RIGIDITY AND GEOMETRICITY FOR SURFACE GROUP ACTIONS ON THE CIRCLE

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Abstract. We prove that rigid representations of $\pi_1\Sigma_g$ in $\mathrm{Homeo}^+(S^1)$ are geometric, thereby establishing a converse statement of a theorem by the first author in [14].

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1. Introduction

1.1. **Character spaces and rigidity.** Let Γ be a discrete group and G a topological group. If $G \subset \mathrm{Homeo}(X)$ for some space X, then the *representation space* $\mathrm{Hom}(\Gamma, G)$, equipped with the compact-open topology, parameterizes actions of Γ on X with image in G. Typically, G is used to specify the regularity of the action – for instance taking $G = \mathrm{Diff}(X)$ parametrizes smooth actions, whereas if G is a Lie group acting transitively on M, these are *geometric* actions in the sense of Ehresmann. Since conjugate actions are dynamically equivalent, the appropriate moduli space of actions is the quotient $\mathrm{Hom}(\Gamma, G)/G$ under the natural conjugation action of G. However, this quotient space is typically non-Hausdorff.

When G is a Lie group and $\mathrm{Hom}(\Gamma, G)$ is an affine variety, algebraic geometers solve this problem by considering the quotient $\mathrm{Hom}(\Gamma, G)/\!/G$ from geometric invariant theory. In the special case where G is a semi-simple complex reductive Lie group, this GIT quotient is simply the quotient of $\mathrm{Hom}(\Gamma, G)$ by the equivalence relation $\rho_1 \sim \rho_2$ whenever *the closures* of their conjugacy classes intersect [12, 13]. In particular, this relation makes the quotient space Hausdorff. In the well-studied case of $G = \mathrm{SL}(n, \mathbb{C})$, the GIT quotient agrees with the space of *characters* of G-representations, motivating the terminology in the following definition.

Definition 1.1. For any discrete group Γ and topological group G, the *character space* $X(\Gamma, G)$ is the largest Hausdorff quotient[1] of $\mathrm{Hom}(\Gamma, G)/G$. We say that two representations are *χ-equivalent* if they give the same point in $X(\Gamma, G)$.

A representation $\rho : \Gamma \to G$ is *rigid*, loosely speaking, if all deformations of $\rho(\Gamma)$ in G are trivial. This notion can be made precise in the setting of character spaces, as follows.

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[1] Recall the largest Hausdorff quotient X_H of a topological space X is a space with the universal property that any continuous map $f\colon X \to Y$ from X to a Hausdorff topological space factors canonically through the projection $X \to X_H$. One construction of X_H is as the quotient of X by the intersection of all equivalence relations $\sim$ such that $X/\sim$ is Hausdorff.

Definition 1.2. A representation $\rho \in \operatorname{Hom}(\Gamma, G)$ is *rigid* if the image of ρ is an isolated point in the character space $X(\Gamma, G)$.

This is quite a strong condition on ρ, and we may loosen it to some weaker, and more explicit conditions. In particular, we will say that ρ is *path-rigid* if the path component of ρ in $\operatorname{Hom}(\Gamma, G)$ is contained in a single χ-equivalence class.

The case of interest in this article is when $G = \operatorname{Homeo}^+(S^1)$, the group of orientation-preserving homeomorphisms of the circle, and $\Gamma = \Gamma_g = \pi_1(\Sigma_g)$ is the fundamental group of an orientable surface of genus $g \geqslant 2$. In this case $\operatorname{Hom}(\Gamma, G)$ has an important interpretation as the space of *flat* or *foliated* topological circle bundles over Σ. As will be explained in Section 2.3, in this setting the character space $X(\Gamma, G)$ is the space of *semi-conjugacy* classes of actions of Γ on S^1, and path-rigid representations are those ρ such that every path can be obtained by a continuous family of semi-conjugacies. Both rigid and path-rigid representations can be thought of as corresponding to foliated bundles that admit only trivial types of deformations.

A second motivation for the study of $X(\Gamma_g, \operatorname{Homeo}^+(S^1))$ comes from Goldman's seminal work on $X(\Gamma_g, \operatorname{PSL}(2, \mathbb{R}))$ and its relations with Teichmüller spaces. In [9], Goldman showed that the connected components of $X(\Gamma_g, \operatorname{PSL}(2, \mathbb{R}))$ are classified by the *Euler number.* The Euler number is a characteristic integer, which Milnor [22] showed takes values in $[-2g+2, 2g-2] \cap \mathbb{Z}$ on (equivalence classes of) representations in $X(\Gamma_g, \operatorname{PSL}(2, \mathbb{R}))$. This is the famous *Milnor–Wood* inequality; to which Wood's contribution was an extension of Milnor's result to representations into $\operatorname{Homeo}^+(S^1) \supset \operatorname{PSL}(2, \mathbb{R})$ [24]. However, as was shown in [14], the Euler number does not classify connected components of $X(\Gamma_g, \operatorname{PSL}(2, \mathbb{R}))$, and extending Goldman's work to representations into $\operatorname{Homeo}^+(S^1)$ appears to be a difficult task. We will comment further on this in the next section.

1.2. **Geometric representations.**

The first known example of a rigid representation of a surface group into $\operatorname{Homeo}^+(S^1)$ comes from a celebrated theorem of Matsumoto [20]. He showed that the set of representations with maximal Euler number, i.e. Euler number equal to $2g-2$, in $X(\Gamma_g, G)$ consists of a single point. As the Euler number is a continuous function on $\operatorname{Hom}(\Gamma_g, G)$, this implies that representations of maximal Euler number are rigid. (The same statement also holds for representations with Euler number $-2g+2$.)

Phrased otherwise, Matsumoto's result says that all representations with Euler number $\pm(2g-2)$ are χ-equivalent to discrete, faithful representations into $\operatorname{PSL}(2, \mathbb{R})$. This hints at an underlying phenomenon for rigidity, namely that these are representations coming from a geometric structure.

Definition 1.3 ([15])**.** Let M be a manifold, and Γ a countable group. A representation $\rho \colon \Gamma \to \operatorname{Homeo}(M)$ is called *geometric* if it is χ-equivalent to a faithful representation with image a cocompact lattice in a transitive, connected Lie group $G \subset \operatorname{Homeo}(M)$.

It is not difficult to classify the geometric representations of surface groups in $\mathrm{Homeo}^+(S^1)$: up to χ-equivalence, all are either discrete, faithful representations into $\mathrm{PSL}(2,\mathbb{R})$, or obtained by lifting such a representation to a finite cyclic extension of $\mathrm{PSL}(2,\mathbb{R})$. See [15] for details.

The main result of [14] is the following.

Theorem 1.4 (Mann [14])**.** *In the space* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$*, all geometric representations are rigid.*

In fact, the main theorem of [14] is stated in a weaker form; it says that the connected component of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ is a single semi-conjugacy, or χ-equivalence, class (we will see soon that these two notions coincide). However, the proof of the theorem is carried out on the level of semi-conjugacy invariants of representations, so actually proves the stronger result that geometric representations descend to isolated points in $X(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

1.3. **Results.** This article is devoted to proving the converse of Theorem 1.4. We show the following.

Theorem 1.5. *Every rigid representation in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ *is geometric.*

In other words, the *only* source of rigidity for actions of Γ_g on S^1 is the existence of an underlying geometric structure.

Our main technical result in the course of proving Theorem 1.5 is stronger for representations of non-zero Euler class, as we need to assume only path-rigidity in this case.

Theorem 1.6. *Let* $\rho\colon \pi_1\Sigma_g \to \mathrm{Homeo}^+(S^1)$ *be a path-rigid representation. If* ρ *is not geometric, then its Euler class is zero, and there exists a one-holed, genus* $g-1$ *subsurface* $\Sigma' \subset \Sigma_g$ *such that* $\rho|_{\pi_1\Sigma'}$ *has a finite orbit.*

The condition of having a large subsurface with a finite orbit makes it very unlikely that such a representation cannot be deformed along a path. This gives strong evidence for the fact that all path-rigid representations should in fact be geometric. However, at the time of writing we are unable to prove this.

The proof of Theorem 1.6 is quite long and involved. A much simpler argument, with some of the same spirit, can be carried out under the additional assumption that the relative Euler number on some genus 1 subsurface is equal to 1; this is the case in particular for representations of Euler class $\geqslant g$. This simpler, though much weaker, proof is presented in the companion article [17]. Although the present article is self-contained, the reader may prefer to take [17] as a starting point.

We conclude this introduction by putting our result in the perspective of the following ambitious problem (which remains wide open) in the natural continuation of Goldman's work on $X(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$.

Question 1.7. *What are the connected components of* $X(\Gamma_g, \mathrm{Homeo}^+(S^1))$*? What are its path components?*

One of the implications of Theorem 1.4 was that the space $X(\pi_1\Sigma_g, \mathrm{Homeo}^+(S^1))$ has strictly more connected components than $X(\pi_1\Sigma_g, \mathrm{PSL}(2,\mathbb{R}))$. At this time, we do not even know if the former has finitely many connected components. The present article, more than simply proving Theorem 1.5, aims at providing some technical tools towards this question; we hope to address it in a future work.

1.4. **Strategy of the proof and organization of the article.** The main ingredient in the proof of Theorem 1.6 is the effect of *bending deformations* on the periodic sets of simple closed curves. Bending deformations are classical in (higher) Teichmüller theory, (see paragraph 2.2.2 for a reminder); and we extend their study to representations to $\mathrm{Homeo}^+(S^1)$.

As a first step, we make a (strong) additional technical hypothesis on representations that forces them to look "locally" (i.e. on the level of some pairs of curves) like representations into $\mathrm{PSL}^k(2,\mathbb{R})$. Specifically, we say that the action of two elements $a, b \in \Gamma_g$ representing standard generators of a one-holed torus subsurface of Σ_g satisfies $S_k(a,b)$ if $\rho(a)$ and $\rho(b)$ are separately conjugate to hyperbolic elements of $\mathrm{PSL}^k(2,\mathbb{R})$, and their fixed points alternate around the circle. We show the following.

Theorem 1.8. *Let ρ be a path-rigid, minimal representation, and suppose furthermore that there exists $k \geqslant 1$ such that $S_k(a,b)$ holds for all standard generators of one-holed torus subsurfaces. Then ρ is geometric.*

The reader can think of this as a dynamical "local-to-global" result.

The proof of Theorem 1.8 starts by using bending deformations of ρ to move the periodic points of generators of $\pi_1\Sigma_g$. Provided ρ is path-rigid, we are then able to conclude the periodic points of many simple closed curves are in the same cyclic order as if ρ were geometric. In the companion article [17], (whose additional hypothesis guarantees that $k = 1$) this same process was sufficient to demonstrate that ρ has maximal Euler number, hence is geometric. Here in the general case, we need to use a more sophisticated tool, and introduce Matsumoto's theory of *Basic Partitions* (see Section 3.5).

