M)$ is simple (given that it is perfect, which we will show tomorrow)? What general properties of a group $G \subset \mathrm{Homeo}_0(M)$ will guarantee that G is simple?

Fragmentation for $\mathrm{Diff}_0^r(M)$**.** Above, we mentioned that $\mathrm{Diff}_0^r(M)$ was an infinite-dimensional (Banach/Frechet) manifold modeled on the space of smooth vector fields on M. One consequence of this fact is that $\mathrm{Diff}_0^r(M)$ is locally connected. We exploit this to give a quick proof of fragmentation.

Proof of fragmentation for $\mathrm{Diff}_0^r(M)$*, following [2].* Let $\{U_1, ...U_n\}$ be an open cover of M. As in the proof of fragmentation for Homeo, it suffices to show we can "fragment" a diffeomorphism g that is close to the identity. By local connectedness, we can take an isotopy g_t from $g_0 = id$ to $g_1 = g$ that stays close to the identity for all t.

Take a partition of unity λ_i subordinate to $\{U_i\}$ and define $\mu_k := \sum_{i \leq k} \lambda_i$. Now define $\psi_k(x) := g_{\mu_k(x)}(x)$. This is a C^r map, and close to the identity, although not a priori invertible. However, the space of C^r diffeomorphisms is *open*, so being suffieintly close to the identity *implies* that it is a diffeomorphism. By definition, ψ_k agrees with ϕ_{k-1} outside of U_k, and hence $g = (\psi_0^{-1}\psi_1)(\psi_1^{-1}\psi_2)...(\psi_{n-1}^{-1}\psi_n)$ is the desired decomposition of g, with each diffeomorphism $\psi_{k-1}^{-1}\psi_k$ supported on U_k. □

Exercise 1.12. Prove that the group of Hamiltonian diffeomorphisms of a symplectic manifold has the fragmentation property. Use the fact that the group is generated by the set of time-dependent Hamiltonians in a small neighborhood of identity in $C^\infty(M \times [0, 1])$. Take a partition of unity subordinate to an open cover, and attempt a similar "cut-off" strategy – you will need to be a little more careful with the decomposition and supports.
This proof strategy also works for $\mathrm{Diff}_0(M)$, since it is generated by small time-dependent smooth vector fields.

1.4 Further exercises

1. (Standard fact) Let G be a connected topological group, and U a neighborhood of the identity in G. Prove that U generates G.
2. Let M be a compact manifold with nonempty boundary.
   (a) Prove that the identity component of $\mathrm{Homeo}(M)$ has the fragmentation property. (Remark: fragmentation for $\mathrm{Homeo}(M)$ also holds, thanks to Edwards-Kirby, which works for manifolds with boundary. If you are familiar with the torus trick, try to prove this.)
   (b) Prove that the identity component of $\mathrm{Homeo}(M)$ is perfect. Is it simple? What about the group of homeomorphisms that fix the boundary pointwise?
   (c) Give examples of normal subgroups of $\mathrm{Diff}_0(M)$ and $\mathrm{Homeo}_0(M)$. **Attempt a complete classification, at least for the one-dimensional case $M = [0, 1]$.
3. The "Lie group exponential map" from $\mathfrak{X}(M)$ (the space of smooth vector fields on M) to $\mathrm{Diff}(M)$ is given by sending a vector field X to the time-1 map of the flow generated by X.
   (a) * Show that, when $M = S^1$, this map is not surjective onto any neighborhood of the identity in $\mathrm{Diff}(S^1)$. Hint: if f has a periodic orbit and is the time one map of a flow, what can you conclude about f?
   (b) * Using part a), show that the Lie group exponential is not surjective onto any neighborhood of the identity in $\mathrm{Diff}(M)$ for any compact manifold M.
4. A group G is called *uniformly perfect* if there is some integer k such that every element can be written as a product of at most k commutators. (The minimal such k is called the "commutator width" of G).
   (a) * Show that $\mathrm{PSL}(2, \mathbb{R})$ is uniformly perfect.
   (b) * Show that $\mathrm{Homeo}_+(S^1)$ is uniformly perfect.

(c) ** Show that $\mathrm{Homeo}_+(S^n)$ is uniformly perfect. (See Ghys' *Groups acting on the circle* for an answer)

(d) ** Show that $\mathrm{Diff}_+(S^1)$ is uniformly perfect. (This is a 0-star question if you know the right theorem to cite, and a *** question otherwise).

(e) ** Let Σ_g be the genus g surface. Is $\mathrm{Homeo}_0(\Sigma_g)$ or $\mathrm{Diff}_0(\Sigma_g)$ uniformly perfect? Does it matter what g is?

5. For the symplectic topologists: Let M be a closed manifold, and let $\mathrm{Symp}(M)$ denote the group of symplectomorphisms of M. Is $\mathrm{Symp}_0(M)$ perfect? (Hint: *flux*. See Chapter 10 of McDuff–Salamon *Introduction to Symplectic Topology* for an introduction to the structure of $\mathrm{Symp}_0(M)$.)

2 Simplicity and perfectness of $\mathrm{Diff}_0(M)$

The main content of this lecture is the expository paper *A short proof that* $\mathrm{Diff}_0(M)$ *is perfect* [39], proving the following theorem originally due to Thurston [64].

Theorem 2.1. $\mathrm{Diff}_0^\infty(M)$ is a perfect group.

Our strategy follows a new proof (30 years after Thurston's!) due to Haller and Teichmann in [29], and Haller–Rybicki–Teichman in [30]. As [39] is already expository, we refer the reader there for the proof.

2.1 Remarks on the Thurston and Mather proofs

In [64], Thurston announced a proof of perfectness of $\mathrm{Diff}_0(M)$. In fact, he announces much more, involving the relationship between the classifying space of $\mathrm{Diff}_0(M)$ as a discrete group (what you use to compute group cohomology) and the classifying spaces for Haefliger structures of codimension $\dim(M)$. (A Haefliger structure is a generalization of a foliation, and the right way to make a notion of classifying spaces and characteristic classes for foliations.) Thurston never wrote the details of his proof, but the special case of 1-dimensional manifolds was proved earlier by Mather [43], which gives a thorough proof. The general (Thurston) case is written in [2].

For us, the important consequence of Thurston's work – what gives perfectness of $\mathrm{Diff}_0(M)$ – is the following.

Theorem 2.2 (Thurston [64])**.** Let $\mathrm{Diff}_c(\mathbb{R}^n)$ denote the group of diffeomorphisms of $\mathbb{R}^n$ with compact support that are isotopic to the identity through a compactly supported isotopy. For any compact n-manifold M,

$$H_1(\mathrm{Diff}_c(\mathbb{R}^n); \mathbb{Z}) \cong H_1(\mathrm{Diff}_0(M); \mathbb{Z}).$$

This is combined with a (analytical) theorem of Herman for the n-torus – we used the $n = 1$ case of this theorem in the proof of Theorem 2.1:

Theorem 2.3 (Herman [31])**.** $\mathrm{Diff}_0(\mathbb{T}^n)$ is perfect. In fact, there is a neighborhood U of the identity such that every $g \in U$ can be written

$$g = R_\lambda [g_0, R_\theta]$$

where R_λ and R_θ are homeomorphisms that act as rotations in each S^1 coordinate.

As a consequence, $H_1(\mathrm{Diff}_0(\mathbb{T}^n)) = 0$, but Thurston's theorem says that this is isomorphic to $H_1(\mathrm{Diff}_c(\mathbb{R}^n))$, which is then isomorphic to $H_1(\mathrm{Diff}_0(M))$ for any compact n-manifold M.

The C^r case. In 1973-74, Mather showed that $\mathrm{Diff}_0^r(M)$ is perfect, in the two cases $1 \leq r \leq \dim(M)$, and $\dim(M) + 1 \leq r < \infty$ ([44], [45]). Combined with an argument as in Corollary 1.10, this also shows that these groups are simple. Mather's proofs use a version of the Schauder-Tychonoff fixed point theorem, to show that each element of $\mathrm{Diff}_0^r(M)$ lies in the commutator subgroup without giving an explicit way of writing it as a commutator. The case of $r = \dim(M)+1$ remains open, even in the case where $M = S^1$ and $r = 2$.

Other groups The first homology of other natural groups of homeomorphisms – e.g. volume-preserving (Thurston, Banyaga), $\mathrm{Symp}_0(M)$ (Banyaga), Lipschitz homeomorphisms (Abe-Fukui),... has been computed. A nice summary table appears in the introduction to [38].

2.2 Exercises

1. Further properties of diffeomorphism groups: k-transitivity

   (a) A group $G \subset \mathrm{Homeo}(M)$ is called k-transitive if for any two k-tuples of distinct points $(x_1, x_2, ...x_k)$ and $(y_1, y_2, ...y_k)$, there exists $g \in G$ such that $g(x_i) = y_i$ for all i. Show that $\mathrm{Diff}_0(M)$ is k-transitive for all k, provided $\dim(M) > 1$

   (b) What happens in the case where $\dim(M) = 1$? (treat $\mathbb{R}$ and S^1 separately).

   (c) * Suppose M has a Riemannian metric. Is the group of volume-preserving diffeomorphisms k-transitive for all k? What about hamiltonian symplectomorphisms on a symplectic manifold? Diffeomorphisms that preserve a foliation? Can you find interesting examples of groups that are (not) k-transitive?

   (d) Give a reasonable definition of what it *should* mean for a group of homeomorphisms of the circle to be n-transitive. Can you characterize groups that are 3 but not 4-transitive?

2. A group which is not perfect (following Mather [46])

   Let $\mathrm{Diff}_c^{1+bv}(\mathbb{R})$ be the group of C^1 diffeomorphisms of $\mathbb{R}$ whose derivatives are functions of *bounded variation.* (Recall that v has bounded variation if there is a finite radon measure $d\mu_v$ such that $\int g \, d\mu_v = \int v g' dx$ holds for each compactly supported C^1 function g on $\mathbb{R}$. Define

   $$\Phi : \mathrm{Diff}_c^{1+bv}(\mathbb{R}) \to \mathbb{R}$$

   by defining $\Phi(f)$ to be the total mass of the *regular* (nonsingular) part of the measure $\mu_{log(f')}$.

   (a) As a warm-up, calculate μ_v when v is the characteristic function of the interval $[0, \infty)$

   (b) Show that Φ is a homomorphism

   (c) Show that Φ is trivial when restricted to the subgroup of C^2 diffeomorphisms. (Note: you can't cheat and just say that $\mathrm{Diff}_c^2(\mathbb{R})$ is perfect – that's an open question!)

   (d) ** Show that Φ is nontrivial.

3. (Another group which is not perfect) The identity component of $\mathrm{Diff}([0, 1])$ is not perfect (why not?). ** What about the subgroup of diffeomorphisms that are infinitely tangent to the identity at the endpoints? (see [62] for much discussion on this group).

4. (Closely related to the question on Symp_0 from the last lecture): Let μ be a measure (assume absolutely continuous with respect to Lebesgue measure) on M, and let $\mathrm{Homeo}_\mu(M)$ denote the identity component of the group of μ-preserving homeomorphisms of M – homeomorphisms such that $\mu(f(A)) = \mu(A)$ for all measurable sets.

   (a) Construct a map $\phi : \mathrm{Homeo}_\mu(\mathbb{T}^2) \to \mathbb{T}^2$ as follows. Think of $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$, the measure μ can be lifted to a measure on $\mathbb{R}^2$. Take a path f_t from id to $f = f_1$, lift it to a path $\tilde{f}_t$ based at id in $\mathrm{Homeo}_0(\mathbb{R}^2)$. Define

   $$\tilde{\phi}(f_t) = \int_{[0,1]\times[0,1]} f_t(x) - x \; d\mu$$

   and let $\phi(f) := \tilde{\phi}(f_t) \mod \mathbb{Z}^2$.
   Show that this is well defined, and a homomorphism.