The next step is to arrive at the property $S_k(a,b)$ from weaker hypotheses on periodic sets of curves. We prove the following statements.

Proposition 1.9. *If a representation $\pi_1\Sigma_g \to G$ is path-rigid then all non-separating simple closed curves have rational rotation number.*

Theorem 1.10. *Suppose ρ is path-rigid and minimal. Then, for all a, b with $i(a,b) = \pm 1$, we have the implication*

$$\mathrm{Per}(\rho(a)) \cap \mathrm{Per}(\rho(b)) = \varnothing \Rightarrow S_k(a,b) \textit{ for some } k.$$

The proofs again make extensive use of bending deformations.

The upshot of these results is that, if a path-rigid and minimal representation *fails* to be geometric, then many curves are forced to have common periodic points. Common periodic points hint at the existence of a finite orbit for ρ, so our strategy becomes to look for a finite orbit in order to derive a contradiction (indeed, representations with a finite orbit are easily seen to be non-path-rigid). This idea turns out to be difficult to implement, so we search first for curves with rotation number zero, as the dynamics of these are easier to control. This search can be performed separately in every one-holed torus in the surface, where the action of the mapping class group is simple to work with. Accordingly, a one-holed torus in Σ_g will be called a *good torus* if it contains a nonseparating simple loop with rotation number zero; otherwise we say it is a *bad torus*. Further, a one-holed torus will be called a *very good torus* if its fundamental group has a finite orbit in S^1. We can prove:

Proposition 1.11. *Let ρ be path-rigid. Suppose that Σ_g contains a bad torus Σ'. Then its complement Σ'' contains only very good tori.*

By studying the evolution of periodic sets under specific bending deformations, we are able to prove the following two statements:

Proposition 1.12. *Let ρ be path-rigid, and non-geometric. Then there cannot exist two disjoint good tori that are not very good.*

Theorem 1.13. *Let ρ be a path-rigid representation. Let $\Sigma_{g',1}$ be a subsurface in which all tori are very good. Then $\pi_1\Sigma_{g',1}$ has a finite orbit.*

These three last statements prove that if ρ is a path-rigid and non-geometric representation then it has a subsurface of genus $g-1$ with a finite orbit; the statement about the Euler class in Theorem 1.6 is then an easy consequence.

Provided $g \geqslant 3$, Theorem 1.13 implies that if ρ is a path-rigid but non-geometric representation, then there exist curves a, b, generating a torus subsurface of Σ_g, such that $\rho(a)$ and $\rho(b)$ have a common fixed point. It then follows from a recent theorem of Alonso, Brum and Rivas [1] that ρ cannot be rigid. However, path-rigidity and the genus $g = 2$ case do not follow. So we pursue a different route, taking their work as inspiration. We prove an independent, simple lemma on rigid representations that shows (after semiconjugacy) all torus subsurfaces have only finitely many finite orbits. This applies to all genera of surfaces, and allows us conclude the proof of Theorem 1.5.

The article is organized as follows. Section 2 introduces tools that will be frequently used in the proof. While some of the material is standard, we also prove new results on complexes of based curves, and prove a series of results on the movement of periodic sets under specific bending deformations. We also give more discussion to character spaces, semi-conjugacy, and the Euler class. In Section 3 we prove Theorem 1.8. In Section 4 we prove Proposition 1.9 and Theorem 1.10. The proof of Theorem 1.6 is then completed in Section 5. Finally, in Section 6 we complete the proof of Theorem 1.5 and state some open questions and directions for further work.

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2. Preliminaries

This section sets notation and develops a toolkit for use in the main proofs. The first part treats curves on surfaces. This subsection will feel familiar to low dimensional topologists, except that we will give much more attention to *based* curves than is usually present in the literature. The second subsection deals with actions of surface groups on S^1 and their deformations. We introduce new material on the behavior of periodic sets under deformations, and the topology of sets of persistent (and non-persistent) periodic points; this will be crucial in later sections of the work. The third subsection covers character spaces, semi-conjugacy, and the Euler number in more detail, including the proof that χ-equivalence coincides with semi-conjugacy in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$.

2.1. **Based curves on surfaces.**

Warning 2.1. Although our notation $\pi_1\Sigma_g$ omits mention of a basepoint, it is crucial to keep in mind that all elements of $\pi_1\Sigma_g$ are based. This is because, very often, given two simple curves γ_1, γ_2, we will need to know whether the product $\gamma_2\gamma_1$ is also (up to homotopy rel endpoints) a simple curve, and this may change if γ_2 is replaced by some conjugate of itself. Note also that the set of simple *based* curves is itself not invariant under conjugation.

As in the introduction, we use the notation $\Gamma_g = \pi_1\Sigma_g$.

Definition 2.2 (Based intersection number)**.** Let $a, b \in \Gamma_g$. We write $i(a,b) = 0$ if we can represent a and b by differentiable maps $a, b\colon [0,1] \to \Sigma_g$ with $\{0,1\}$ mapped to the basepoint, whose restrictions to $[0,1)$ are injective, and such that the cyclic order of their tangent vectors at the base point is either $(a'(0), -a'(1), b'(0), -b'(1))$ or $(a'(0), -a'(1), -b'(1), b'(0))$, or the reverse of one of these.
If instead the cyclic order of tangent vectors is $(a'(0), b'(0), -a'(1), -b'(1))$ or the reverse, we write $i(a,b) = 1$ and $i(a,b) = -1$ respectively.

Note that this is an *ad hoc* definition, for these are the only two configurations of pairs of curves that we will be interested in. For typical pairs of curves a, b, the number $i(a,b)$ is not defined.

Convention 2.3 (Read words from right to left)**.** Since we will often be working with a given representation, we will often abuse notation and identify an element $a \in \Gamma_g$ with a homeomorphism of the circle. Thus, following the convention of function composition, we write words in Γ_g (i.e. products of loops by concatenation) from right to left.

We also fix commutator notation to be $[a,b] := b^{-1}a^{-1}ba$.

Notation 2.4 (Tori)**.** Given two simple nonseparating loops $a, b \in \Gamma_g$ with $i(a,b) = \pm 1$, their commutator $[a,b]$ bounds a genus 1 subsurface containing a and b, which we will denote by $T(a,b)$. Figure 1 illustrates $T(a_1, b_1)$.

Although $T(a,b)$ is only defined up to based homotopy, we may still speak reasonably of curves that intersect it, as follows.

Definition 2.5. We say that a simple, nonseparating curve γ is *disjoint* from $T(a,b)$ if γ has intersection number zero with each of a, b and $[a,b]$, and that γ *enters* or *intersects* $T(a,b)$ otherwise. We say that two tori $T(a,b)$ and $T(a',b')$ are *disjoint* if a, b are disjoint from $T(a',b')$ and if a', b' are disjoint from $T(a,b)$.

In the same line as Warning 2.1, note that $T(a,b) \neq T(b,a)$.

Definition 2.6 (Based standard generators)**.** We say a system of based loops $(a_1, \ldots, b_g)$ is *standard* if the surface, together with these curves, are as in Figure 1. They give the following standard presentation of the fundamental group:

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \,|\, [a_g, b_g] \cdots [a_1, b_1] = 1 \rangle.$$

In such a picture, an easy mnemonic way not to confuse a_i and b_i with $a_i^{\pm 1}$ or $b_i^{\pm 1}$, is to remember that the curve $[a_i, b_i]$ begins with the letter a_i and ends with b_i^{-1}.

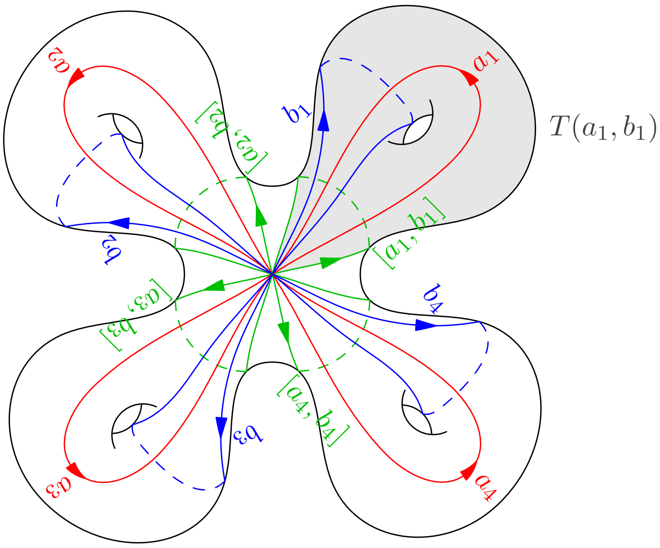

FIGURE 1. Standard generators on the genus g surface ($g = 4$)

Recall that a *fat-graph* is a graph together with the data of a total cyclic order on ends of edges at each vertex. If every edge also comes with a preferred orientation, it is called a *fat-quiver*. We define the *directed standard k-chain* to be the fat-quiver shown in Figure 2. As a graph it is simply the standard rose with k petals, and as a fat-quiver, the ends of edges are in the cyclic order $(\gamma_1'(0), \gamma_2'(0), -\gamma_1'(1), \gamma_3'(0), -\gamma_2'(1), \gamma_4'(0), \ldots, -\gamma_k'(1))$. In particular, $i(\gamma_i, \gamma_{i+1}) = +1$, and $i(\gamma_i, \gamma_j) = 0$ whenever $|j-i| \geqslant 2$. By walking along the sides of the curves γ_i, and computing the Euler characteristic, we observe that the surface obtained by thickening the standard chain of length $2k$ is a genus k surface with one boundary component, and the one from the chain of length $2k+1$ is a genus k surface with two holes.

Definition 2.7 (Chains)**.** Let Σ be a surface, and $\gamma_1, \ldots, \gamma_k$ loops based at the base point. We say that $(\gamma_1, \ldots, \gamma_k)$ forms an *oriented, directed k-chain* if these curves arise from an orientation-preserving embedding of the

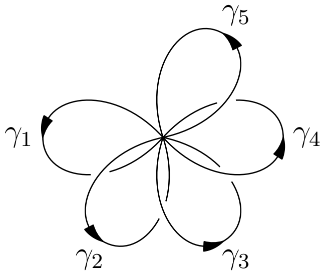

FIGURE 2. A directed chain of length 5

directed standard k-chain into Σ, and a *directed k-chain* if the embedding may reverse the orientation. We say it forms simply a *k-chain* if there exists a family of signs $(\epsilon_1, \ldots, \epsilon_k)$ such that $(\gamma_1^{\epsilon_1}, \ldots, \gamma_k^{\epsilon_k})$ is a directed k-chain. We say that a (directed) k-chain is *completable* if its sits in the middle of some $k+2$-chain.