   (b) Generalize this definition to give a map from *paths* in $\mathrm{Homeo}_\mu(M)$ to $H^1(M; \mathbb{R})$. Show that this descends to a homomorphism $\mathrm{Homeo}_\mu(M) \to \mathbb{H}^1(M; \mathbb{R})/\Gamma$, where Γ is a discrete subgroup. (This homomorphism is called the *mass flow* by Fathi [19], who showed that its kernel is a perfect group. It is also known as the *homological rotation vector* e.g. in various dynamical applications, pioneered by John Franks.)

3 The algebraic-topological correspondence

The goal of this lecture is to convince you that the algebraic structure of homeomorphism or diffeomorphism groups is amazingly rich. As a first example, it *completely determines the manifold.*

Theorem 3.1 (Whittaker, 1963 [67])**.** Let M and N be topological manifolds, and suppose there is an (abstract) isomorphism $\Phi : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$. Then M and N are homeomorphic, and Φ is induced by a homeomorphism $M \to N$.

Whittaker's theorem applies to homeomorphism groups of more general topological spaces, but we are interested in the manifold case. This theorem was generalized to diffeomorphism groups by Fillipkiewicz.

Theorem 3.2 (Filipkiewicz, 1982 [21])**.** Let M and N be compact manifolds and suppose there is an isomorphism $\Phi : \mathrm{Diff}_0^r(M) \to \mathrm{Diff}_0^s(N)$. Then $r = s$, M and N are diffeomorphic, and Φ is induced by a C^r-diffeomorphism $f : M \to N$.

A proof of both theorems (following the same strategy, essentially due to Whittaker but refined by Filipkiewicz) is outlined in the exercises. Although it looks like this is a unified proof strategy that covers both cases, the proof relies on fragmentation, which was different for $\mathrm{Homeo}_0(M)$ and $\mathrm{Diff}_0(M)$!

Other structures.
Filipkiewicz's theorem inspired a broader research program:

> *Let $G(M, X)$ be the group of homeomorphisms of a manifold M preserving some structure X. Show that $G(M, X) \cong G(N, X)$ implies that M and N are homeomorphic via an X-preserving homeomorphism.*

This has been carried out in many cases – smooth volume forms, and symplectic structures (Banyaga); foliations, contact structures (Rybicki). An interesting open question is the case of *real analytic diffeomorphisms.* Fragmentation fails here, so the proof would require a fundamentally different approach.

Whittaker's theorem was also generalized to homeomorphism groups of general (non-manifold) topological spaces. The best result in this line is perhaps the work of Rubin [59], who unified all previous results using a novel model-theoretic approach.

3.1 A dictionary?

Given that $\mathrm{Homeo}_0(M)$ determines M, there should be a correspondence

$$\text{algebraic property of } \mathrm{Homeo}_0(M) \longleftrightarrow \text{topological property of M}$$

As an easy example (for which one does not need Whittaker's theorem), connectedness of M corresponds to simplicity of $\mathrm{Homeo}_0(M)$ – if M is the disjoint union of M_1 and M_2, then $\mathrm{Homeo}_0(M) \cong \mathrm{Homeo}_0(M_1) \times \mathrm{Homeo}_0(M_2)$.

However, in practice very of this little is known. In 1991 Ghys asked the following question (for diffeomorphism groups, but it applies just as well to Homeo or Diff^r.)

Question 3.3 (Ghys)**.** Let M and N be closed manifolds, and suppose that there is a homomorphism $\mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$. Does it follow that $\dim(M) \leq \dim(N)$?

This question remained open until 2013, when it was answered in the affirmative by Hurtado in [32]. Hurtado shows two remarkable things.

Theorem 3.4. Let $\Phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ be a homomorphism. Then Φ is continuous.

Theorem 3.5. Let $\Phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ be a continuous homomorphism. Then $\dim(M) \leq \dim(N)$, and in the case of equality, N is a cover of M, and Φ is a lift.

Hurtado's theorems generalize to non-compact manifolds. In this case there are more options for continuous homomorphisms between homeomorphism groups of manifolds of the same dimension, arising from embeddings of M in N as an open submanifold.

The proof of Hurtado's theorems make essential use of smoothness in many ways. Theorem 3.5 uses the Montgomery–Zippin theorem on smooth actions of finite dimensional groups (an outline is given in the exercises). Theorem 3.4 uses *distorted elements* of $\mathrm{Diff}_0(M)$, which we'll discuss in Lecture 5. Today, we'll discuss a fundamentally different proof for homeomorphism groups.

3.2 Automatic continuity

In this second half of the lecture, we'll see that the algebraic structure of $\mathrm{Homeo}_0(M)$ not only determines the topological structure of M (that was Whittaker's theorem), but determines the topological structure of $\mathrm{Homeo}_0(M)$ in a very strong sense.

Theorem 3.6 (Mann, [42])**.** Let M be a compact manifold, and H a separable topological group. Any homomorphism $\Phi : \mathrm{Homeo}_0(M) \to H$ is necessarily continuous.

An immediate consequence is that any homomorphism $\mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$ is continuous. The isomorphism case was a consequence of Whittaker's theorem– but see Exercises 2. and 3. below. Another immediate consequence is the following result, which was mentioned in an exercise in lecture 1.

Corollary 3.7. $\mathrm{Homeo}_0(M)$ has a *unique* topology that makes it a complete, separable group.

Proof. Put some mysterious (complete, separable) topology on $\mathrm{Homeo}_0(M)$, and let G denote the resulting topological group. The identity map $\mathrm{Homeo}_0(M) \to G$ is a continuous isomorphism of Polish groups. It is easy to show using a Baire category argument that such a map is necessarily also open (a stronger result is Pettis' theorem, 9.10 in [35]), hence a homeomorphism. □

The proof that we give of Theorem 3.6 follows [42], which builds on the work of C. Rosendal in [56]. For an introduction to automatic continuity, see [57].

3.3 Exercises

1. ** (Open, but perhaps not hard!) Using automatic continuity, give an easier proof of Whittaker's theorem:
   If $\Phi : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$ *is an isomorphism, then* Φ *is induced by a homomorphism* $M \to N$

2. * Give examples of interesting homomorphisms between diffeomorphism or homeomorphism groups. Attempt to construct "pathological" examples. How bad can they be? [I would be very interested to hear any examples you come up with!!]

3. *** (Open problem) Let M and N be compact n-manifolds, and suppose that $\Phi : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$ is a homomorphism. Is N a cover of M?
The following variant of this problem is a conjecture in [51]:
Let N be a closed surface of genus $g \geq 1$, and let $M = N - D^2$. Show that every homomorphism $\Phi : \mathrm{Homeo}_c(M) \to \mathrm{Homeo}_0(N)$ comes from an embedding of M in N.

4. ** In a recent paper [51], E. Militon gave a complete classification of all homomorphisms $\mathrm{Homeo}_+(S^1) \to \mathrm{Homeo}(A)$, where $A = S^1 \times [0, 1]$ is the closed annulus. Recall that such a homomorphism is called an *action of* $\mathrm{Homeo}_+(S^1)$ *on* A. Can you figure out Militon's classification?
(Hint to start: Which actions of $\mathrm{Homeo}_+(S^1)$ preserve a foliation of A by circles? Now can you come up with an action that does not?)

5. *** (Open) Let M be a compact manifold without boundary. Prove that $\mathrm{Diff}_0(M)$ has automatic continuity.

6. This sequence of exercise gives an outline of the proof of Filipkiewicz's theorem.

   (a) Prove the following lemma, using the hint.

   Lemma 3.8. Let M and N be manifolds, and $w : M \to N$ a bijection such that $whw^{-1} \in \mathrm{Homeo}(N)$ for all $h \in \mathrm{Homeo}(M)$. Then w is continuous.

   (Hint: $\{N \backslash \mathrm{fix}(h) \mid h \in \mathrm{Homeo}(M)\}$ is a collection of open sets that forms a basis for the topology of M. Show that the image of such a set under w^{-1} is open.)

   Remark/challenge. Takens (1979) generalized this lemma for diffeomorphisms. He showed that if $w : M \to N$ is a bijection such that $whw^{-1} \in \mathrm{Diff}^r(N)$ for all $h \in \mathrm{Diff}^r(M)$, then w is a C^r diffeomorphism. Can you reproduce his argument?

   (b) Let $\Phi : \mathrm{Homeo}_0(M) \to \mathrm{Homeo}_0(N)$ be an isomorphism. For $x \in M$, let $S_x \subset \mathrm{Homeo}_0(M)$ denote the *stabilizer of* x, i.e. the set $\{f \in \mathrm{Homeo}_0(M) : f(x) = x\}$. *Suppose* that for each $x \in M$ the image $\Phi(S_x)$ was the stabilizer of a point $w(x) \in N$. Show that $x \mapsto w(x)$ is a bijection satisfying the conditions of Lemma 3.8.

   (c) Now we prove that $\Phi(S_x)$ is a point stabilizer. First show that $\{B \subset N \mid G_B \subset \Phi(S_x)\}$ is a $\Phi(S_x)$-invariant set, and is *not* all of M. (hint: fragmentation) Hence, its complement is a nonempty, closed, $\Phi(S_x)$-invariant set.

   (d) * Let C denote the nonempty, closed $\Phi(S_x)$-invariant set you found above. Show that $C \neq M$. (Filipkiewicz's proof uses a trick, which doesn't use any hard tools, but seems nonintuitive to me!)

   (e) * Show that C consists of a singe point, and that $\Phi(S_x)$ is the stabilizer of this point.

   (f) Given the remark/challenge after Lemma 3.8, prove Filipkiewicz's theorem using the same argument.

7. This exercise gives an outline of the proof of Hurtado's theorem: *If $\phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ is continuous, then $\dim(M) \leq \dim(N)$ (and in the case of equality, the map comes from a cover)*

   (a) Given $\phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$, attempt *ala Filipkiewicz* to produce a map $M \to N$ as follows. For $B \subset M$, let G_B denote the group of diffeomorphisms supported on B. Let $B_r(x)$ denote the ball of radius r about x. Note that, for $x \in M$,

$$x = \bigcap_{r>0} B_r(x) = \bigcap_{r>0} \mathrm{interior}\left(\mathrm{supp}(G_{B_r(x)})\right)$$

so define

$$S(x) = \bigcap_{r>0} \text{interior}\left(\text{supp}(\Phi(G_{B_r(x)})\right).$$

Verify that

- ** $S(x)$ is a nonempty set
- $x \mapsto S(x)$ is "equivariant": $\Phi(f)S(x) = S(f(x))$

(b) ** Show that S is "injective" in the sense that $x \neq y \to S(x) \cap S(y) = \emptyset$. Hurtado's strategy is to first show that for any distinct $n+1$ points (where $n = dim(N)$), $\cap_{i=1}^{n+1} S(x_i) = \emptyset$ by supposing for contradiction that this was not the case, and looking at the images of flows of commuting vector fields supported in small neighborhoods of the x_i. Use Montgomery-Zippin.

Given this, take a maximal $k < n+1$ such that there exists disjoint $x_1, ...x_k$ with $\cap_{i=1}^{k} S(x_i) \neq \emptyset$, and show that $k = 1$.

(c) * Assuming equality of dimension, show that $S(x)$ consists of isolated points, and $S(x) \mapsto x$ is a covering map.

8. *** Suppose $\phi : \text{Homeo}_0(M) \to \text{Homeo}_0(N)$ is a homomorphism. Is it true that $\dim(M) \leq \dim(N)$? What if equality holds?

4 Additional structure in the 1-manifold case

This lecture is motivated by a very broad question.

Question 4.1. Let M be a manifold. Characterize the finitely generated/countable subgroups of $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}_0(M)$. What countable groups act nontrivially or faithfully on M (i.e. admit nontrivial/faithful homomorphisms to $\mathrm{Homeo}(M)$)? Given a group, can one describe/classify/encode the possible actions?

The Zimmer program. One refinement of Question 4.1 leads to a family of conjectures and results known as the *Zimmer program*. The Zimmer program is organized around the principle that "large groups shouldn't act on small manifolds". *Large* traditionally means that the group is an irreducible lattice in a semisimple Lie group of large rank, *small* refers to the dimension of the manifold. Progress on this program has been made in two separate scenarios – one is restricting to the class of volume-preserving actions, i.e. homomorphisms $\Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$, the other is case where the manifold has dimension 1. While these seem like very restrictive hypotheses, the main results and techniques in both cases are highly nontrivial. To indicate just how difficult the general problem is, here is a remark from the 2008 paper [23]:

> *Except for $M = S^1$, there is no known example of a torsion-free finitely generated group that doesn't act by homeomorphisms on M.*

Fisher's paper [22] is a beautiful survey on the Zimmer program, treating mostly the differentiable and measure-preserving case. For an interesting take on the Zimmer program for actions by homeomorphisms – with remarks on the use of torsion tricks, and enough other interesting nuggets to feed a research program for years – see [66]. [2]

Much of the success of the Zimmer program for 1-manifolds comes from a (remarkable!) positive answer to Question 4.1 in the case where M is one-dimensional. The answer comes from *orderability*, which will be the content of the rest of this lecture.

4.1 Left-invariant orders on groups

We start with a purely algebraic definition.

Definition 4.2. A group G is *left-orderable* if there is a total order $\leq$ on G that is invariant under left multiplication. Such a total order is called a *left-invariant order*, or *left-invariant linear order* on G[3] .

The study of left-orderable groups and left invariant orders on groups has deep connections with algebra, dynamics, and topology. You'll see some of this in Ying's talk this afternoon.

Example 4.3. (Examples of L.O. groups)

- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R} \times \mathbb{Z}$, ...

[2] This paper contains one of my personal favorite published wide-open questions: *"What groups are discrete subgroups of* $\mathrm{Homeo}(D^n$ *rel* $\partial)$*? Needless to say, they are torsion free (and they are subgroups of* $\mathrm{Homeo}(M)$ *for any* m*-manifold, for* $m \geq n$*)"*

[3] Of course, we could just as well work with right-invariant orders on G. At the moment, left-invariance seems to be more popular, but this was not always the case.

- Finitely generated torsion-free nilpotent groups (Malcev, 1960s)
- Free groups
- Surface groups *($\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ proof, another proof is given in the exercise below)*
- Braid groups *(hyperbolic geometry proof)*

Exercise 4.4. (Constructing new L.O. groups from old ones)

a) Suppose H is a normal subgroup of G, and both H and G/H have left-invariant orders. Use these to construct a left-invariant order on G.

b) Use this to give a proof that the fundamental group of a closed, orientable surface is left orderable.
Hint: a subgroups of a surface group is fundamental group of a (possibly noncompact) surface, so an infinite index subgroup is free[4]*...*

The next proposition and theorem are our motivation for studying left-orderable groups.

Proposition 4.5. $\mathrm{Homeo}_+(\mathbb{R})$ is left-orderable.

Proof. Given $f, g \in \mathrm{Homeo}_+(\mathbb{R})$, say that $f < g$ if $f(0) < g(0)$. Notice that this condition is invariant under left-multiplication. Unfortunately, we don't know what to do if $f(0) = g(0)$. To remedy this, instead of just looking at 0, enumerate a countable dense subset $\{x_1, x_2, x_3, ...\}$ of $\mathbb{R}$. Given $f, g \in \mathrm{Homeo}_+(\mathbb{R})$ let k be the minimum integer such that $f(x_k) \neq g(x_k)$. Now say that $f < g$ if $f(x_k) < g(x_k)$, and $f > g$ otherwise. ☐

It turns out that left-orderability completely characterizes countable subgroups of $\mathrm{Homeo}_0(\mathbb{R})$.

Theorem 4.6. Let G be a countable group. Then G is left-orderable if and only if there is an injective homomorphism $G \to \mathrm{Homeo}_+(\mathbb{R})$. Moreover, given an order on G, there is a canonical (up to conjugacy in $\mathrm{Homeo}_+(\mathbb{R})$) injective homomorphism $G \to \mathrm{Homeo}_+(\mathbb{R})$.

Proof. The "if" direction comes from the order inherited from G as a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. For the "only if", suppose that we have a left-invariant order on G. We start by mapping G into $\mathbb{R}$ (as a set) in an order-preserving way. Enumerate the elements of G as $g_0, g_1, g_2, ...$ and label $0 \in \mathbb{R}$ with g_0. If $g_1 > g_0$, then label 1 with g_1, if not, label -1 with g_1. Inductively, having identified points with g_0 through g_k, if $g_{k+1} > g_i$ for all $i \leq k$, assign g_{k+1} to the point 1 to the right of all the previously placed points (similarly if $g_{k+1} < g_i$, use a point one to the left). Otherwise, find the unique i and j such that $g_i < g_{k+1} < g_j$ appears in the total ordering of the group elements up to g_{k+1}, and label the midpoint of g_i and g_j with g_{k+1}.

Now G has a natural order-preserving action on the set of labeled points (by left-multiplication). Our construction implies that limit points are mapped to limit points, so the action extends to the closure. Call the closure A. We can also think of elements of G as permuting the connected components of $\mathbb{R} \setminus A$. To extend this to an action by homeomorphisms of $\mathbb{R}$, we specify that $g \in G$ map an interval in the complement of A to its image by the unique affine map between intervals. ☐

[4] for a rigorous proof of this fact, see section 4.2.2 of [63]

Note that countability was an essential ingredient in this proof. In fact, the "only if" direction is not true for uncountable groups! See [41] for discussion and examples.

Question 4.7. i) Is there a "nice" characterization of left-ordered groups that act on $\mathbb{R}$?
ii) More generally, given a totally ordered space, characterize the ordered groups that act on the space in an order-preserving way.
iii) Given an (uncountable) left-ordered group, find the "simplest possible" ordered space on which it acts.

4.2 Circular orders

Motivated by the definition of left-invariant orders and the relationship with homeomorphisms of the line, we try to replicate this for the circle. Whatever our definition of "circular order," we would like the following to be true.

"Theorem" 4.8. (Wishful thinking)

i) $\mathrm{Homeo}_0(S^1)$ is a circularly ordered group

ii) A countable group G is circularly orderable if and only if there is an injective homomorphism $G \to \mathrm{Homeo}_0(S^1)$.

Our model for a circular order on a *set* is the cyclic ordering of points on S^1. This is most easily described as a *cocycle.*

The order cochain on S^1. Define a function $\mathbf{ord} : S^1 \times S^1 \times S^1 \to \{\pm 1, 0\}$ by

$$\mathbf{ord}(x,y,z) = \begin{cases} 0 & \text{if } x = y,\ y = z \text{ or } x = z \\ 1 & \text{if } (x,y,z) \text{ is positively oriented} \\ -1 & \text{if } (x,y,z) \text{ is negatively oriented} \end{cases}$$

Exercise 4.9 (Cocycle condition)**.** There are only 6 nondegenerate "orderings" (combinatorial configurations) of a quadruple of distinct points x, y, z, w on the circle (check!), but 8 ways to assign a value if 1 or -1 to each triple. This should convince you that there is some relationship between $\mathbf{ord}(x,y,z), \mathbf{ord}(x,y,w), \mathbf{ord}(x,z,w)$ and $\mathbf{ord}(y,z,w)$. What is it?

As well as the cocycle condition, the function **ord** also satisfies an "orientation compatibility" condition – for any permutation σ, we have $\mathbf{ord}(x,y,z) = \mathrm{sign}(\sigma)\,\mathbf{ord}(\sigma(x),\sigma(y),\sigma(z))$. Thus, it's natural to think of **ord** as a *oriented 2-cochain* on the complete simplex on the set S^1. Notice also that **ord** is invariant under the left (diagonal) action of S^1 – for $g \in S^1$, $\mathbf{ord}(x,y,z) = \mathbf{ord}(gz, gy, gz)$. (It's also invariant under the action of $\mathrm{Homeo}_0(S^1)$, but more on that later.) In this sense, **ord** gives a S^1-left-invariant order on S^1.

There is a way to *de-homogenize* this function to get a function on $S^1 \times S^1$ that is not invariant under left-multiplication, but contains the same data – we simply define $c(x,y) := \mathbf{ord}(1, x, xy)$. If you are familiar with both the standard inhomogeneous and homogeneous n-cochain models for group cohomology, you'll recognize that **ord** is a an *homogeneous* 2*-cocycle*, and c is its inhomogeneous counterpart.

Definition 4.10 (Circular order, definition #1)**.** A *circular order* on a set Γ is an oriented integral 2-cocycle **ord** on Γ taking the values $0, 1$ and -1, and such that $\mathbf{ord}(x, y, z) \neq 0$ whenever (x, y, z) is a nondegenerate simplex. (i.e. a distinct triple).

If Γ is a group, a *left-invariant circular order* (often confusingly just called a *circular order* when the fact that Γ is a group is understood) is a Γ-invariant such cocycle.

Exercise 4.11 (Linear orders as cochains, see [65])**.** Suppose X is a totally ordered set. Define the *linear order cochain* $L : X \times X \to \{\pm 1, 0\}$ by

$$L(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x < y \\ -1 & \text{if } y < x \end{cases}$$

a) What relationship is there between $L(x, y), L(x, z)$ and $L(y, z)$? (i.e. what corresponds to transitivity of $<$?) Antisymmetry can be dealt with by thinking of L as a function not on $X \times X$ but on *oriented* 1-simplices in the complete simplex on X. Hence the terminology "cochain" again.

b) *Define* a linear order on X to be a 1-cochain satisfying the condition for transitivity above (and convince yourself this is an ok definition). Show that the coboundary of a linear order is a circular order, and conversely that every circular order on X is the coboundary of some linear order.

c) Now suppose that X is a group. Show that left-invariant circular orders on X are *not* necessarily the coboundaries of *left-invariant* linear orders on X.

Exercise 4.12 (Cut points definition)**.** The following alternative definition of circular order appears in [10]. Prove that it is equivalent to our definition.

> *Let S be a set with at least 4 elements. A circular ordering S is a choice of total ordering $<_p$ on $S \setminus \{p\}$ for each $p \in S$, such that if p and q are two distinct elements, the total orderings $<_p$, $<_q$ differ by a* cut *on their common domain of definition. That is, for any x, y distinct from p, q, the order of x and y with respect to $<_p$ and $<_q$ is the same unless $x <_p q <_p y$, in which case we have $y <_q p <_q x$.*

By imitating the proof of Theorem 4.6, we can show

Theorem 4.13. Let Γ be a countable circularly ordered group. Then there is a faithful homomorphism $\phi : \Gamma \to \mathrm{Homeo}_+(S^1)$. Moreover, ϕ can be constructed so that some point $x_0 \in S^1$ has trivial stabilizer in Γ.

Proof. Exercise. $\square$

Conversely, we also have

Theorem 4.14. Let Γ be a countable subgroup of $\mathrm{Homeo}_+(S^1)$. Then Γ has a circular order.

This is an immediate consequence of the following answer to our "wishful thinking" theorem above.

Theorem 4.15. $\mathrm{Homeo}_+(S^1)$ has a (left-invariant) circular order.

We prove this below, but first give a clever alternative proof of Theorem 4.14 due to Calegari. This proof uses Proposition 4.5 to modify an action of Γ on the circle to produce a new action where some point x has trivial stabilizer, in which case Γ inherits the cyclic order of the set $\Gamma x \subset S^1$.