For example, $(a_1^{-1}b_1a_1, a_1, b_1^{-1})$ is a non-completable 3-chain in Σ_g, and the collection $(a_1, \delta_1, a_2, \delta_2, \ldots, \delta_{g-1}, a_g, b_g^{-1})$ (as well as its sub-chains), where we have set $\delta_i = a_{i+1}^{-1}b_{i+1}a_{i+1}b_i^{-1}$, forms a directed chain. Also, the family $(a_1^{-1}b_1a_1, a_1, \delta_1, a_2, b_2^{-1})$ forms a (non-completable) 5-chain that will be handy in Section 5.3.

We conclude this paragraph with some considerations on complexes of pairs of based curves.

Lemma 2.8. *Let G_0 denote graph whose vertices are the pairs $(a, b) \in \Gamma_g^2$ with $i(a, b) = \pm 1$, with an edge between two pairs (a, b) and (b, c) whenever (a, b, c) is a 3-chain. Then G_0 is connected.*

The proofs of the main results of this article do not depend on this lemma, as we will simply need to work on a connected component of this graph – in fact, our proof in the companion article [17] is done along this line. However, the lemma is quite elementary, so here we take the honest approach of giving the proof and using the whole connected graph instead of making reference to a connected component.

The proof of Lemma 2.8 is divided into two main observations. It essentially copies the proof of Proposition 6.7 of [18], but corrects a minor mistake there, where the complex of *based* curves should have been used instead of the standard curve complex.

Observation 2.9. *Let G_1 be the graph whose vertices are the elements of Γ_g represented by simple, non-separating curves, and with edges between a and b if and only if $i(a, b) = \pm 1$. Then G_1 is connected.*

Proof. Let G_2 be the graph with the same vertices, but with edge between a and b if $i(a, b) = 0$, -1 or 1 (i.e. whenever $i(a, b)$ is well defined). Let G_3 be the graph with vertex set consisting of the elements of Γ_g represented by simple curves (possibly separating), with an edge between a and b whenever $i(a, b)$ is well defined. By drilling a puncture in Σ_g at the base point, this graph G_3 can be identified with the arc complex of the surface Σ_g^1, and this is well-known to be connected (see eg [Hatcher91]). We now show that its subgraph G_2 is also connected. Suppose $a_1 - a_2 - a_3$ is a path in G_3, and a_2 is separating. If a_1 and a_3 lie in distinct components of $\Sigma' \smallsetminus a_2$, then $a_1 - a_3$ is a shortcut. If a_1 and a_3 do not lie in distinct components, then we may replace a_2 in this path by any essential arc lying in the other side of $\Sigma' \smallsetminus a_2$ (such an arc exists since a_2 is essential). Thus, we can make the separating elements disappear from any path connecting two points of G_2 in G_3.

Now we can show that G_1 is connected by inserting terms in paths of G_2. It suffices to show that, if $a_1 - a_2$ is an edge of G_2 and $i(a_1, a_2) = 0$, then there always exists a curve b such that $i(a_1, b) = \pm 1$ and $i(b, a_2) = \pm 1$. So suppose $i(a_1, a_2) = 0$. Then a neighborhood of the curves a_1 and a_2 in Σ_g is a pair of pants P, with three boundary components, freely homotopic to a_1, a_2 and $a_1 a_2^{\pm 1}$. If Σ, Σ' and Σ'' are, respectively, the connected components of $\Sigma_g \smallsetminus P$ separated from P by a_1, a_2 and $a_1 a_2^{\pm 1}$, then we have $\Sigma = \Sigma'$ or $\Sigma = \Sigma''$, since a_1 is non separating. Similarly, $\Sigma' = \Sigma$ or $\Sigma' = \Sigma''$ as a_2 is non separating. Hence $\Sigma = \Sigma'$ in all cases, thus, there does exist a connecting curve b as needed. ☐

Observation 2.10. *Let a, b, a' such that $i(a, b) = \pm 1$ and $i(a', b) = \pm 1$. Then (a, b) is connected to (a', b) in the graph G_0 from Lemma 2.8.*

Proof. Let $\sim$ denote the equivalence relation on vertices of G_0 of being in the same connected component. Let a, b, a' be as in the statement of the observation, and let N be the (geometric) minimum number of disjoint intersections, besides of the base point, between the based curves a and a'. We will proceed by induction on N, starting with the base case $N = 0$. In this case $i(a, a') \in \{0, \pm 1\}$. If $i(a, a') = 0$, then (a, b, a') is a 3-chain and $(a, b) \sim (b, a')$. If $i(a, a') = \pm 1$, then for some $\epsilon \in \{-1, 1\}$, we have $i(b^\epsilon a, a') = 0$ (this is seen by looking at a neighborhood of the base point), hence $(b^\epsilon a, b, a')$ is a 3-chain and $(b^\epsilon a) \sim (b, a')$. Now $(b^\epsilon a, b) \sim (a, b)$, because there exists a curve c such that $(b^\epsilon a, b, c)$ and (a, b, c) are both 3-chains. This proves the base case.

Now, suppose $N \geqslant 1$. Orient the curves a and a' so that their tangent vectors at $t = 0$ are on the same side of b at the base point. Let $(x_1, \ldots, x_N)$ be the intersection points of a and a', as ordered along the path a. Let a'' be the path obtained from following a' from its starting time, until we hit x_N (actually, any of the x_i would do), and then following the end of the path a. Then we have $i(a, b) = \pm 1$, $i(a', b) = \pm 1$, $i(a'', b) = \pm 1$ and the intersections of a and a' with a'' outside the base point are strictly less than N; this concludes our induction. ☐

Proof of Lemma 2.8. Let (a, b) and (c, d) be such that $i(a, b) = \pm 1$ and $i(c, d) = \pm 1$. There exists a path between b and c in G_1, which can be extended to a path $\gamma_1 - \gamma_2 - \cdots - \gamma_n$ in G_1 with $(a, b, c, d) = (\gamma_1, \gamma_2, \gamma_{n-1}, \gamma_n)$. By Observation 2.10, for all $j \in \{1, \ldots, n-2\}$, (γ_j, γ_{j+1}) is connected to $(\gamma_{j+1}, \gamma_{j+2})$ in G_0, hence (a, b) is connected to (c, d). ☐

Finally, here is an easy variation of Lemma 2.8.

Lemma 2.11. *Let G denote graph whose vertices are the pairs $(a, b) \in \Gamma_g^2$ with $i(a, b) = \pm 1$, with an edge between two pairs (a, b) and (b, c) whenever (a, b, c) is a* completable *3-chain. Then G is connected.*

Proof. First, observe that whenever $T(a,b)$ and $T(c,d)$ are disjoint, (a,b) and (c,d) are in the same connected component of G. Now, observe that if (a,b,c) is a directed 3-chain, then it is completable if and only if ca is nonseparating. (The reader may find it helpful to draw a picture.) It follows that, if (a,b,c) is a non-completable 3-chain in Σ_g, then there exists a pair (d,e) such that a,b,c do not enter $T(d,e)$. Hence, (a,b) and (b,c) are connected to (d,e) in G, and it follows that G is connected. $\square$

2.2. **Actions on the circle.**

2.2.1. *Basic dynamics of circle homeomorphisms.* We quickly review some definitions for the purpose of setting notation. For more detailed background on this material, the reader may consult [6, 15, 7, 23] for example. Recall that $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ denotes the group of homeomorphisms of $\mathbb{R}$ commuting with translation by 1. Viewing S^1 as $\mathbb{R}/\mathbb{Z}$ gives the central extension $\mathbb{Z} \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$.

The principal dynamical invariant of elements of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ and $\mathrm{Homeo}^+(S^1)$ is the *rotation number* of Poincaré.

Definition 2.12 (Rotation number)**.** Let $f \in \mathrm{Homeo}^+(S^1)$ and $\tilde{f}$ is a lift of f in $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. The *rotation numbers* (sometimes called the translation number in the case of $\tilde{f}$) are defined by $\widetilde{\mathrm{rot}}(\tilde{f}) := \lim_{n\to\infty} \frac{\tilde{f}^n(0)}{n} \in \mathbb{R}$, and $\mathrm{rot}(f) := \widetilde{\mathrm{rot}}(\tilde{f}) \bmod \mathbb{Z}$.

We assume the reader is familiar with the essential properties of the rotation number. Those that we will use most frequently are that rot and $\widetilde{\mathrm{rot}}$ are homomorphisms when restricted to abelian (eg. cyclic) subgroups, that $\mathrm{rot}(f) = p/q \in \mathbb{Q} \bmod \mathbb{Z}$ if and only if f has a periodic orbit of period q, and that $\widetilde{\mathrm{rot}}$, and hence rot, are invariant under semi-conjugacy. (The definition of semi-conjugacy is recalled in Section 2.3 where we will be using it.)

As is standard, we use $\mathrm{Per}(f)$ to denote the set $\{x \in S^1 \mid \exists n \in \mathbb{Z},\ f^n(x) = x\}$ of periodic points of f. If $n = 0$, we also denote this by $\mathrm{Fix}(f)$. For $\tilde{f} \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, we use $\mathrm{Per}(\tilde{f})$ to denote the set of all lifts of points of $\mathrm{Per}(f)$ to $\mathbb{R}$.

Notation 2.13. For $f \in \mathrm{Homeo}^+(S^1)$ with $\mathrm{Per}(f) \neq \varnothing$, let $q(f)$ denote the smallest non-negative integer such that $\mathrm{Fix}(f^{q(f)}) \neq \varnothing$ (equivalently, the smallest non-negative integer such that $\mathrm{rot}(f^{q(f)}) = 0$), and let $p(f)$ be the least non-negative integer such that f has rotation number equal to $\frac{p(f)}{q(f)}$ mod $\mathbb{Z}$.

An *attracting periodic point* for f is a point $x \in \mathrm{Per}(f)$ with a neighborhood I of x such that $f^{nq(f)}(I) \to x$ as $n \to \infty$. A *repelling* periodic point of f is defined as an attracting periodic point of f^{-1}. The sets of attracting and repelling periodic points will be denoted $\mathrm{Per}^+(f)$ and $\mathrm{Per}^-(f)$ respectively.

2.2.2. *One-parameter families and bending deformations.* Let $\gamma \in \Gamma_g$ be a based, simple loop. Cutting Σ_g along γ decomposes Γ_g into an amalgamated product $\Gamma_g = A *_{\langle\gamma\rangle} B$, or an HNN-extension $A*_{\langle\gamma\rangle}$, depending on whether γ is separating. Thus, deforming a representation $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ (or indeed to any topological group) amounts to deforming the restriction(s) of

ρ on A (and B, if γ separates), subject to the constraint that these deformations agree on γ. The following deformation is the analog of a bending deformation from the theory of quasi-Fuchsian or Kleinian groups, in a very special case as we bend only along one simple curve.