Proof sketch of Theorem 4.14. (Following 2.2.14 in [10]).
Let $\Gamma \subset \mathrm{Homeo}_+(S^1)$ be countable. If $x \in S^1$ has trivial stabilizer in Γ, then Γ can be identified with the orbit of x, which inherits a natural (left Γ-invariant) circular order as a subset of S^1. Otherwise, let H be the stabilizer of x. Then H acts faithfully on $S^1 \setminus \{x\} \cong \mathbb{R}$, so is left-orderable, and proposition 4.5 now lets you build an action ϕ of H on $\mathbb{R} \cong [0,1]$ so that some point has trivial stabilizer. The idea now is to "blow up" the (countable) orbit $G \cdot x$, replacing each point of the orbit with an interval, to produce a new circle and new G-action where H acts by ϕ on each inserted interval. $\square$

To show now that $\mathrm{Homeo}_+(S^1)$ has a circular order, we use the fact that it's (universal) central extension $\widetilde{\mathrm{Homeo}}_+(S^1)$ can be identified with the subgroup of homeomorphisms of $\mathbb{R}$ that commute with integer translations. The proof can be generalized to show that, if G is any group, and $\tilde{G}$ a central extension

$$0 \to \mathbb{Z} \to \tilde{G} \to G \to 1$$

such that $\tilde{G}$ is linearly ordered, and the image of $\mathbb{Z}$ in $\tilde{G}$ is cofinal, then G has a left-invariant circular order. This is a theorem originally due to [68].

Proof of Theorem 4.15. Let $\tilde{G} = \widetilde{\mathrm{Homeo}}_+(S^1)$, and let T denote the generator of $\mathbb{Z} \subset \tilde{G}$. Put a left-invariant order on $\tilde{G}$ induced from a left-invariant order on $\mathrm{Homeo}_+(\mathbb{R})$ as in Proposition 4.5. Note that, for each $\tilde{g} \in \tilde{G}$, there is a unique $n \in \mathbb{Z}$ such that $T^n \leq \tilde{g} < T^{n+1}$, or equivalently, a unique n such that $\mathrm{id} \leq T^n \tilde{g} < T$.

We define a circular order on $\mathrm{Homeo}_+(S^1)$ as follows. Given $g \in \mathrm{Homeo}_+(S^1)$, let $\tilde{g} \in \tilde{G}$ denote the lift of g such that $\mathrm{id} \leq \tilde{g} < T$. Since different lifts of g differ by elements of $\mathbb{Z}$, our observation above implies that such a lift exists and is unique. For a nondegenerate triple g_1, g_2, g_3 in S^1, define

$$c(g_1, g_2, g_3) = \begin{cases} 1 & \text{if } \tilde{g}_1 < \tilde{g}_2 < \tilde{g}_3 \text{ up to a cyclic permutation of } 1,2,3 \\ -1 & \text{if } \tilde{g}_1 < \tilde{g}_3 < \tilde{g}_2 \text{ up to a cyclic permutation of } 1,2,3. \end{cases}$$

and for a degenerate triple where $g_i = g_j$ for some j, define $c(g_1, g_2, g_3) = 0$.

It is an easy exercise to check that this satisfies the order cocycle condition. We now show that it is invariant under left-multiplication. Let $h \in \mathrm{Homeo}_+(S^1)$, and let $\tilde{h}$ be any lift of h. Suppose for concreteness that, $\tilde{g}_1 < \tilde{g}_2 < \tilde{g}_3$. Then we in fact have

$$\ldots \leq T^{-1}\tilde{g}_1 < T^{-1}\tilde{g}_2 < T^{-1}\tilde{g}_3 < \mathrm{id} \leq \tilde{g}_1 < \tilde{g}_2 < \tilde{g}_3 < T \leq T\tilde{g}_1 < T\tilde{g}_2 < T\tilde{g}_3 \ldots$$

and, by left-invariance and the fact that T is central, we have

$$\ldots < T^{-1}\tilde{h}\tilde{g}_2 < T^{-1}\tilde{h}\tilde{g}_3 < \tilde{h} \leq \tilde{h}\tilde{g}_1 < \tilde{h}\tilde{g}_2 < \tilde{h}\tilde{g}_3 < T\tilde{h} \leq T\tilde{h}\tilde{g}_1 \ldots$$

Note that such a sequence gives a complete list of all lifts of each hg_i. Since consecutive appearances of lifts of hg_1, hg_2 and hg_3 in this sequence always have a positive order (i.e. after cyclic permutation of subscripts on g, they appear in the order 1,2,3), the triple of lifts that lie between id and T will have a positive cyclic order. This proves that the cocycle c is left-invariant.

$\square$

4.3 The Euler class for circularly ordered groups

Recall that $H^2(\mathrm{Homeo}_+(S^1);\mathbb{Z}) = \mathbb{Z}$, generated by the *Euler class* corresponding to the central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathrm{Homeo}_+(S^1) \to 1.$$

Moreover, this class has a *bounded representative* – in fact there is a cocycle representative taking only the values 0 and 1.

A two minute crash course on bounded cohomology of groups.
Let G be a group, and A an abelian coefficient group (think $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$). The cohomology $H^*(G;A)$ is the cohomology of the ("homogeneous") complex $(C^n)^G := \{f : G^{n+1} \to A \mid f(g_0, ...g_n) = f(hg_0, ...hg_n) \forall h \in G\}$ with coboundary

$$\delta f(g_0, ...g_{n+1}) = \sum (-1)^i f(g_0, ...\hat{g}_i, ...g_{n+1})$$

Let $(C_b^n)^G \subset (C^n)^G$ denote the subset of bounded functions. It is easy to see that $\delta(C_b^n)^G \subset (C_b^{n+1})^G$, so this is also an exact cocomplex. Its cohomology is denoted $H_b^*(G;A)$.

One can similarly use a bounded version of the *inhomogeneos* cocomplex to compute the cohomolgy. Here the n-cochains are functions $f : G^n \to A$, with no G-invariance assumption, but the coboundary operator is more complicated.

We list a few key facts to convince you that bounded cohomology is rich an interesting and contains a lot of information about the algebraic structure of the group. For a further introduction, see [7] or [10], and for an introduction to bounded cohomology in diverse areas of current research, see Monod's excellent survey paper [53].

Proposition 4.16 (Bounded cohomology – key facts and examples)**.**

1. Key example (Brooks, Epstein-Fujiwara, Bestvina-Fujiwara,...) If G is Gromov-hyperbolic, then $H_b^2(G;\mathbb{R})$ is infinite dimensional.
2. (Theorem of Trauber) If G is amenable, then $H_b^k(G;\mathbb{R}) = 0$ for all $k > 0$
3. There is a "comparison map" $H_b^*(G;A) \to H^*(G;A)$ given by forgetting that a cocycle representative is bounded. It is an open question in many cases whether this is injective and/or surjective. The kernel of this map on $H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ corresponds to *quasimorphisms* on G, and is closely related to *stable commutator length.*
4. (Dynamical interpretation, Ghys) As we'll see in the next section, certain elements of $H_B^2(G;\mathbb{Z})$ correspond to actions of G on the circle.
5. (Open questions) There is no countable group G for which $H_b^*(G,\mathbb{R})$ is known, except for cases where it is known to vanish in all degrees!

The Euler class in bounded cohomology. If Γ is a countable circularly ordered group, Theorem 4.13 gives us a (faithful) homomorphism $\phi : \Gamma \to \mathrm{Homeo}_+(S^1)$, so we can pull back the Euler class to obtain an element $\phi^*(e) \in H_b^2(\Gamma;\mathbb{Z})$. Coincidentally (or not...) the definition of circular order was also in terms of a bounded 2-cocycle.

Theorem 4.17 (Ghys, Barge-Ghys [3], Thurston [65]...)**.**

$$[\mathbf{ord}] = 2\phi^*(e) \text{ in } H_b^2(\Gamma; \mathbb{Z}).$$

Proof; details left as exercise. The theorem follows from an equality of (bounded) cocycles. Fix $x \in S^1$, and define an (inhomogeneous) "order cochain" on $\mathrm{Homeo}_+(S^1)$ by

$$c(f, g) := \mathbf{ord}(x, f(x), fg(x)).$$

One can check that this has coboundary 0, so defines an element of $H_b^2(\mathrm{Homeo}_+(S^1))$. If $\phi : \Gamma \to \mathrm{Homeo}_+(S^1)$ is the homomorphism of Theorem 4.13 and $x \in S^1$ is a point *with trivial stabilizer*, then $\phi^*(c)$ is exactly – as cochains – the original order cochain on Γ.

To prove Theorem 4.17, we will show that $[c] = 2\phi^*(e)$ in $H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{Z})$. One way to do this is to exhibit the difference $c - 2\phi^*(e)$ as an explicit coboundary. This is left as an exercise; see the "Further exercises" section for a hint. A related construction is given in [3].

As an alternative strategy, one can show first that $H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{Z})$ is one-dimensional – see Theorem 4.18 below, so one only needs to check which *multiple* of e corresponds to **ord**. ☐

Theorem 4.18 (Matsumoto–Morita [47])**.** $H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{Z}) \cong \mathbb{Z}$

Proof outline. By Thurston's work on cohomolgoy of homeomorphism groups in [64], the map $H^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \to H^2(B\,\mathrm{Homeo}_+(S^1); \mathbb{R}) \cong \mathbb{R}$ is injective. Since the Euler class, considered as a real cocylce, is nontrivial, we have $H^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \cong \mathbb{R}$.

The kernel of the comparison map $H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \to H^2(\mathrm{Homeo}_+(S^1); \mathbb{R})$ consists of the *homogeneous quasimorphisms* $\mathrm{Homeo}_+(S^1) \to \mathbb{R}$. (See the exercises for the definition of quasimorphism). It is a fact that there are no homogeneous quasimorphisms on a uniformly perfect group. By Herman's theorem, $\mathrm{Homeo}_+(S^1)$ is uniformly perfect, so it follows that the kernel of $H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \to H^2(\mathrm{Homeo}_+(S^1); \mathbb{R})$ is trivial. Since the Euler class has a bounded representative, we have

$$H_b^2(\mathrm{Homeo}_+(S^1); \mathbb{R}) \cong \mathbb{R}.$$

Finally, we move from $\mathbb{R}$ to $\mathbb{Z}$ coefficients by considering the exact sequence (applicable to any group G)

$$0 = H_b^1(G; \mathbb{R}) \to H_b^1(G; \mathbb{R}/\mathbb{Z}) = H^1(G; \mathbb{R}/\mathbb{Z}) \to H_b^2(G; \mathbb{Z}) \to H_b^2(G; \mathbb{R})$$

coming from the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$. ☐

Dynamical interpretation of the Euler class. The following theorem of Ghys says that the bounded Euler class completely captures the dynamics of a group action.

Theorem 4.19 (Ghys [27])**.** *Let $\phi_i : \Gamma \to \mathrm{Homeo}_+(S^1)$, $i = 1, 2$ be homomorphisms. If $\phi_1^*(e) = \phi_2^*(e)$ in $H_b^2(\Gamma; \mathbb{Z})$, then ϕ_1 and ϕ_2 are semiconjugate.*

To be *semiconjugate* means that ϕ_1 is obtained from ϕ_2 by "blowing up" orbits – the operation performed in the proof of Theorem 4.14 – and/or the inverse operation of collapsing wandering intervals For a precise definition see [27].

There is also an analoge of Theorem 4.13 phrased in terms of the Euler cocycle.