Definition 2.14. (Bending)

1. *Separating curves.* Let $\gamma = c \in \Gamma_g$ represent a separating simple closed curve with $\Gamma_g = A *_{\langle c \rangle} B$. Let c_t be a one-parameter family of homeomorphisms commuting with $\rho(c)$. Define ρ_t to agree with ρ on A, and to be equal to $c_t \rho c_t^{-1}$ on B.
2. *Nonseparating curves.* Let $\gamma = a$, and $b \in \Gamma_g$ with $i(a,b) = -1$, and let $c = [a,b]$, writing again $\Gamma_g = A *_{\langle c \rangle} B$. Take a 1-parameter family a_t commuting with $\rho(a)$, and define ρ_t to agree with ρ on B, and on T_{ab} define $\rho_t(a) = \rho(a)$ and $\rho_t(b) = a_t b$.

In both cases, we call this deformation a *bending along* γ.

While we will typically use "1-parameter family" to mean a one-parameter subgroup, for these bending constructions to define a path of representations one only needs c_t and a_t to be continuous paths based at id in the centralizers of c and a in $\mathrm{Homeo}^+(S^1)$, respectively.

As a special case of bending, if γ_t is a one-parameter family with $\gamma_1 = \gamma$, then the deformation given above is the precomposition of ρ with $\tau_{\gamma *}$, where τ_γ is the Dehn twist along γ. However, as we are working with explicit computations involving based curves, it is important to us that these deformations are well defined, not just up to inner automorphisms. In order for the Dehn twist along γ to make proper sense as an automorphism of Γ_g, we need to choose a way to freely homotope γ away from the base point. The arbitrary choice made in the definition, in the case (1) above, consists in pushing γ to the side of B, and in the case (2) it fits in the following general convention.

Convention 2.15. Suppose we are given a directed k-chain $(\gamma_1, \dots, \gamma_k)$, and wish to write a Denh twist along the loop γ_i. Then we will always do so by pushing γ_i outside the base point in such a way that it intersects only γ_{i-1} and γ_{i+1} (if these curves exist), in a neighborhood of the chain. Accordingly, if ρ is a given representation and γ_i^t is a one-parameter family commuting with $\rho(\gamma_i)$, then the deformation leaves γ_j unchanged for $|j - i| \geqslant 2$ and $j = i$, and changes $\rho(\gamma_{i-1})$ into $\gamma_i^{-t} \rho(\gamma_{i-1})$ and $\rho(\gamma_{i+1})$ into $\rho(\gamma_{i+1}) \gamma_i^t$.

Not all elements of $\mathrm{Homeo}^+(S^1)$ embed in a one parameter subgroup. In fact, if $\mathrm{rot}(f)$ is irrational, then f embeds in such a subgroup if and only if $\mathrm{Per}(f) = S^1$, in which case f is conjugate to a rotation. However, elements with rational rotation number do have large centralizers, giving us some flexibility in the use of bending deformations. We formalize this in the next lemma. Here, and later on, it will be convenient to fix a section of $\mathrm{Homeo}^+(S^1)$ in $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

Notation 2.16. For $f \in \mathrm{Homeo}^+(S^1)$, let $\widehat{f} \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be the (unique) lift of f with $\widetilde{\mathrm{rot}}(\widehat{f}) = p(f)/q(f) \in [0,1)$; we will call it the *canonical lift* of f. Later, we will also need to refer to the lift of f with translation number in $(-1, 0]$, this we denote by $\check{f}$. Note that $\widehat{f}^{-1} = \widecheck{f^{-1}}$.

Lemma 2.17. *(Positive 1-parameter families) Let* $f \in \mathrm{Homeo}^+(S^1)$ *have rational rotation number. Then there exists a one-parameter group* $(f_t)_{t\in\mathbb{R}}$*, which commutes with* f*, such that* $\forall t \neq 0$*,* $\mathrm{Fix}(f_t) = \partial\mathrm{Per}(f)$*, and for all* $t > 0$ *and* $x \in \mathbb{R} \smallsetminus \partial\mathrm{Per}(\widetilde{f})$*, we have* $\widehat{f_t}(x) > x$*.*

Here and in what follows ∂X denotes the *frontier* of a subset X of $\mathbb{R}$ or S^1.

Proof. The set $S^1 \smallsetminus \partial\mathrm{Per}(f)$ consists of a union of open intervals permuted by f. Choose a single representative interval I_α from each orbit. Note that $f^{q(f)}(I_\alpha) = I_\alpha$ for any such interval, and the restriction of $f^{q(f)}$ to $S^1 \smallsetminus \mathrm{Per}(f)$ is either fixed point free or the identity. Thus, we may identify each I_α with $\mathbb{R}$ such that $f^{q(f)}$, in coordinates, is $x \mapsto x + C$, for some $C \in \{-1, 0, 1\}$. Define s_t on I_α to be $x \mapsto x + t$. Since these I_α are in different orbits of the action of f on S^1, we may extend s_t equivariantly to a 1-parameter family of homeomorphisms of S^1. ☐

This construction will be used frequently enough to merit a definition.

Definition 2.18. Let $f \in \mathrm{Homeo}^+(S^1)$. A *positive one-parameter family commuting with* f is any family f_t as in Lemma 2.17. If f is understood, or if we wish to vary f, we will often simply refer to such an f_t as a *positive one-parameter family.*

2.2.3. *Periodic sets under deformations.* We now make some observations on how periodic sets change under bending deformations using positive one-parameter families. The main application of these comes in Section 5.2, but they will also make a few earlier appearances.

Let f and $g \in \mathrm{Homeo}^+(S^1)$ have rational rotation numbers. It follows immediately from the definition of canonical lift that

$$x \in \mathrm{Per}(\widehat{f}) \Leftrightarrow \widehat{f}^{q(f)}(x) = x + p(f).$$

Let f_t be a positive one-parameter family commuting with f. Let $g_t := f_t \circ g$, and let $\widetilde{g_t} = \widehat{f_t} \circ \widehat{g}$. Note that $\widetilde{g_t} = \widehat{g_t}$, provided the rotation number of g_t is constant as t varies.

For all $(x, t) \in S^1 \times \mathbb{R}$, we set

$$\Delta_{f,g}(x, t_1, \dots, t_{q(g)}) = \widetilde{g_{t_{q(g)}}} \circ \cdots \circ \widetilde{g_{t_1}}(\tilde{x}) - \tilde{x} - p(g),$$

$$\text{and } \delta_{f,g}(x,t) = \Delta_{f,g}(x, t, \cdots, t) = (\widetilde{g_t})^{q(g)}(\tilde{x}) - \tilde{x} - p(g).$$

This does not depend on the lift $\tilde{x} \in \mathbb{R}$ of x, but does depend on the choice of the one-parameter family f_t (so we are somewhat abusing notation). Further, we set

$$P(f,g) = \left\{x \in S^1 \mid \forall t \in \mathbb{R},\ \delta_{f,g}(x,t) = 0\right\},$$

$$N(f,g) = \left\{x \in S^1 \mid \forall t \in \mathbb{R},\ \delta_{f,g}(x,t) \neq 0\right\},$$

$$\text{and } U(f,g) = \left\{x \in S^1 \mid \exists! t \in \mathbb{R},\ \delta_{f,g}(x,t) = 0\right\}.$$

Unlike $\delta_{f,g}$, these sets do not depend on the choice of the positive one-parameter family (provided that it is chosen as in Lemma 2.17).

Assuming $\mathrm{rot}(g_t)$ is constant, then $P(f,g) = \bigcap_{t\in\mathbb{R}} \mathrm{Per}(g_t)$ is the set of *persistent* periodic points; $N(f,g)$ is the set of points that are *never* periodic for any g_t, and $U(f,g)$ is the set of points that lie in $\mathrm{Per}(g_t)$ for a *unique* time t.

Let $T_{f,g}\colon U(f,g) \to \mathbb{R}$ be the map that assigns to each $x \in U(f,g)$, the unique time $t \in \mathbb{R}$ for which $\delta_{f,g}(x,t) = 0$.

Lemma 2.19. *Suppose g_t has constant rotation number. Then we have the following properties.*

1. *The set $P(f,g)$ is closed, moreover*
$$P(f,g) = \mathrm{Per}(g) \cap \bigcap_{k=0}^{q(g)-1} g^k(\partial \mathrm{Per}(f));$$
in particular, if $\mathrm{rot}(f) = 0$ then every element of $P(f,g)$ has a finite orbit under the group $\langle f, g\rangle$.
2. *The sets $P(f,g)$, $N(f,g)$ and $U(f,g)$ partition the circle.*
3. *The set $U(f,g)$ is open, and the map $T_{f,g}\colon U(f,g) \to \mathbb{R}$ is continuous.*
4. *For any $\varepsilon > 0$, there exists t_0 such that $\mathrm{Per}(f_t \circ g)$ lies in the ε-neighborhood of $P(f,g) \cup \partial N(f,g)$ for all $t > t_0$.*

Proof. By construction, the map $\Delta_{f,g}(x,\cdot)$ is (separately, in each variable t_j) constant if $\widetilde{g_{t_{j-1}}} \circ \cdots \circ \widetilde{g_{t_1}}(\tilde{x}) \in \partial \mathrm{Per}(f)$, and strictly increasing otherwise. Monotonicity implies that the subsets $\Delta_{f,g}(x, \mathbb{R}^{q(g)})$ and $\delta_{f,g}(x,\mathbb{R})$ of $\mathbb{R}$ coincide. The affirmations (1) and (2) are easy consequences of these observations. Let us prove (3). Let $x_0 \in U(f,g)$, and write $t_0 = T(x_0)$, so $\delta(x_0,t_0) = 0$. Fix $\varepsilon > 0$. Since $x_0 \in U(f,g)$, we have $\delta(x_0, t_0+\varepsilon) > 0$, and $\delta(x_0, t_0-\varepsilon) < 0$. Since the maps $x \mapsto \delta(x, t_0+\varepsilon)$ and $x \mapsto \delta(x, t_0-\varepsilon)$ are continuous, there exists $\eta > 0$ such that, for all $x \in (x_0-\eta, x_0+\eta)$ we have $\delta(x, t_0+\varepsilon) > 0$ and $\delta(x, t_0-\varepsilon) < 0$. Thus, for each $x \in (x_0-\eta, x_0+\eta)$, the map $t \mapsto \delta(x,t)$ takes positive and negative values, hence has a (unique) zero in the interval $(t_0-\varepsilon, t_0+\varepsilon)$. In other words, $(x_0-\eta, x_0+\eta) \subset U(f,g)$, and for all $x \in (x_0-\eta, x_0+\eta)$, we have $|T(x) - T(x_0)| < \varepsilon$.

For statement (4), fix $\varepsilon > 0$. Let $I_1, \ldots, I_n$ denote the (finitely many) connected components of $U(f,g)$ of length $> \varepsilon$. Let $K \subset U(f,g)$ be the set of points of $U(f,g)$ that are distance at least ε from $P \cup \partial N$. Then, $K \subset \bigcup_i I_i$, and it follows that K is compact. Since T is continuous, its restriction to K takes values in some segment $[-t_0, t_0]$, this gives the t_0 from the statement. □

The next propositions describe the topology of the sets $P(f,g), N(f,g)$ and $U(f,g)$ in more detail.

Proposition 2.20. *Suppose g_t has constant rotation number. Then all accumulation points of $\partial N(f,g)$ lie in $P(f,g)$.*

The bulk of the proof of this is accomplished by the following lemma.

Lemma 2.21. *Let $x_0 \in S^1 \setminus \mathrm{Per}(g)$ and let I be a small interval containing x_0. Suppose there exists $u_k \in U(f,g) \cap I$ converging to x_0 from the right. Then there exists $\varepsilon > 0$ such that $(x_0, x_0+\varepsilon) \subset U(f,g)$.*

Of course the symmetric statement, with sequences converging to x_0 at the left, holds as well, with a symmetric proof.

Proof. Let $x_0 \notin \mathrm{Per}(g)$, so we have $d := d(x_0, g^{q(g)}(x_0)) > 0$. First, suppose for contradiction that for all $j \in \{1, \ldots, q(g)\}$, $g^j(x_0)$ is accumulated on the right by points of $\partial \mathrm{Per}(f)$. Choose $z_{q(g)} \in \partial \mathrm{Per}(f) \cap (g^{q(g)}(x_0), g^{q(g)}(x_0) + \frac{d}{2})$, and, inductively for $j = q(g) - 1, q(g) - 2, ..., 1$ define $z_j \in \partial \mathrm{Per}(f) \cap (g^j(x_0), g^{-1}(z_{j+1}))$ for $j \in \{1, \dots, q(g) - 1\}$, and set $\delta = g^{-1}(z_1) - x_0$. Then, for all $t > 0$ we have $(f_t g)^j(x_0, x_0 + \delta) \subset (g^j(x_0), z_j)$, hence $(f_t g)^{q(g)}(x_0, x_0 + \delta) \subset (g^{q(g)}(x_0), g^{q(g)}(x_0) + \frac{d}{2})$. Now let $k \geqslant 0$ be such that $u_k \in (x_0, x_0 + \delta)$, and choose inductively $y_1 \in (g(x_0), g(u_k)) \cap \partial \mathrm{Per}(f)$, and $y_j \in (g^j(x_0), g(y_{j-1})) \cap \partial(f)$, for $j \geqslant 2$. Then, for all $t \in \mathbb{R}$ we have $(f_t g)^{q(g)}(u_k) \in (y_{q(g)}, z_{q(g)})$, hence $(f_t g)^{q(g)}(u_k) \in (g^{q(g)}(x_0), g^{q(g)}(x_0) + \frac{d}{2})$; this contradicts that $u_k \in U(f, g)$.

Thus, if a sequence $u_k \in U(f, g)$ converges to x_0 from the right, then there exists some $j \in \{1, \dots, q(g)\}$ such that $g^j(x_0)$ is not accumulated on the right by points of $\partial \mathrm{Per}(f)$. Let j be the minimum such index, and let y be such that $(g^j(x_0), y] \subset S^1 \smallsetminus \partial \mathrm{Per}(f)$. Let k be large enough so that $g \circ (f_t \circ g)^{j-1}(u_k) \subset (g^j(x_0), y]$ holds for all $t \in \mathbb{R}$. (Such k exists using the argument above, since $g^i(x_0)$ is accumulated on the right by $\partial \mathrm{Per}(f)$ for all $i < j$.) Let $z \in (x_0, u_k)$. We will now show that $z \in U(f, g)$.

Since f_t acts transitively on $(g^j(x_0), y]$, for T sufficiently large we have $f_T \circ g \circ (f_T \circ g)^{j-1}(z) > g \circ (f_T \circ g)^{j-1}(u_k)$. If $T > T(u_k)$, this guarantees that $\delta_{f,g}(z, T) > 0$. Similarly, if T' is small enough, we will have $f_{T'} \circ g \circ (f_{T'} \circ g)^{j-1}(z) < g \circ (f_T \circ g)^{j-1}(u_{k'})$ for any given $u_{k'} \in (x_0, z)$, and choosing $T' < T(u'_k)$ ensures that $\delta_{f,g}(z, T') < 0$. This shows that $z \in U(f, g)$, as desired. ☐

Proof of Proposition 2.20. Let x_0 be an accumulation point of $\partial N(f, g)$. If $x_0 \notin \mathrm{Per}(g)$, then by Lemma 2.21, on any side of x_0 containing a sequence of points in $\partial N(f, g)$, there is a neighborhood of x_0 containing no points of $U(f, g)$. Since $P(f, g), N(f, g)$ and $U(f, g)$ partition S^1, it follows that there is also a sequence of points in $P(f, g)$ approaching x_0 from this side. Since $P(f, g)$ is closed, $x_0 \in P(f, g) \subset \mathrm{Per}(g)$, a contradiction.

It follows that $x_0 \in \mathrm{Per}(g)$. If also $x_0 \notin P(f, g)$, then $x_0 \in U(f, g)$ since x_0 is a periodic point for $f_0 \circ g = g$. But $U(f, g)$ is open, a contradiction. ☐

All the discussion above describes the variation of $\mathrm{Per}(g)$ upon deforming g by composition with f_t on the left. However, one may equally well replace g by $g f_t$ and arrive at sets P, N, and U with the same properties (indeed, replacing g by $g f_t$ is equivalent to replacing g^{-1} by $f_{-t} g^{-1}$). There is no reason to privilege left-side deformations in the definition of bending, and we will occasionally make use of such deformations on the right during the proof.

2.3. **The character space for** $\mathrm{Homeo}^+(\mathbf{S^1})$.

In Section 1 of [4], Calegari and Walker introduce what they call a *character variety for* $\mathrm{Homeo}^+(S^1)$. They propose that one should study rotation numbers of elements of $\mathrm{Homeo}^+(S^1)$ as the natural analog of trace functions on linear representations. A theorem of Ghys states that, for any group Γ, the rotation numbers of elements $\rho(\gamma)$ essentially parametrize the space of representations $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ up to semi-conjugacy, so it makes sense to think of representations up to semi-conjugacy as the character variety for $\mathrm{Homeo}^+(S^1)$. This perspective was adopted implicitly in [14].

Here, we observe that one can recover this analogy from our more general definition of character spaces. Recall that $X(\Gamma, G)$ is the largest Hausdorff quotient of $\mathrm{Hom}(\Gamma, G)/G$ for *any* group Γ and *any* topological group G. Letting, further, $G/\!/G$ denote the space $X(\mathbb{Z}, G)$, there is, for each $\gamma \in \Gamma$ a natural, continuous map $X(\Gamma, G) \to G/\!/G$, which sends the class of a representation ρ to the class of $\rho(\gamma)$. When $G = \mathrm{SL}(2, \mathbb{C})$ for example, these are precisely the trace functions. And, as we will see below, when $G = \mathrm{Homeo}^+(S^1)$, these are the *rotation numbers*, and the space $X(\Gamma, G)$ is, as a set, exactly the set of semi-conjugacy classes of representations.

Following this analogy, the "character variety" for $\mathrm{Homeo}^+(S^1)$ not only comes with its "ring of functions" (namely the rotation number functions), but with an underlying topological space as well; i.e. $X(\Gamma, G)$. This gives the most natural setting to speak of rigidity, or to pose Question 1.7.

Definition 2.22 (Ghys [6])**.** Let Γ be any group. Two homomorphisms ρ_1 and $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}))$ are *semi-conjugate* if there exists a monotone (possibly non-continuous or non-injective) map $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(x+1) = h(x) + 1$ for all $x \in \mathbb{R}$, and $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$ for all $\gamma \in \Gamma$.
Similarly, ρ_1 and $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are semi-conjugate if there is such a map $h\colon \mathbb{R} \to \mathbb{R}$ such that for all γ, there are lifts $\widetilde{\rho_1(\gamma)}$ and $\widetilde{\rho_2(\gamma)}$ which are semi-congugate by this map h.

Ghys [6] proves that, under this definition, semi-conjugacy is an equivalence relation (see also [15] for an expository account). We note that this definition is *not* the usual notion of semi-conjugacy from topological dynamical systems (eg. as in [11]), which is not a symmetric relation. This was perhaps the cause of a typo in the original article [6], which has led to some amount of confusion. However, as it is now standard for actions on S^1, we use the term semi-conjugacy for the definition above.

We now make a few dynamical remarks. Recall that an action of a group on a topological space is *minimal* if the closure of every orbit is dense. In the special case of groups acting on S^1, there is a trichotomy: either the action is minimal, or has a finite orbit, or it has a closed invariant set (called the *exceptional minimal set*) homeomorphic to a Cantor set, to which the restriction of the action is minimal. The following observations are easy consequences of the definition of semi-conjugacy.

Observation 2.23. *Every action ρ_1 with an exceptional minimal set is semi-conjugate to a minimal action ρ_2, by a* continuous *semi-conjugacy map h satisfying $h \circ \rho_1(\gamma) = \rho_2(\gamma) \circ h$; this map simply collapses each complimentary component of the exceptional minimal set to a point. Furthermore, if ρ_2 is minimal, and ρ_1 arbitrary, then any h satisfying this equation is necessarily continuous. In particular, a semi-conjugacy h between two minimal actions is invertible, and hence a* conjugacy*.*

Observation 2.24. *Let $\rho_2 \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ be minimal, and let ρ_1 be any action semi-conjugate to ρ_2, as in the previous observation. Then for any $\gamma \in \Gamma$, $\mathrm{Per}(\rho_2(\gamma)) = h\mathrm{Per}(\rho_1(\gamma))$, hence $|\mathrm{Per}(\rho_2(\gamma))| \leqslant |\mathrm{Per}(\rho_1(\gamma))|$.*

Another useful consequence of Observation 2.23 is that certain bending deformations preserve the conjugacy class of rigid actions, as follows.

Observation 2.25. *Suppose that ρ is minimal and path-rigid, and let a, b have $i(a,b) = -1$. Since $b^{q(b)}$ lies in a 1-parameter family, there is a bending deformation replacing $\rho(a)$ with $\rho(b^{Nq(b)}a)$ for any $N \in \mathbb{Z}$. This bending is realized by precomposition with a Dehn twist (see Section 2.2.2), so the new representation has the same image as ρ; in paritcular it is minimal, hence* conjugate *to ρ.*

Our next goal is to show that semi-conjugacy is exactly the quotient relation needed to produce the character space $X(\Gamma, \mathrm{Homeo}^+(S^1))$ as defined in the introduction. To motivate this (and give one tool for the proof) we show first that this is certainly *not* the case for $\mathrm{Homeo}^+(\mathbb{R})$. Here there are many dynamically distinct actions but the character space is a single point:

Proposition 2.26. *For any discrete group Γ, the space $X(\Gamma, \mathrm{Homeo}^+(\mathbb{R}))$ consists of a single point.*

Proof. Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(\mathbb{R}))$. Let S be a finite symmetric subset of Γ. Given $\varepsilon > 0$, we will conjugate ρ so that $|\rho(s)(x) - x| < \epsilon$ holds for all $s \in S$ and $x \in \mathbb{R}$, hence show that conjugates of ρ approach the trivial representation in the compact-open topology.