Theorem 4.20 (Ghys [27])**.** *Let Γ be a finitely generated group. If $c \in H_b^2(\Gamma; 0)$ has a cocycle representative taking only the values 0 and 1, then there is a homomorphism $\phi : \Gamma \to \mathrm{Homeo}_+(S^1)$ with $\phi^*(e) = c$.*

The relationship between Ghys' theorem and Theorem 4.13 is slightly subtle – the subtlety comes from the *nondegeneracy condition* on the cocycle **ord** in Definition 4.10. This condition amounts to forcing the action of Γ on S^1 to be highly nontrivial, in fact to have a point with trivial stabilizer. The circular order on Γ is then recovered from the circular order on the orbit of this point as a subset of S^1. There is no such condition built into the definition of the Euler cocycle; Ghys' theorem characterizes *all* actions of Γ on S^1 in terms of the second bounded cohomology of Γ. In particular, Γ need not even inject into $\mathrm{Homeo}_+(S^1)$ as ϕ is permitted to be a non-faithful action.

Below is a schematic summary.

1. Faithful actions:

$$\begin{tikzcd} \boxed{\begin{array}{c} \phi : \Gamma \hookrightarrow \mathrm{Homeo}_+(S^1) \\ \text{(arbitrary)} \end{array}} \arrow[r, "\text{blow up orbit}"] & \boxed{\begin{array}{c} \hat{\phi} : \Gamma \hookrightarrow \mathrm{Homeo}_+(S^1) \\ (x \text{ trivial stabilizer}) \end{array}} \arrow[r, leftrightarrow, "\mathbf{ord}"] & \boxed{\begin{array}{c} \text{C.O. (order cocycle) on } \Gamma \\ \text{from orbit of } x \end{array}} \end{tikzcd}$$

ϕ and $\hat{\phi}$ are semiconjugate. $\qquad \phi^*(e) = \hat{\phi}^*(e) = \mathbf{ord}$.

2. Arbitrary actions:

$$\begin{tikzcd} \boxed{\begin{array}{c} \phi : \Gamma \to \mathrm{Homeo}_+(S^1) \\ \text{(arbitrary)} \end{array}} \arrow[r, leftrightarrow, "\phi^*(e)"] & \boxed{\begin{array}{c} \{0,\, 1\}\text{-valued, bounded cocycle} \\ \text{in } H^2_b(\Gamma; \mathbb{Z}) \end{array}} \end{tikzcd}$$

Exercise 4.21. Give an algebraic condition on a cocycle taking values 0 and 1 equivalent to the nondegeneracy condition on the circular order cocycle.

4.4 Further exercises

1. ***(Open question) Give an example of a compact manifold $M \neq S^1$ and a finitely generated, torsion-free group Γ, with no faithful homomorphism $\Gamma \to \mathrm{Homeo}(M)$.
2. Let Γ be a group. Prove that the following conditions are each equivalent to being left-orderable
   (a) (Positive cone condition). Γ admits a decomposition into semigroups $\Gamma = \{id\} \sqcup P \sqcup N$, where $N = \{g^{-1} : g \in P\}$.
   (b) *(Finite condition, see [16] for proof). For every finite collection of nontrivial elements $g_1, ..., g_k$, there exist choices $\epsilon_i \in \{-1, 1\}$ such that the identity is not an element of the semigroup generated by $\{g_i^{\epsilon_i}\}$. (hint: one direction is easy: suppose Γ is orderable, and choose exponents to make g_i positive...)
   (c) **(Burns–Hale theorem, see proof sketch in [10]). Every finitely generated subgroup H of Γ admits a surjective homomorphism to a nontrivial left-orderable group.
3. (Dynamical realization of bi-orderable groups, suggested by Ying Hu)
   A group is called *bi-orderable* if it has a total order which is preserved under both left- and right-group multiplications. Prove that a group Γ is bi-orderable if and only if it admits an embedding into $\mathrm{Homeo}_+(\mathbb{R})$, such that, for any $g \in \Gamma$ either $gx \geq x$ or $gx \leq x$ for any $x \in \mathbb{R}$.
4. *Does $\mathrm{Homeo}_+(S^1)$ have a left-invariant circular order in the sense of Definition 4.10?
5. Show that $c - 2\phi^*(e)$ is the coboundary of a bounded 1-cochain.
   Hint: following [33], use the "dirac mass" function $\delta_0(f) = \begin{cases} 1 \text{ if } f(0) = 0 \\ 0 \text{ otherwise} \end{cases}$ (and perhaps modify by a constant).

6. (Bounded cohomology and quasimorphisms) A function $\phi : G \to \mathbb{R}$ is a *quasimorphism* if there exists D such that

$$|\phi(g_1 g_2) - \phi(g_1) - \phi(g_2)| \text{ for all } g_1, g_2 \in G.$$

You can think of it as being "a homomophism up to bounded error."

In fancier language, ϕ is a (inhomogeneous) 1-cochain on G, whose coboundary is bounded.

   (a) Let $QM(G)$ denote the set of all quasimorphisms on G. Show that

$$QM(G)/C^1_b(G) \oplus \mathrm{Hom}(G; \mathbb{R})$$

   is isomorphic to the kernel of the comparison map $H^2_b(G; \mathbb{R}) \to H^2(G; \mathbb{R})$. (The isomorphism is given by the usual coboundary).

   (b) * A quasimorphism $\phi : G \to \mathbb{R}$ is called *homogeneous* if $\phi(g^n) = n\phi(g)$ for all n. Show that each $\phi \in QM(G)$ has a homogeneous representative in $QM(G)/C^1_b(G)$. In other words, a quasimorphism can be "homogenized" by adding a bounded function.

   (c) **(Example due to Brooks) Let $G = \langle a, b \rangle$ be the free group on two generators. For a word w, define $\phi(w)$ to be the number of appearances of the string aab in the word w, minus the number of the appearances of aab in w^{-1}. Show that ϕ is a quasimorphism. Hint: work on the cayley graph.

   (d) ***(The Poincaré rotation number) Let $G = \widetilde{\mathrm{Homeo}}_+(S^1)$, thought of as the group of homeomorphisms of $\mathbb{R}$ that commute with integer translation. Define a function $\phi : G \to \mathbb{R}$ by

$$\phi(f) = \lim_{n \to \infty} \frac{f^n(0)}{n}.$$

   Show that ϕ is a quasimorphism. What is its relationship with the Euler class?

7. *** (Open question, Calegari) Let $\mathrm{Homeo}(D^2, \partial)$ be the group of homeomorphisms of the 2-dimensional disc that fix the boundary pointwise. Is $\mathrm{Homeo}(D^2, \partial)$ left-orderable?

   Calegari has an enlightening blog post on this topic: lamington.wordpress.com/2009/07/04/orderability-and-groups-of-homeomorphisms-of-the-disk/

   In particular, the group of *diffeomorphisms* $\mathrm{Homeo}(D^2\, \partial)$ is left-orderable! The proof of this uses *Thurston Stability* – the main technique developed by Thurston to generalize the Gray stability theorem in foliation theory.

5 Metrics on homeomorphism and diffeomorphism groups

Lectures 1-3 discussed the interplay between the algebraic structure of $\mathrm{Diff}_0^r(M)$ and the topology of M and $\mathrm{Diff}_0^r(M)$. Today, we turn to a new topic – new also in the sense that most research on it is in its infancy – and study the *geometry* of $\mathrm{Diff}_0^r(M)$. However, we'll see familiar techniques re-appear in this context.

Motivated by the algebra $\leftrightarrow$ topology relationship, one of our main goals will be to understand the relationship between the *geometry* of $\mathrm{Diff}_0^r(M)$, with respect to a suitable metric, and the topology of the underlying manifold M. Of course, the geometry of $\mathrm{Diff}_0^r(M)$ depends on the choice of metric, so we begin with a discussion of candidates for good metrics on $\mathrm{Diff}_0^r(M)$. We would like the metric to reflect the fact that $\mathrm{Diff}_0^r(M)$ is a *group*, so for today we consider only *left-invariant* metrics[5]. Recall that the Birkhoff-Kakutani theorem implies that any metrizable topological group admits a left-invairant metric that generates its topology, so restricting our attention to left-invariant metrics is not unreasonable.

We start this lecture with a zoo of diverse examples of metrics. Eventually, we turn our focus to word metrics, and then motivate the study of the large-scale geometry of such metrics by introducing an *extension problem*. Lecture 6 is a detailed study of large-scale geometry of homeomorphism groups.

5.1 Constructing metrics 1: "Riemannian" metrics

If G is a Lie group, there exist many left-invariant Riemannian metrics on G, obtained by taking any inner product on the tangent space $T_{\mathrm{id}}G \cong \mathfrak{g}$ and then left-translating it to give an inner product at the tangent space to each other point.

We mentioned in Lecture 1 that, for $r \geq 1$, $\mathrm{Diff}_0(M)$ is a Banach or Frechet manifold, whose tangent space at the identity can be identified with the Lie algebra of C^r vector fields on M. The same process gives us a *weak-Riemannian structure* on $\mathrm{Diff}_0(M)$.

Definition 5.1. A *weak-Riemannian structure* on a manifold is a smooth choice of inner product on each tangent space.

By contrast a *strong Riemannian structure* is a smooth choice of inner product on each tangent space *that induces an isomorphism of the tangent space at each point with the cotangent space at that point.* In finite dimensions this comes for free, but in general, strong Riemannian structures on a manifold only exist if the manifold is modeled on a Hilbert space. However, one can still use a weak-Riemannian structure to define path length and a (possibly degenerate) metric.

Definition 5.2. Let G be a manifold, and $\langle , \rangle_g$ an inner product on $T_g(G)$. The *length* of a path $\{g_t\}_{t=0}^1$ is

$$\ell(g_t) = \int_{t=0}^{1} \langle \dot{g}_t,\, \dot{g}_t \rangle_{g_t} \; dt$$

where $\dot{g}_t = \frac{d}{dt}(g_t)$

[5] Of course, we could just as well consider right-invariant metrics – if you haven't checked already, verify that d_L is a left-invariant metric on a group G if and only if $d_R(g_1, g_2) := d_L(g_1^{-1}, g_2^{-1})$ is a right-invariant metric.

Remark 5.3. Definition 5.2 works just as well given only a norm $\|\cdot\|_g$ on $T_g(G)$; we define $\ell(g_t) := \int \|\dot{g}_t\|_{g_t}\, dt$

Given a notion of path-length, we get a distance function

$$d(f,g) =: \inf\{\ell(g_t) \mid g_t \text{ is a path from } f \text{ to } g\}.$$

In some cases, the path-distance defined by this weak-Riemannian structure gives a genuine metric. (However, weak-Riemannian structures are not generally so well behaved – certain choices of inner product on vector fields on S^1 give a path-distance that is $\equiv 0$ [4]!) When one does have a metric, one can ask whether it is complete, geodesic, locally geodesic, etc. Little is known about such metrics on $\mathrm{Diff}_0(M)$, except when $M = S^1$. See the introduction of [18], or the paper [13] for a brief survey of recent results.

A success story: Hofer's metric. Let (M,ω) be a symplectic manifold. The *Hofer metric* on the space $\mathrm{Ham}_c(M)$ of compactly supported Hamiltonian symplectomorphisms is the path-metric distance given by the L_∞ norm $\|H_t\| = \sup_{x\in M} H_t(x) - \inf_{x\in M} H_t(x)$ on the space of hamiltonian vector fields (the tangent space at the identity to $\mathrm{Ham}_c(M)$). Nondegeneracy of this metric is a difficult theorem, proved for $\mathbb{R}^{2n}$ by Hofer, and in the most general case by Lalonde and McDuff. See Chapter 12 of [48] for more on the geometry of Ham with the Hofer metric.

It is easy to check that Hofer's metric is not only left, but also right–invariant. Remarkably, it is the *only* such metric on $\mathrm{Ham}_c(M)$!