As a first case, assume also that the subgroup generated by S has no global fixed points in $\mathbb{R}$. Then define $h(0) = 0$, and iteratively, for $n \in \mathbb{Z}$ define $h(n\varepsilon/2) = \max_{s \in S} s(h((n-1)\varepsilon/2)))$ if $n > 0$, and $h(n\varepsilon/2) = \min_{s \in S} s(h((n+1)\varepsilon/2)))$ if $n < 0$. Extend h over the interior of each interval $[n\varepsilon/2, (n+1)\varepsilon/2]$ as an affine map. Since S has no global fixed point, this map h is surjective, hence it is an orientation-preserving homeomorphism. Furthermore, we have $hsh^{-1}(n\varepsilon/2) \in [(n-1)\varepsilon/2, (n+1)\varepsilon/2]$ for all $s \in S$. Thus, $|hsh^{-1}(x) - x| < \varepsilon$ holds for all $x \in \mathbb{R}$.

If instead the subgroup generated by S does have a global fixed point, we may define h to be the identity on the set F of global fixed points, and define it as above on each connected component of $\mathbb{R} \smallsetminus F$. □

Note that the same result holds more generally for spaces of continuous representations when Γ is a topological group, with the modification that S should be a compact set.

We now prove the main result of this section. In one direction of the proof we will use a theorem of Ghys and Matusmoto, but we defer the statement of this until after the proof as it leads naturally into the next section.

Proposition 2.27. *Let Γ be any group. Two representations ρ_1 and ρ_2 in* $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ *are semi-conjugate if and only if they are χ-equivalent.*

Proof. For one direction, it suffices to prove that the quotient of the space $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ by semi-conjugacy is Hausdorff. This follows from Theorem 2.29 below (due to Ghys and Matsumoto), since the maps rot and τ in the theorem are continuous, well defined on semi-conjugacy classes, take values in the (Hausdorff) spaces S^1 and $\mathbb{R}$, and distinguish semi-conjugacy classes. It follows that distinct semi-conjugacy classes are separated by invariant open sets in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$.

For the converse, we use the dynamical trichotomy from the remarks following Definition 2.22. If ρ has a finite orbit, then we can employ a similar strategy to the proof of Proposition 2.26 to conjugate it arbitrarily close to an action on the circle by rigid rotations. Hence, there is a unique element of the character space corresponding to the semi-conjugacy class of ρ.

Now suppose instead that ρ has an exceptional minimal set. By Observation 2.23 there is a minimal action ρ' and continuous map h such that each $\gamma \in \Gamma$ has lifts satisfying $\widetilde{\rho'(\gamma)} \circ h = h \circ \widetilde{\rho(\gamma)}$ as in Definition 2.22. Let S be a finite subset of Γ, and fix $\varepsilon > 0$. Let $\delta \in (0, \varepsilon)$ be small enough so that for all $s \in S$ and all $x, y \in S^1$, $|x - y| < \delta$ implies $|\rho'(s)(x) - \rho'(s)(y)| < \varepsilon$.

Since h is continuous and commutes with $x \mapsto x + 1$, we can approximate it by a homeomorphism $h' \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ at C^0 distance at most δ from h. Let $s \in S$ and $x \in \mathbb{R}$, and take the lifts $\widetilde{\rho'(s)}$ and $\widetilde{\rho(s)}$ as above. Then we have

$$|\widetilde{\rho'(s)}(x) - \widetilde{\rho'(s)} \circ (h \circ h'^{-1})(x)| < \varepsilon$$

and

$$|h \circ \widetilde{\rho(s)} \circ h'^{-1}(x) - h' \circ \widetilde{\rho(s)} \circ h'^{-1}(x)| < \varepsilon,$$

hence the definition of semi-conjugacy and the triangle inequality gives

$$|\widetilde{\rho'(s)}(x) - h' \circ \widetilde{\rho(s)} \circ h'^{-1}(x)| < 2\varepsilon.$$

This proves that every representation without finite orbit is χ-equivalent to the minimal representation in its semi-conjugacy class. ☐

The content of the theorem of Ghys and Matsumoto used above is the statement that representations to $\mathrm{Homeo}^+(S^1)$ are essentially determined by rotation numbers of elements. To make this precise we need a definition.

Definition 2.28 (Translation cocycle)**.** For $f, g \in \mathrm{Homeo}^+(S^1)$, define $\tau(f, g) := \widetilde{\mathrm{rot}}(\tilde{f}\tilde{g}) - \widetilde{\mathrm{rot}}(\tilde{f}) - \widetilde{\mathrm{rot}}(\tilde{g})$, where $\tilde{f}$ and $\tilde{g}$ are any lifts of f and g to $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Note that the value of $\tau(f, g)$ is independent of the choice of lifts.

Theorem 2.29 (Ghys [6], Matsumoto [19])**.** *Let Γ be any group, and let S be a generating set for Γ. Two representations ρ_1 and ρ_2 in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}^+(S^1))$ are semi-conjugate if and only if the following two conditions hold:*

i) $\mathrm{rot}(\rho_1(s)) = \mathrm{rot}(\rho_2(s))$ *for each* $s \in S$*,*
ii) $\tau(\rho_1(a), \rho_1(b)) = \tau(\rho_2(a), \rho_2(b))$ *for all* a *and* b *in* Γ*.*

As an illustration of the power of Theorem 2.29, let us prove a lemma that will be very useful later on. For $f_1, \ldots, f_n \in \mathrm{Homeo}^+(S^1)$, let

$$\tau(f_1, \ldots, f_n) = \widetilde{\mathrm{rot}}(\widetilde{f_n} \circ \cdots \circ \widetilde{f_1}) - \sum_i \widetilde{\mathrm{rot}}(\widetilde{f_i})$$

(which obviously does not depend on the choices of lifts). This is notationally convenient, but actually contains no more information than the two-variable τ, since

$$\tau(f_1, \ldots, f_n) = \tau(f_1, f_n \circ \cdots \circ f_2) - \sum_{j=2}^{n-1} \tau(f_j, f_n \circ \cdots \circ f_{j+1}).$$

Lemma 2.30. *Let $f, g \in \mathrm{Homeo}^+(S^1)$ be two homeomorphisms with rational rotation number. The following assertions are equivalent.*

(1) f and g share a periodic point.
(2) For every $\ell \geqslant 1$ and all integers $n_1, m_1, \ldots, n_\ell, m_\ell$, we have

$$\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \cdots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) = 0.$$

Together with Theorem 2.29, this lemma asserts that given γ_1, γ_2 in a group Γ, the property of whether or not $\rho(\gamma_1)$ and $\rho(\gamma_2)$ share a periodic point only depends on the semi-conjugacy class of ρ. It is in this form that Lemma 2.30 will be used throughout this text.

Proof. The implication (1) $\Rightarrow$ (2) is trivial. For the converse, suppose $\mathrm{Per}(f) \cap \mathrm{Per}(g) = \varnothing$. Then $S^1 \smallsetminus (\mathrm{Per}(f) \cup \mathrm{Per}(g))$ is a union of intervals. As $\mathrm{Per}(f)$ and $\mathrm{Per}(g)$ are closed, disjoint sets, only finitely many of these complementary intervals have one boundary point in each of $\mathrm{Per}(f)$ and $\mathrm{Per}(g)$. The intervals bounded at their right by a point of $\mathrm{Per}(f)$ and at their left by a point of $\mathrm{Per}(g)$ alternate with the others, in particular there are an even number of such complementary intervals. Let $I_1, \dots, I_{2\ell}$ denote these intervals, in their cyclic order on the circle, and let $I_j = (x_j, y_j)$. Up to shifting the indices cyclically, we have $x_i, y_{i+1} \in \mathrm{Per}(g)$ and $x_{i+1}, y_i \in \mathrm{Per}(f)$ for all i even.

Choose a point x in I_1. The interval (x_1, y_2) contains only points of $\mathrm{Per}(g)$, hence there exists some power n_1 of $f^{q(f)}$, such that $f^{n_1 q(f)}(x) \in I_2$. Similarly, there exists a power n_2 of $g^{q(g)}$ which maps $f^{n_1 q(f)}(x)$ into I_3, and so on. The last operation can be done so that the image of x, under a suitable word $g^{n_\ell q(g)} f^{n_\ell q(f)} \cdots g^{n_2 q(g)} f^{n_1 q(f)}$, lies at the right of x in I_1. Then, choosing the canonical lifts of $f^{n_i q(f)}$ and $g^{m_i q(g)}$, we observe that $\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \cdots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) \geqslant 1$. ☐

Remark 2.31. In the case $\mathrm{Per}(f) \cap \mathrm{Per}(g) = \varnothing$, the integer ℓ in the proof above also only depends on τ – in fact, it is the *minimal* integer such that there exist $m_i, n_i \in \mathbb{Z}$ with $\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \cdots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) \geqslant 1$.

To see this, fix $h < \ell$. For any x_j even, and any $M, N \in \mathbb{Z}$ we have $\widehat{g^{Mq(g)}}\,\widehat{f^{Nq(f)}}(\widetilde{x_j}) < \widetilde{x_{j+2}}$, where $\widetilde{x_j}$ and $\widetilde{x_{j+2}}$ are consecutive lifts of x_j and x_{j+2}. Thus, for any choice of integers n_i, m_i we have

$$\widehat{g^{n_h q(g)}}\,\widehat{f^{n_h q(f)}} \cdots \widehat{g^{n_2 q(g)}}\,\widehat{f^{n_1 q(f)}}(\widetilde{x_j}) < \widetilde{x_{j+2h}}$$

(again, taking consecutive lifts), and since $h < \ell$ this implies that

$$\widehat{g^{n_h q(g)}}\,\widehat{f^{n_h q(f)}} \cdots \widehat{g^{n_2 q(g)}}\,\widehat{f^{n_1 q(f)}}(x) < x + 1$$

for all $x \in \mathbb{R}$, whence $\tau(f^{n_1 q(f)}, g^{m_1 q(g)}, \cdots, f^{n_\ell q(f)}, g^{m_\ell q(g)}) < 1$.

2.4. **The Euler class.**

We now recall the Euler class for $\mathrm{Homeo}^+(S^1)$ as a discrete group (equivalently, for flat topological S^1 bundles) and state the related results needed later in this work.