Theorem 5.4 (Buhovsky – Ostrover [8])**.** Any continuous, bi-invariant, nondegenerate Finsler metric on $\mathrm{Ham}_c(M)$ is equivalent to the Hofer metric.

Eliashberg and Polterovich earlier showed that using the L_p rather than L_∞ norm on hamiltonian vector fields gives a totally degenerate metric, i.e. with $d \equiv 0$. Another way in which a metric can be considered "degenerate" is if it is bounded. The following is a well known open question.

Question 5.5 (open)**.** Does there exist a symplectic manifold (M,ω) such that the Hofer norm on $\mathrm{Ham}(M)$ is bounded?

For many examples (Lalonde, McDuff, Schwartz, Entov, Polterovich...) the Hofer norm is known to be unbounded, and as far as I know, the expected answer to Question 5.5 is "no"

5.2 Constructing metrics 2: Word metrics

If G is any group, and S a symmetric generating set for for G, define a *word norm* $\|\cdot\|_S$ on G by

$$\|g\|_S = \min\{k \mid g = s_1 s_2 ... s_k \text{ where } s_i \in S\}.$$

There is a standard procedure to turn a norm on a group into a left-invariant metric: define

$$d_S(g,h) = \|h^{-1}g\|_S.$$

Exercise 5.6 (standard facts)**.** Check that the word norm defined above is indeed symmetric, nondegenerate, and subadditive (i.e. $\|gh\| \le \|g\| + \|h\|$). Check that d_S satisfies the triangle inequality. Show how one can use a norm to define a *right*-invariant metric in a similar way.

Example: Fragmentation norm. Let $\mathcal{U}$ be an open cover of M. The *$\mathcal{U}$-fragmentation norm* on $\mathrm{Diff}_0^r(M)$ is the word norm with respect to the generating set

$$\{f \in \mathrm{Diff}_0^r(M) \mid \mathrm{supp}(f) \subset U \text{ for some } U \in \mathcal{U}\}.$$

The fragmentation metric will play an important role in the second half of this talk, in particular, we'll see how it changes as we vary $\mathcal{U}$.

We mention briefly another application of the fragmentation norm. Let $\mathrm{Homeo}_c(D^2, \text{area})$ denote the group of area-preserving homeomorphisms of the disc that fix a neighborhood of the boundary. It is an open question whether this group is simple. In [37], F. Le Roux shows that simplicity is equivalent to boundedness of a certain fragmentation norm.

Theorem 5.7 (Le Roux [37])**.** The following fragmentation condition is equivalent to simplicity of $\mathrm{Homeo}_c(D^2, \text{area})$. *There exists a constant m such that any area-preserving homeomorphism of the plane supported on a disc of area 1 is the composition of m area-preserving homeomorphisms supported on topological discs each of area 1/2.*

Example: Autonomous metric.[6] In Lecture 1, we mentioned that every $f \in \mathrm{Diff}_0(M)$ is the time-1 map of the flow of a time-dependent vector field; similarly every $f \in \mathrm{Ham}(M)$ is the time-1 map of the flow of a time-dependent Hamiltonian vector field. This is not true if one removes the "time-dependent" condition – in fact there are diffeomorphisms of S^1 (or any manifold) arbitrarily close to the identity that are not the time one map of any flow. In other words, the Lie group *exponential map* from $\mathfrak{X}(M)$ to $\mathrm{Diff}_0(M)$ is not surjective. See Chapter 1 of [36] for more details.

However, $\mathrm{Diff}_0(M)$ is *generated* by the set of flows, and $\mathrm{Ham}(M)$ is generated by the set of autonomous (time independent) Hamiltonians. The *autonomous metric* on $\mathrm{Diff}_0(M)$ and $\mathrm{Ham}(M)$ are defined using the word norm with respect to these generating sets. This has been studied (although is not well understood) in the case of $\mathrm{Ham}(M)$, see [5] [6].

Example: commutator length Let G be a perfect group (for example, $\mathrm{Diff}_0^r(M)$ for $r \neq \dim(M)+1$). Then the *commutator subgroup* $[G, G]$ generates G and the *commutator length* of an element, $\mathrm{cl}(g)$, is the norm with respect to this generating set. If cl, and hence the induced norm on G, is bounded, then G is called *uniformly perfect*. See Exercise **??** from Lecture 1. Commutator length is closely related to quasi-morphisms on $\mathrm{Diff}_0^r(M)$, see e.g. [25].

Exercise 5.8. (Conjugation-invariant metrics.)

a) A metric is called *conjugation-invariant* if $d(a, b) = d(gag^{-1}, gbg^{-1})$ for all $a, b, g \in G$. Suppose d is a left-invariant metric. Verify that d is also right-invariant iff it is conjugation-invariant.

b) Verify that the autonomous metric and the commutator length metric are conjugation-invariant.

c) Define the *conjugation-invariant fragmentation norm* on $\mathrm{Diff}_0(M)$ by

$$\|f\| = \min\{k \mid f = f_1 f_2 ... f_k \text{ where } \mathrm{supp}(f_i) \text{ is contained in some embedded ball in } M\}.$$

Verify that the induced left-invariant metric is conjugation-invariant.

[6] "autonomous" comes from the fact that the vector fields are autonomous, or *time–independent*.

In [9], Burago, Ivanov and Polterovich discuss general conjugation-invariant norms on diffeomorphism groups. They prove

Theorem 5.9 ([9])**.** Let M be a compact manifold. Any conjugation invariant norm $\|\cdot\|$ on $\mathrm{Diff}_0(M)$ is "discrete" in the sense that 0 is not an accumulation point of $\{\|g\| : g \in G\}$.
If M is a 3-manifold, or $M = S^n$, then any conjugation-invariant norm on $\mathrm{Diff}_0(M)$ is discrete and bounded.

In particular, the first part of this theorem shows that no sub-Riemannian metric on $\mathrm{Diff}_0(M)$ can be conjugation invariant, in contrast with, the Hofer metric on Ham, or the metric induced by the killing form on a compact semi-simple Lie group. One step of the proof is to reduce the theorem to one about a particular conjugation-invariant norm.

Theorem 5.10 ([9])**.** If the conjugation-invaraint fragmentation norm is bounded on $\mathrm{Diff}_0(M)$, then any conjugation invariant norm on $\mathrm{Diff}_0(M)$ is bounded.

Question 5.11 (Open question, [9])**.** Does there exist an unbounded conjugation-invariant norm on $\mathrm{Diff}_0(\Sigma_g)$?

5.3 Motivation: A new take on extension problems

In Bena's lecture on Wednesday, we saw the following

Theorem 5.12. Let $\Gamma = \pi_1(\Sigma_g) \xrightarrow{\phi} \mathrm{Diff}_+(S^1)$ be the standard Fuchsian representation (with image a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$). Let S be a surface with $\chi(S) < 0$, and at least one S^1 boundary component. Then there is no *extension* of ϕ to $\mathrm{Homeo}_0(S)$, i.e. no homomorphism ρ such that the diagram below commutes.

$$\begin{tikzcd} & \mathrm{Diff}_0(S) \arrow[d, "restrict"] \\ \Gamma \arrow[ur, "\rho"] \arrow[r, "\phi"'] & \mathrm{Diff}_+(S^1) \end{tikzcd}$$

The essence of the proof was an argument that the Euler class $\phi^*(e) \in H^2(B\Gamma;\mathbb{Z})$ is nonzero, but that $H^2(B\,\mathrm{Diff}_0(S)) = 0$.

Motivated by this success, we ask a seemingly similar question:

Question 5.13. Let D^2 denote the 2-disc. Does there exist a finitely generated group Γ, and homomorphism $\phi : \Gamma \to \mathrm{Diff}_+(S^1)$ with no extension to $\mathrm{Diff}_0(D^2)$?

Unfortunately there can be no cohomological obstruction in this case, as $B\,\mathrm{Diff}_0(D^2)$ is homotopy equivalent to $B\,\mathrm{Diff}_+(S^1)$, and the restriction map is a homotopy equivalence (exercise: prove), so induces an isomorphism on cohomology of classifying spaces. Note also that the corresponding question for groups of *homeomorphisms* has a positive answer – any group of homeomorphisms of S^1 can be extended to a group of homeomorphisms of D^2 by "coning off" the action. However, a clever argument of Ghys gives a negative answer.

Theorem 5.14 (Ghys [26])**.** There is a finitely generated group Γ, and homomorphism $\phi : \Gamma \to \mathrm{Diff}_+(S^1)$ with no extension to $\mathrm{Diff}_0(D^2)$

The theorem Ghys states in [26] is weaker – he states that the whole group $\mathrm{Diff}_+(S^1)$ does not extend – but his proof produces a finitely generated example. The following outline of Ghys' proof is adapted from [40].

Proof. Let Γ be the group with the following presentation

$$\langle f, g, t \mid t^6 = \mathrm{id}, [f, g] = t,\ [f, t^2] = \mathrm{id},\ [g, t^2] = \mathrm{id} \rangle$$

We now construct an action of Γ on S^1 by diffeomorphisms. Recall from Lecture 2 that any rotation of the circle can be written as a commutator of hyperbolic elements in $\mathrm{PSL}(2, \mathbb{R}) \subset \mathrm{Diff}_+(S^1)$. So let $\bar{f}$ and $\bar{g}$ be such that $[\bar{f}, \bar{g}]$ is a rotation of order 2. Let f and g be lifts of $\bar{f}$ and $\bar{g}$ to diffeomorphisms of the threefold cover of the circle – and identify this cover with S^1. If we choose the (unique) lifts that have fixed points, then the commutator $[f, g]$ will be rotation of S^1 by $\pi/3$. Let $t = [f, g]$. Then $t^6 = \mathrm{id}$, and t^2 is a deck transformation of our threefold cover, so commutes with the lifts f and g. This gives the desired action of Γ on S^1.

Now assume for contradiction that there is an extension $\rho : \Gamma \to \mathrm{Diff}_0(D^2)$. Since $\rho(t)$ has order 6, Kerekjarto's theorem (see [12] for a nice proof) implies that $\rho(t)$ is conjugate to an order 6 rotation, hence has a unique fixed point in the interior of D^2. Let x denote this fixed point. Since $\rho(f)$ and $\rho(g)$ commute with $\rho(t)$, they also fix x. This gives us a homomorphism $\Gamma \to \mathrm{GL}(2, \mathbb{R})$ by taking derivatives at x:

$$\gamma \mapsto D(\rho(\gamma))_x.$$

Now $D(\rho(t^2))_x$ is a rotation by $2\pi/3$, and it follows from the relations in Γ that $D(\rho(f))_x$ and $D(\rho(g))_x$ lie in its centralizer. But the centralizer of such a rotation in $\mathrm{GL}(2, \mathbb{R})$ is abelian, hence $[D(\rho(f))_x, D(\rho(g))_x] = \mathrm{id}$. This contradicts the fact that $[D(\rho(f))_x, D(\rho(g))_x] = D(\rho(t))_x$. □

With a small modification of this argument, Ghys also shows that there is a homomorphism of a finitely generated group $\Gamma \to \mathrm{Diff}_0(S^n)$ that does not extend to $\mathrm{Diff}_0(D^{n+1})$, for all n *odd*. In all cases, the main trick is using torsion to force a global fixed point.

Question 5.15. Does there exist a finitely generated *torsion free* group Γ, and homomorphism $\phi : \Gamma \to \mathrm{Diff}_+(S^1)$ with no extension to $\mathrm{Diff}_0(D^2)$?

The easiest way to adapt Ghys' proof would be to find a direct substitute for torsion – an algebraic condition on a group $\Gamma \subset \mathrm{Diff}_0(D^2)$ that would force an element to have a unique fixed point, or at least a predictable fixed set. Fortunately, there is such a property – *distortion*.