In fact, the real Euler class has already made a brief appearance – it is the element $e_{\mathbb{R}} \in H^2(\mathrm{Homeo}^+(S^1); \mathbb{R}) \cong \mathbb{R}$ represented by the inhomogeneous 2-cocycle τ from Definition 2.28. The *(integer) Euler class* is the generator e of $H^2(\mathrm{Homeo}^+(S^1); \mathbb{Z}) \cong \mathbb{Z}$ which maps to $e_{\mathbb{R}}$ under the natural inclusion. For the experts, we recall that Ghys' original version of Theorem 2.29 above is the statement that the semi-conjugacy class of a representation $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$ is determined by the *bounded, integral* Euler class $\rho^*(e) \in H^2(\Gamma; \mathbb{Z})$; Matsumoto translated this into the language of rotation numbers.

The *Euler number* of a representation $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$ is the integer $\langle \rho^*(e), [\Gamma_g] \rangle$, where $[\Gamma_g]$ denotes the fundamental class, i.e. a generator of

$H_2(\Gamma_g, \mathbb{Z})$. Although this definition only makes sense for fundamental groups of closed surfaces, (a surface with boundary has free fundamental group, and $H_2(F_n; \mathbb{Z}) = 0$) there is a *relative* Euler number for surfaces with boundary, which is additive when such subsurfaces are glued together. This can be made precise in the language of bounded cohomology, and, following [3, § 4.3], we will use the following definition. (Compare also Goldman [8] and Matsumoto [20].)

Definition 2.32 (Relative Euler number for pants.)**.** Let $P \subset \Sigma_g$ be a subsurface homeomorphic to a pair of pants. If P contains the base point of Σ_g, equip it with three curves a, b, c as in Figure 3. If not, choose a point in P, and a path in Σ_g from its base point to the chosen point in P, and use it to define three such curves a, b and c.

Let $\rho\colon \pi_1\Sigma_g \to \mathrm{Homeo}^+(S^1)$, and let $\widetilde{\rho(a)}$, $\widetilde{\rho(b)}$ be any lifts of $\rho(a)$ and $\rho(b)$ to $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, and let $\widetilde{\rho(c)} = \left(\widetilde{\rho(b)}\widetilde{\rho(a)}\right)^{-1}$. Then the contribution of P to the Euler number of ρ is

$$\mathrm{eu}_P(\rho) = \widetilde{\mathrm{rot}}\left(\widetilde{\rho(a)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(b)}\right) + \widetilde{\mathrm{rot}}\left(\widetilde{\rho(c)}\right).$$

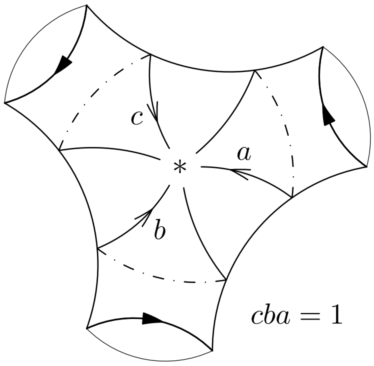

FIGURE 3. A pair of pants with standard generators of its fundamental group

This extends naturally to one-holed tori.

Definition 2.33 (Relative Euler number for one-holed tori.)**.** Let $T = T(a, b) \subset \Sigma_g$ be a one-holed torus subsurface, where a and b are oriented curves representing a standard pair of generators as in Figure 1. Let $c = [a, b]$. Then the triple $(b^{-1}, a^{-1}ba, c)$ forms a set of boundary curves of a pants subsurface obtained by cutting T along a loop freely homotopic to b. Let $\widetilde{\rho(b)}$ and $\widetilde{\rho(a)}$ be lifts of $\rho(b)$ and $\rho(a)$ respectively. Then, as $\widetilde{\mathrm{rot}}$ is conjugacy invariant, we have $\widetilde{\mathrm{rot}}(\widetilde{\rho(b)}^{-1}) = -\widetilde{\mathrm{rot}}(\widetilde{\rho(a)}^{-1}\widetilde{\rho(b)}\widetilde{\rho(a)})$, so consistent with the definition above, the *relative Euler number of ρ on T* is given by

$$\mathrm{eu}_T(\rho) = \widetilde{\mathrm{rot}}\left(\widetilde{\rho(b)}^{-1}\widetilde{\rho(a)}^{-1}\widetilde{\rho(b)}\widetilde{\rho(a)}\right).$$

If the surface Σ_g is cut into pairs of pants (and/or tori), the Euler class of ρ is the sum of the contributions of these subsurfaces. See [3, § 4.3] for a detailed discussion, and [17] for a short exposition and a proof. With the consistent orientation of the boundary curves of each pant, $\mathrm{eu}(\rho)$ is a sum of $6g-6$ rotation numbers, in which each curve appears (up to conjugacy) with a positive and a negative power, hence the resulting number is an integer. We leave it as an exercise to the reader to verify that this agrees with the standard definition of Euler number.

3. A FIRST STATEMENT

This section proves the main theorem under a strong additional hypothesis. We will show that if ρ is path-rigid and if for every $a, b \in \Gamma_g$ with $i(a,b) = \pm 1$, $\rho(a)$ and $\rho(b)$ dynamically "look like" a geometric representation, then ρ is in fact geometric. In other words, the local condition that ρ "looks geometric" on pairs a, b with $i(a,b) = \pm 1$ implies global geometricity. To formalize this, we introduce some definitions.

Definition 3.1. Say that an element $f \in \mathrm{PSL}^k(2,\mathbb{R})$ is *hyperbolic* if its projection to $\mathrm{PSL}(2,\mathbb{R})$ is hyperbolic. Equivalently, all its periodic points are hyperbolic in the sense of classical smooth dynamics.

Definition 3.2. Let $a, b \in \Gamma_g$ and $\rho : \Gamma_g \to \mathrm{Homeo}^+(S^1)$. Denote by $S_k(a,b)$ (the notation ρ is suppressed) the property that

i. $i(a,b) = \pm 1$ and $\rho(a)$ and $\rho(b)$ are each separately conjugate to a hyperbolic element of $\mathrm{PSL}^k(2,\mathbb{R})$, and
ii. their periodic points *alternate* around the circle, meaning that each pair of points of $\mathrm{Per}(a)$ are separated by $\mathrm{Per}(b)$, and vice versa.

If all pairs a, b with $i(a,b) = \pm 1$ have $S_k(a,b)$, then we say that ρ has property S_k.

With this notation we can state the main result of this section.

Theorem 3.3. *Let ρ be a path-rigid, minimal representation, and suppose ρ satisfies S_k for some k. Then ρ is geometric.*

Before embarking on the proof, we discuss some other variations on hyperbolicity to be used later in the section.

Let $f \in \mathrm{Homeo}^+(S^1)$. We say that an open interval $I \subset S^1$ is *attracting for* f if $f(\overline{I}) \subset I$. We say that I is *repelling for* f if it is attracting for f^{-1}. Matsumoto [20] calls homeomorphisms that do not admit attracting intervals *tame*. In line with his terminology, we call those homeomorphisms which do *savage*. More specifically, we have:

Definition 3.4. A homeomorphism $f \in \mathrm{Homeo}^+(S^1)$ is *n-savage* if there exist $2n$ open intervals with pairwise disjoint closures, indexed in cyclic order by $I_1^-, I_1^+, \ldots, I_n^-, I_n^+$ such that

$$f(S^1 \smallsetminus (\cup_{j=1}^n \overline{I_j^-})) = \cup_{j=1}^n I_j^+.$$

In this sense, savage means 1-savage.

The next observation is an immediate consequence of the definition, we leave the proof to the reader.

Observation 3.5. *If f is n-savage, then f^k is also n-savage for any $k \in \mathbb{Z} \smallsetminus \{0\}$. Furthermore,* $\mathrm{rot}(f^n) = 0$ *and f has least one periodic point in each interval I_j^+ and I_j^-.*

As a concrete example, note that if f is conjugate to a hyperbolic element in $\mathrm{PSL}^k(2,\mathbb{R})$, then f is n-savage if and only if $n \leqslant k$.

The intervals I_j^+ and I_j^- in the definition of savage are by no way unique, but it will be convenient to use the notation $I^+(f) := \cup_{j=1}^n I_j^+$ and $I^-(f) := \cup_{j=1}^n I_j^-$, even if this these sets depend on choices. We also set $I(f) := I^+(f) \cup I^-(f)$.

Definition 3.6. Two n-savage homeomorphisms $f, g \in \mathrm{Homeo}^+(S^1)$ are *in n-Schottky position* if their respective attracting and repelling intervals $I_j^\pm$ can be chosen so that $I(f)$ and $I(g)$ have disjoint closures.

Note that, if f and g are n-Schottky, then f^{-1} and g are n-Schottky as well. Note also that the condition $S_k(a,b)$ is not equivalent to k-Schottky, although $S_k(a,b)$ does imply that a^N and b^N are k-Schottky for sufficiently large N. One of the challenges we face is to show that hypothesis S_k on a path-rigid representation ρ implies that a and b are indeed k-Schottky whenever $i(a,b) = \pm 1$.

3.1. **Outline of Proof of Theorem 3.3.** We start in Section 3.2 with a series of lemmas that use rigidity and property S_k to show the cyclic order of periodic points of various non-separating curves agrees with that of a geometric representation, and that certain pairs of curves are k-Schottky. Following this, we show in Section 3.3 that the Euler number of a path-rigid, minimal, S_k representation agrees with a geometric one, i.e. is equal to $\pm\frac{2g-2}{k}$. From there, we need to improve this essentially combinatorial result to the fact that the representation is actually geometric. Our main tool is existing work of Matsumoto on *Basic Partitions.* His technique uses a decomposition of the surface into one-holed tori and pairs of pants, involving many separating curves, which on the level of fundamental groups corresponds to a decomposition of Γ_g into a tree of groups. However, our hypothesis S_k is a condition on *non-separating* curves (which is much more natural for us, given that we are working with standard generators, chains, and bending deformations). Thus, before invoking Matsumoto, we must pass from non-separating to separating curves; this is done in Section 3.4. (A more direct approach would be to adapt Matsumoto's techniques to a general decomposition of the surface group into a graph of groups, but this Pyrrhic victory would actually make our proof longer and more difficult.)

We are now ready to embark on the proof. Throughout, we make the following assumption.

Assumption 3.7. For the rest of this section, ρ denotes a path-rigid minimal representation of Γ_g that satisfies S_k. To simplify notation, we often omit ρ, identifying $a \in \Gamma_g$ with $\rho(a) \in \mathrm{Homeo}^+(S^1)$. Thus, we will speak of $\mathrm{Per}(a)$, denote an attracting point of $\rho(a)$ by a^+, etc.

3.2. **Order of periodic points.** Property S_k makes it much easier to understand periodic points under deformations. We start with several lemmas to this effect.