Definition 5.16. Let Γ be a finitely generated group, with generating set S, and let $\| \cdot \|_S$ denote the word norm with respect to S. An element $g \in \Gamma$ is *distorted* if

$$\liminf_{n \to \infty} \frac{\|g^n\|_S}{n} = 0$$

The following theorem of Franks and Handel now gives us exactly what we would like:

Theorem 5.17 (Consequence of Franks–Handel [24])**.** Let Γ be a finitely generated subgroup of $\mathrm{Diff}_0(D^2)$, and $t \in \Gamma$ a distorted element. If t acts on ∂D^2 as a nontrivial rotation, then t has a *unique* interior fixed point.

Using this, it is possible to modify Ghys' example and give a positive answer to Question 5.15. See [40] for a proof.

Distortion in groups of diffeomorphisms. Franks and Handel's work in [24] proves a much more general result about distorted elements in groups of diffeomorphisms of surfaces – Theorem 5.17 is only one specific instance. The general theme is that distortion places strong restrictions on the dynamics of a group element. Hurtado's proof of Theorem 3.4 also relies heavily on distorted elements in groups of diffeomorphisms, and even more recent results along this theme are contained in [24], [11], [50], and [54]. In many cases, distortion has provided a fruitful approach to questions related to the Zimmer program.

One unsatisfactory point about the definition of distortion given above is that it references a particular choice of generating set S for Γ. Fortunately, this is not necessary:

Exercise 5.18. Let Γ be a finitely generated group, and S and T finite generating sets. Show that g is distorted with respect to $\|\cdot\|_S$ if and only if it is distorted with respect to $\|\cdot\|_T$.

In the next lecture, we will see this as a specific instance of a more general theme: distortion is a *quasi-isometry invariant* and any two word metrics on Γ are *quasi-isometric.*

However, there is another unsatisfactory point in our definition of distortion – the reference to a particular *finitely generated subgroup* $\Gamma \subset \mathrm{Diff}_0(M)$. Ideally, one would like a notion of distortion for elements in homeomorphism or diffeomorphism groups akin to the following:

Definition 5.19 (Tentative definition?)**.** Let $G = \mathrm{Diff}_0(M)$ or $\mathrm{Homeo}_0(M)$, and let S be a generating set for G, with word norm $\|\cdot\|_S$. An element $g \in \Gamma$ is *distorted* if

$$\liminf_{n\to\infty} \frac{\|g^n\|_S}{n} = 0$$

However, as we'll see in the next lecture, since G is not finitely (nor compactly) generated, this notion is no longer independent of the generating set. Historically, the solution has been to *define* an element of $\mathrm{Diff}_0(M)$ to be distorted if there exists a finitely generated subgroup in which it is distorted. In the next lecture, we propose a better solution – at least for $\mathrm{Homeo}_0(M)$ – by identifying a *preferred* word metric[7] on $\mathrm{Homeo}_0(M)$. We'll also see a theorem of Rosendal that shows that this metric, despite being discrete, captures the topology of $\mathrm{Homeo}_0(M)$ at least on a large scale, and we'll study the large-scale geometry of homeomorphism groups.

5.4 Further exercises

1. (a) *Modify the argument of Theorem 5.14 to produce a finitely generated group Γ and homomorphism $\Gamma \to \mathrm{Diff}_0(S^n)$ that does not extend to $\mathrm{Diff}_0(D^{n+1})$, for n *odd.*
   (b) ***(open). Do the same for n even.
2. Answer Bena's question from problem set 3:
   For $n \geq 2$, give an example of a group action $\Gamma \to \mathrm{Homeo}_0(Sn \cup \ldots \cup S^n)$ such that
   - Γ is countable and torsion free
   - There exists W with $\partial W = Sn \cup \ldots \cup S^n$ for which the action does not extend.

[7]technically, a preferred quasi-isometry class. This is completely analogous to the situation for finitely generated groups, where all finite generating sets give quasi-isometric word metrics, but an *infintie* generating set might not.

3. **(reprove my theorem) Given an example of a countable, torsion-free group Γ and an action $\Gamma \to \mathrm{Homeo}_0(S^1)$ that does not extend to $\mathrm{Homeo}_0(D^2)$. *** Can you do this for S^2 and D^3?

4. (The autonomous metric on $\mathrm{Diff}_0(M)$). Let $S \subset \mathrm{Diff}_0(M)$ be the set of diffeomorphisms that are the time 1-map of the flow of some vector field. (Recall that in lecture 1 we said $S \neq \mathrm{Diff}_0(M)$).

   (a) Prove that S generates $\mathrm{Diff}_0(M)$. (use simplicity of $\mathrm{Diff}_0(M)$)

   (b) Show, if you didn't already that the word metric d_S is conjugation-invariant.

   (c) **Can you give an example of a manifold where the autonomous metric is unbounded?

5. *** (open) Let $g \geq 1$. Find an unbounded conjugation-invariant metric on $\mathrm{Diff}_0(\Sigma_g)$

6 Large-scale geometry of homeomorphism groups

6.1 The basics of large-scale geometry

Large-scale or *coarse* geometry is the study of properties of metric spaces that can be identified by someone with poor eyesight – properties that can be formulated in terms of large distances only. These properties are usually named “quasi–” or “coarse–” (e.g. coarse connectedness, coarse simple connectedness, quasi-geodesic, quasi-isometry...)

Definition 6.1. Let X, d_X and Y, d_Y be metric spaces. A *quasi-isometric embedding* $X \to Y$ is a function $f : X \to Y$ such that there exist k, c with

$$\frac{1}{k} d_Y(f(p), f(q) - c \leq d_X(p, q) \leq k d_Y(f(p), f(q)) + c$$

for all points $p, q \in X$.

If, additionally each point $y \in Y$ is a uniformly bounded distance from a point in the image of f, we say that f is a *quasi-isometry*

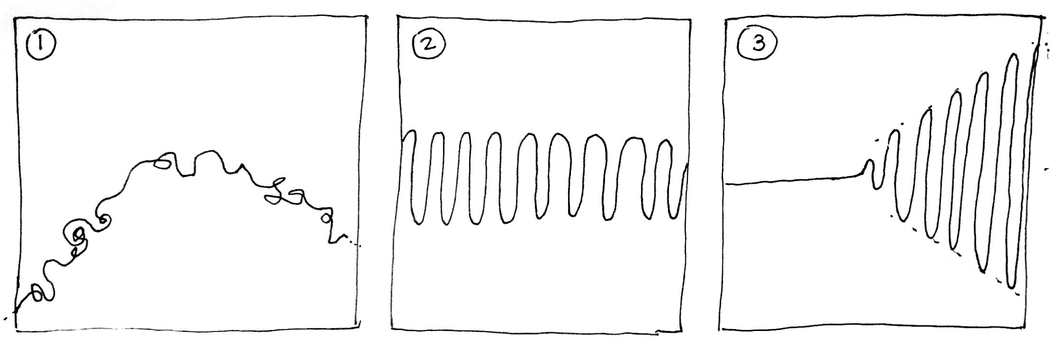

Figure 2: Assume these are length-preserving embeddings $\mathbb{R} \to \mathbb{R}^2$ Which are quasi-isometric embeddings?

Exercise 6.2. (For those unfamiliar with large-scale geometry.) Show that *quasi-isometric* is an equivalence relation on metric spaces. Show it is nontrivial by giving an example of metric spaces that are not quasi-isometric.

Example 6.3. The following are examples of quasi-isometries.

1. The inclusion of $\mathbb{Z}^n$ into $\mathbb{R}^n$
2. The “inverse” map $\mathbb{R}^n \to \mathbb{Z}^n$ given by $(x_1, ...x_n) \mapsto (\lfloor x_1 \rfloor, ... \lfloor x_n \rfloor)$
3. Let Γ be a finitely generated group, and S and T symmetric, finite, generating sets with word metrics d_S and d_T. The identity map $(\Gamma, d_S) \to (\Gamma, d_T)$. is a quasi-isometry

4. If Γ' is a finite index subgroup of Γ, then inclusion $\Gamma' \to \Gamma$ is a quasi-isometry (with respect to any word metrics)

5. (Generalizing 5.2.1) If M is a compact manifold, then $\tilde{M}$ is quasi-isometric to $\pi_1(M)$. This is a special case of:

6. (Milnor-Schwartz[8]) Let Γ be a finitely generated group, and suppose Γ acts properly discontinuously on a space X by isometries. If X/Γ is compact, then any *orbit map* $\gamma \mapsto \gamma(x_0)$ is a quasi-isometry $\Gamma \to X$. [9]

Item 3 in the example above gives us a first indiction why quasi-isometry is the "right" notion of equivalence for metrics on groups, at least in the finitely-generated case. A finitely generated group Γ does not have a canonical metric, but it *does* have a canonical quasi-isometry type. This statement readily generalizes to locally compact groups.

Exercise 6.4. Let G be a locally compact, compactly generated group. If S and T are compact generating sets, show that (G, d_S) and (G, d_T) are quasi-isometric. [See also problem 1. in the "Further exercises" section.]

For example, you proved as an easy exercise in Lecture 1 that a connected group was generated by any neighborhood of the identity. If your group is locally compact (for example, a Lie group), Exercise 6.4 now says that the corresponding word metric gives a well-defined QI type, independent of the neighborhood of the identity you chose.

Considering metrics up to quasi-isometry dissolves the division between word metrics and metrics that generate the topology of G (such metrics are said to be *compatible*).

Theorem 6.5 (Birkoff-Kakutani, Struble, Rosendal[10])**.** Let G be a locally compact metrisable group generated by a compact symmetric set S. Then G admits a compatible left-invariant metric quasi-isometric to d_S.

Much of the existing literature on large-scale geometry of groups focuses on the finitely generated case. The general locally compact case is treated in a wonderful and very new book of Cornulier and de la Harpe [15]. For the more advanced, Cornulier also has a survey preprint on the quasi-isometric classification of locally compact groups [14], with emphasis on the hyperbolic/negatively curved case and QI rigidity of locally symmetric spaces.

To illustrate the power of the large-scale perspective, we give a quick proof of a toy theorem.

Theorem 6.6 (Toy theorem)**.** If a finite index subgroup of $\mathbb{Z}^n$ is isomorphic to a finite index subgroup of $\mathbb{Z}^m$, then $m = n$.

Proof. Suppose there is an isomorphism of finite index subgroups of $\mathbb{Z}^m$ and $\mathbb{Z}^n$. Then there exists a (k, c) quasi-isometry $\phi : \mathbb{Z}^n \to \mathbb{Z}^m$. We'll use a trick to get rid of the additive constant c and turn this into a k-bilipschitz map.

[8] This is often called the "Milnor-Švarc" lemma, but according to a story I heard from M. Kapovich, this spelling is the result of a translation of "Schwartz" into and back out of Russian. The point of the story being that translation is only a quasi-isometry of languages...

[9] Gromov later proved a theorem that indicates cocompact actions are exactly the right thing to think about: *Two finitely generated groups Γ_1 and Γ_2 are quasi-isometric iff there exist commuting proper cocompact actions of Γ_1 and Γ_2 on some locally compact Hausdorff space X.*

[10] This is Rosendal's modification of Struble's modification of the Birkhoff–Kakutani metrisation theorem

For $\epsilon > 0$, let $\mathbb{Z}^n_\epsilon$ denote $\mathbb{Z}^n$ with the usual metric rescaled by ϵ, so the old unit ball is the new ϵ-ball. You can think of $\mathbb{Z}^n_\epsilon$ as sitting inside $\mathbb{R}^n$ as a lattice generated by $(\epsilon, 0, 0, ...0)$, etc.

We have

$$d_{\mathbb{Z}^m_\epsilon}(\phi(x), \phi(y)) = \epsilon d_{\mathbb{Z}^m}(\phi(x), \phi(y)) \leq \epsilon \left(k d_{\mathbb{Z}^n_\epsilon}(x, y) + c\right) = k d_{\mathbb{Z}^n_\epsilon}(x, y) + \epsilon c$$

So ϕ is a $(K, \epsilon c)$ quasi-isometry as a map $\mathbb{Z}^n_\epsilon \to \mathbb{Z}^m_\epsilon$. Now we "take a limit": define $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}^m$ by taking any sequence $x_j \in \mathbb{Z}^n_{1/j}$ that converges to x, and set $\hat{\phi}(x_j) = \lim_{j\to\infty} \phi(x_j)$. This gives a $(K, 0)$ quasi-isometry, i.e. a k-bi-Lipschitz map. A bi-Lipschitz map is a homeomorphism onto its image – which implies that $m \leq n$. But we could have just as well done this with m and n reversed. $\square$

Remark 6.7. An important application of this "rescaling the metric" technique is a kind of *coarse differentiation* of quasi-isometries: one first turns a quasi-isometry into a bi-Lipshitz map (of a different space...), then applies Rademacher's theorem to conclude this map is almost everywhere differentiable. This strategy has been quite fruitful in problems of classification of quasi-isometries.

6.2 Large-scale geometry of general metrisable groups

Our next goal is to try to apply the techniques of large-scale geometry to homeomorphism and diffeomorphism groups. Unfortunately, these groups are *not* locally compact – so the word metric with respect to a nice generating set may not give a canonical quasi-isometry type as in Exercise 6.4. Here is an example of what can go wrong.

Example 6.8. Consider the (additive) group $\mathbb{R}^{\mathbb{N}}$ with the standard product topology. This group is connected, so generated by any neighborhood of the identity. A neighborhood basis of the identity is given by sets of the form

$$U_{n,\epsilon} := (-\epsilon, \epsilon)^n \times \mathbb{R} \times \mathbb{R} \times ...$$

(with n and ϵ varying). If $n \neq m$, then the word metrics with respect to $U_{n,\epsilon}$ and $U_{m,\epsilon}$ with not be quasi-isometric.

In [58], C. Rosendal develops a framework for dealing with this problem – identifying which properties of a metrisable group imply that it has a canonical quasi-isometry type, and giving us an analog of Theorem 6.5 for a wider class of groups than just the locally compact. We assume all groups are Polish – completely metrizable and separable, although completeness isn't always needed.

The starting point of Rosendal's theory is the right generalization of "compact set."

Definition 6.9. A subset $K \subset G$ is said to have property (OB) relative to G if it has finite diameter in every compatible left-invariant metric on G.[11]

For connected groups, Definition 6.9 is equivalent to the following condition:

(∗) *For every open neighborhood V of* id*, there is $k \in \mathbb{N}$ such that $K \subset V^k$*

[11] The terminology (OB) comes from "orbites bornées" – an equivalent formuation of the condition is that for every continuous isometric action of G on a metric space, every orbit Ax has finite diameter.

Here V^k denotes the set of words of length k in letters of V. If G is not connected, then we allow there to be some finite set F such that $K \subset (FV)^k$. If G is connected, the two conditions are equivalent, since any finite set lies in some V^n. For simplicity, we'll restrict our attention to connected groups from now on.

Condition $(*)$ lets us define a *canonical word metric* for any group with a relatively-(OB) neighborhood of the identity.

Exercise 6.10. Let G be a connected Polish group, and suppose that U and V are symmetric neighborhoods of the identity, each with the relative property (OB). Use property $(*)$ to show that the word metrics d_U and d_V are quasi-isometric.

We say that a group with a relatively-(OB) neighborhood of the identity has *the local property (OB)*.

Rosendal also proves an analog of Theorem 6.5 for groups with the local property (OB), and shows that these word metrics are canonical in an even stronger sense.

Theorem 6.11 (Rosendal, [58])**.** Let G be a Polish group with $V \subset G$ a relatively-(OB) neighborhood of the identity.

i) (Compatibility) G admits a *compatible*, left-invariant metric that is quasi-isometric to the word metric d_V.

ii) (Maximality) The word metric d_V is *maximal* in the following sense: If d is a compatible, left-invariant metric on G, then there exist k, c so that $d(a, b) \leq k d_V(a, b) + c$ for all $a, b \in G$.

iii) (Uniqueness) If d' is any metric satisfying the maximality condition above, then (G, d') is quasi-isometric to (G, d_V).

6.3 Large-scale geometry of homeomorphism groups

Using fragmentation, we now prove that $\mathrm{Homeo}_0(M)$ has a well defined quasi-isometry type.

Theorem 6.12 (Mann–Rosendal)**.** $\mathrm{Homeo}_0(M)$ has the local property (OB).

Proof. Let M be a closed manifold. Let $\{B_1, B_2, ...B_n\}$ be a cover of M by embedded open balls. The proof of the fragmentation theorem (Theorem 1.7) implies that there is a neighborhood U of the identity in $\mathrm{Homeo}_0(M)$ such that any $g \in U$ can be factored as a product

$$g = g_1 g_2 \cdots g_n$$

with $\mathrm{supp}(g_i) \subset B_i$. (This was indicated in our sketch of a proof, the complete proof is Corollary 1.3 in [17]). We'll show that U has the relative property (OB).

Let $V \subset U$ be a neighborhood of the identity. We will find a finite set F with $U \subset FV^nF \subset (FV)^{n+1}$. By our comment regarding condition $(*)$, this is sufficient to show that U has the relative property (OB). Let $\epsilon > 0$ be small enough so that any homeomorphism f with $\mathrm{supp}(f)$ contained in a ball of diameter ϵ is contained in V. For each B_i, choose $h_i \in \mathrm{Homeo}_0(M)$ so that $h_i(B_i)$ is contained in some ball B_i' of diameter ϵ. Then $\mathrm{supp}(h_i^{-1} g_i h_i) \subset B_i'$, so $h_i^{-1} g_i h_i \in V$; equivalently, $g_i \in h_i V h_i^{-1}$. Define F to be the finite set

$$F = \{h_i, (h_j^{-1} h_{j+1}), h_n^{-1} \mid 1 \leq i \leq n, 1 \leq j \leq n-1\}.$$

Then $g \in h_1 V h_1^{-1} h_2 V......h_n V h_n^{-1} \subset (FV)^{n+1}$.

□

Example 6.13. (Some examples of QI types)

1. (Calegari–Freedman [11]) $\mathrm{Homeo}_0(S^n)$ is quasi-isometric to a point.
2. If M has infinite fundamental group, then $\mathrm{Homeo}_0(M)$ is unbounded. (*point-pushing*)
3. $\widetilde{\mathrm{Homeo}_0}(S^1)$ also has the local property (OB), and is quasi-isometric to $\mathbb{Z}$.

6.4 Distortion revisited

The following definition is standard for finitely generated groups, and readily generalizes to any group with the local property (OB).

Let G be a group and $H \subset G$ a subgroup. Assume that both G and H have the local property (OB) – if they are discrete groups, they should be finitely generated – and let d_G and d_H be maximal metrics (in the sense of Theorem 6.11) – if G and H are finitely generated, one may take any word metric.

Definition 6.14. The d_H, d_G *distortion function* is a function $D : H \to \mathbb{Z}$ defined by

$$D(r) := \sup\{d_H(h, \mathrm{id}) \mid h \in H \text{ and } d_G(h, \mathrm{id}) \leq r\}$$

Distortion measures how much of a "shortcut" between elements in H one can take by traveling in G. The subgroup H is said to be *undistorted* if D is bounded, in which case the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding, and is *distorted* otherwise. If we replace d_G (and/or d_H) by quasi-isometrically equivalent metrics and let D' denote the new distortion function, then there exists $c \in \mathbb{R}$ such that $\frac{1}{c} D(r/c) \leq D'(r) \leq cD(cr)$. Thus, whether the distortion function is *linear*, *polynomial*, *exponential*, *superexponential*, etc. depends only on the large scale geometry of G and H.

Exercise 6.15. In Lecture 5, we defined an element g in a finitely generated group Γ to be distorted if

$$\liminf_{n\to\infty} \frac{\|g^n\|_S}{n} = 0.$$

Show that this is equivalent to distortion of the $\mathbb{Z}$ subgroup generated by g in Γ, in the sense of the definition above.

The following proposition, essentially due to Militon, reconciles the notion of distorted elements in $\mathrm{Homeo}_0(M)$, in the historical sense, with distorted cyclic subgroups.

Proposition 6.16 (Militon, [50])**.** Let M be a compact manifold, and $\|\cdot\|_U$ the word norm induced by a neighborhood U of the identity with the relative property (OB). Let $g \in G$ be a distorted element. Then there exists a finitely generated subgroup $S \subset G$ such that

$$\|g^n\|_S \leq Kn\log(n)$$

Application of distortion: Proof of Hurtado's theorem. Uniting nearly all the topics we've covered so far, we give a rough indication of Hurtado's proof of continuity of homomorphisms between diffeomorphism groups, using a theorem of Militon on distortion (which itself uses fragmenation!).

Theorem 6.17 (Hurtado [32])**.** Let M and N be compact manifolds, and $\phi : \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(N)$ be a homomorphism. Then ϕ is continuous.

We will only sketch a proof of the following weaker statement, which is Hurtado's first step.

> *Suppose* $f_n \to \mathrm{id}$ *in* $\mathrm{Diff}_0(M)$*. Then there is a subsequence* f_{n_k} *such that* $\phi(f_{n_k})$ *converges to an isometry of* N*.*

Very loosely, the strategy of proof is as follows. Suppose that g_n is a sequence of diffeomorphisms approaching the identity. The same ingredients as in Militon's Proposition 6.16 (namely, Lemma 1 from [49]) allows one to find a finitely generated subgroup $\langle S \rangle$, containing all g_n, and such that the word norm $\|g_n\|_S$ is controlled – and the control is essentially independent of the sequence g_n.

Now the fact that g_n can be expressed as words of bounded length gives some weak control on the norms of the derivatives of $\phi(g_n)$ – one can give a kind of double-exponential bound. However, *if* the norms of derivatives of $\phi(g_n)$ formed an unbounded sequence, then one could pass to a subsequence g'_n along which they grew arbitrarily fast – say a triple exponential – so that when one runs the same argument with the sequence g'_n (finding a finite set S', etc.) this would contradict the double-exponential control on norm given by the word length of g'_n in S'.

This strategy shows that all k^{th} derivatives of $\phi(g_n)$ remain bounded. One then applies Arzela-Ascoli to find a convergent subsequence converging to some $h \in \mathrm{Diff}_0(N)$ and uses an "averaging a metric along iterates of h" to show that h is an isometry.

6.5 Further exercises

1. (Quasi-isometry types of locally compact groups)
   (a) Verify that a locally compact, compactly generated group G has a well defined quasi-isometry type (i.e. if T and S are compact generating sets, then (G, d_S) and (G, d_T) are quasi-isometric. Remark: note that this is not necessarily true if S is not compact).
   (b) Let Γ be a (finitely generated) cocompact lattice in a Lie group G. (Note that G is locally compact, so has a well-defined QI type.) Show Γ is quasi-isometric to G.
2. Prove that $\mathrm{Homeo}_0(S^1)$ is quasi-isometric to a point as follows:
   (a) Let $x, y \in S^1$. Show that any $f \in \mathrm{Homeo}_0(S^1)$ can be written as $f = f_1 f_2 f_3$ where f_1 and f_3 fix x, and f_2 fixes y.
   (b) *Show that the subgroup of homeomorphisms of S^1 that fix x is quasi-isometric to a point. (i.e. for any small neighborhood of the identity U, there exists k so that any f fixing x can be written as a word of length at most k in U.
   (c) ** Challenge: Can you generalize to S^n?

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