Lemma 3.8. *Let $i(a,b) = 1$, let $F \subset S^1$ be a countable set, and let b_t be a positive one-parameter family commuting with $b = \rho(b)$. Then for some $t \in \mathbb{R}$, we have $\mathrm{Per}(b_t\rho(a)) \cap F = \varnothing$.*

Proof. We use the notation from Section 2.2.3. Path-rigidity of ρ implies that $\mathrm{rot}(b_t a)$ is constant, and Property S_k implies that $P(b,a) = \varnothing$, so we need only worry about points in $U = U(b,a)$. Thus, provided $t \notin T_{b,a}(F)$, we have $\mathrm{Per}(b_t a) \cap F = \varnothing$. $\square$

Lemma 3.9 (Disjoint curves have disjoint Per)**.** *Let (a,b,c) be a completable directed 3-chain. Then $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. In fact, $\mathrm{Per}(c) \cap b^n(\mathrm{Per}(a)) = \varnothing$ for all $n \in \mathbb{Z}$.*

Proof. Fix $n \in \mathbb{N}$. Complete (a,b,c) to a directed 4-chain (a,b,c,d), and apply a bending deformation replacing c with $d_t c$ (leaving the action of a and b unchanged, hence $b^n\mathrm{Per}(a)$ unchanged), for a positive family d_t. By Lemma 3.8, there is some t such that $\mathrm{Per}(d_t c) \cap b^n\mathrm{Per}(a) = \varnothing$. Now the conclusion follows from path-rigidity of ρ, together with Lemma 2.30. $\square$

Note that, if $i(a,b) = \pm 1$, then for any $n \in \mathbb{Z}$ we also have $i(b^n a, b) = \pm 1$, hence $S_k(b^n a, b)$ holds. The next lemma describes the position of the periodic points of $S_k(b^n a, b)$ for large n. This is particularly useful since there exist bending deformations replacing the pair a, b with $b^n a, b$ provided that $q(b)$ divides n (see Observation 2.25).

Lemma 3.10 (Movement of Per by bending)**.** *Suppose $i(a,b) = \pm 1$. Then as $N \to +\infty$, the points of $\mathrm{Per}^+(b^N a)$ approach $\mathrm{Per}^+(b)$, and $\mathrm{Per}^-(b^N a)$ approaches $a^{-1}\mathrm{Per}^-(b)$; similarly, as $N \to -\infty$, $\mathrm{Per}^+(b^N a)$ approaches $\mathrm{Per}^- b$ and $\mathrm{Per}^-(b^N a)$ approaches $a^{-1}\mathrm{Per}^+(b)$.*

Proof. The conclusion of the lemma is an easy exercise, provided that $a^{-1}\mathrm{Per}(b) \cap \mathrm{Per}(b) = \varnothing$. We claim that path-rigidity of ρ implies this extra provision. To see this, suppose for example that $i(a,b) = 1$, and let (c,a,b) be a completable directed 3-chain. By Lemma 3.9, $\mathrm{Per}(c) \cap \mathrm{Per}(b) = \varnothing$. Thus, we can make a positive bending deformation replacing a with ac_t, until $(ac_t)^{-1}\mathrm{Per}(b) \cap \mathrm{Per}(b) = \varnothing$. $\square$

Notation 3.11. Let f and g be homeomorphisms of S^1. When talking about cyclic order of periodic points, we use the notation $(\!(f^+, g^+, g^-, f^-)\!)_k$ to mean that, in cyclic order, there is one attracting point for f, followed by an attracting point for g, followed by a repelling point for g, followed by an attracting point for f, with this pattern repeating k times. The notation $f^{\pm}$ means any point from $\mathrm{Per}(f)$. We also use other obvious variations, such as $(\!(f^{\pm}, g^-, f^{\pm}, g^+)\!)_k$, and extend this naturally to periodic points of three or more homeomorphisms.

When such a cyclic order is given, we call an interval $I \subset S^1$ of *type* (f^+, g^-) if it is bounded on the left (proceeding anti-clockwise, using the natural orientation of S^1) by a point of $\mathrm{Per}^+(f)$ and on the right by a point of $\mathrm{Per}^-(g)$. Of course, we also use other obvious variations.

Lemma 3.12 (Periodic points of 3-chains)**.** *Let (a,b,c) be a completable directed 3-chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c come in the following cyclic order:*

$$\left(\!\!\left(a^-, b^-, a^+, c^{\pm}, b^+, c^{\pm}\right)\!\!\right)_k,$$

Proof. Up to reversing orientation of S^1, we may suppose that the cyclic order of points in $\mathrm{Per}(a) \cup \mathrm{Per}(b)$ is $(\!(a^-, b^-, a^+, b^+)\!)_k$. Choose two consecutive points of $\mathrm{Per}(b)$ (in cyclic order), and denote these by b^- and b^+. (To avoid unnecessary subscripts or superscripts, we will often use $f^{\pm}$ to denote a *specific* attracting/repelling fixed point of a homeomorphism f, and the double-bracket notation above to indicate the cyclic order of a set of fixed points.) Let a^+ be the point of $\mathrm{Per}(a)$ between b^- and b^+, and let $c^{\pm}$ be the periodic point of c in this interval (there is exactly one by hypothesis S_k). We know that the points of $\mathrm{Per}(a)$ in the interval (b^-, b^+) are in cyclic order $(b^-, a^+, b^{q(b)}(a^+), b^+)$.

By Lemma 3.9, $c^{\pm}$ cannot be equal to a^+ or $b^{q(b)}(a^+)$. Suppose for contradiction that $c^{\pm}$ lies in the interval (b^-, a^+), or in the interval $(b^{q(b)}(a^+), b^+)$. Then the closed segment $[a^+, b^{q(b)}(a^+)]$ does not contain any periodic point of c. Let $(c_t)_{t \in \mathbb{R}}$ be a positive 1-parameter family commuting with c, and use this to perform a bending along c as in Section 2.2.3. Using the notation from this section, we have $\delta_{c,b}(a^+, 0) > 0$, but for t sufficiently negative, we have $\Delta_{c,b}(a^+, 0, \ldots, 0, t) < 0$. Thus, for some $t_0 < 0$, we have $\delta_{c,b}(a^+, t_0) = 0$, i.e. $a^+ \in \mathrm{Per}(c_{t_0} b) \cap \mathrm{Per}(a)$. This, together with Lemma 2.30 and the path-rigidity of ρ, yields a contradiction.

The same argument applies to an interval of the form (b^+, b^-), where b^+ and b^- denote two other consecutive points of $\mathrm{Per}(b)$. In that case, the argument shows that the (unique) periodic point of c in this interval lies between points of the form $b^{q(b)}(a^-)$ and a^-, proving the lemma. □

In particular, for all pairs $a, c \in \Gamma_g$ such that there exists a completable 3-chain (a, b, c), Lemma 3.12 provides information about the periodic sets of a and c.

Corollary 3.13. *Let a and c be two non separating curves with $i(a, c) = 0$, and suppose c is not conjugate to a or a^{-1}. Then their periodic points are in cyclic order $(\!(a^{\pm}, a^{\pm}, c^{\pm}, c^{\pm})\!)_k$.*

Proposition 3.14. *Let (a, b, c) be a completable directed 3-chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c and the b-preimages of $\mathrm{Per}(c)$ are in cyclic order*

$$\left(\!\!\left(a^-, b^{-1}(c^{\pm}), b^-, b^{-1}(c^{\pm}), a^+, c^{\pm}, b^+, c^{\pm}\right)\!\!\right)_k .$$

Proof of Proposition 3.14. Apply a bending deformation of ρ replacing b with $c^{Nq(c)} b$, and leaving the action of c and a unchanged. By Lemma 3.10, for N sufficiently large, $\mathrm{Per}^-(c^{Nq(c)} b)$ approaches $b^{-1}\mathrm{Per}^-(c)$, and $\mathrm{Per}^-(c^{-Nq(c)} b)$ approaches $b^{-1}\mathrm{Per}^+(c)$. Since ρ is path-rigid, the cyclic order of periodic points is invariant under these deformations, hence the points $b^{-1}(c^{\pm})$ all must lie in intervals of type (a^-, a^+).

Now up to replacing c with c^{-1} (its orientation is unimportant in this proof) we may assume that the order of periodic points given by Lemma 3.12 is $(\!(a^-, b^-, a^+, c^+, b^+, c^-)\!)_k$. Then $b^{-1}\mathrm{Per}^-(c)$ lies in the intervals of type (b^+, b^-), as b preserves these intervals. Thus, points of $b^{-1}\mathrm{Per}^-(c)$ are between consecutive points of $\mathrm{Per}^-(a)$ and $\mathrm{Per}^-(b)$. Similarly, the points $b^{-1}(c^+)$ are between consecutive points of the form b^- and a^+. □

The following variation is proved using the same style of argument.

Lemma 3.15. *Let $a, b, c \in \Gamma_g$ be three non-separating curves such that $i(a, b) = -1$ and c is disjoint from $T(a, b)$. Up to reversing the orientation of S^1, we may suppose that the periodic points of a and b are in the order $\langle\!\langle a^-, b^+, a^+, b^- \rangle\!\rangle_k$. Then the periodic points of c all lie in intervals of type (b^-, a^-).*

Note that the order in which we prefer to take the periodic points of a and b is different here than in the two preceding statements, because here $i(a, b) = -1$.

Proof. Similar to the proof of Proposition 3.14, we perform bending deformations. Since ρ is path-rigid, the cyclic order of periodic points does not change after the bending deformation replacing b with $a^{Nq(a)}b$ (leaving a and c unchanged). The effect of these deformations is to push $\mathrm{Per}^+(b)$ as close as we want to either $\mathrm{Per}^+(a)$ or $\mathrm{Per}^{-1}(a)$. Applying Lemma 3.10 as in the proof of Proposition 3.14 shows that periodic points of c cannot be in the intervals of type (a^-, b^+) or (b^+, a^+) – as the argument is entirely analogous, we omit the details. The same argument again using the deformation replacing a by $b^{Nq(b)}a$ shows that the periodic points of c cannot be in the intervals of type (a^+, b^-), either. ☐

Proposition 3.16. *Let a, c be two non separating curves with $i(a, c) = 0$, and suppose c is not conjugate to a or a^{-1}. Then $\rho(a)$ and $\rho(c)$ are in k-Schottky position.*

Proof. Up to changing the orientation of c, we may choose non separating curves b and d such that (a, b, c, d) is the beginning of a standard basis of $\pi_1\Sigma_g$.

Using a deformation as in Lemma 3.9, path-rigidity of ρ implies that the points of $\mathrm{Per}^-(d)$, $c^{-1}\mathrm{Per}^+(d)$, $\mathrm{Per}^-(b)$, and $a^{-1}\mathrm{Per}^+(b)$ are all distinct. Fix small disjoint neighborhoods U^+ of $\mathrm{Per}^-(d)$, U^- of $c^{-1}\mathrm{Per}^+(d)$, and also V^+ of $\mathrm{Per}^-(b)$, and V^- of $a^{-1}\mathrm{Per}^+(b)$.

By Lemma 3.10, if n is large enough, $d^{-nq(d)}c(S^1 \smallsetminus U^-) \subset U^+$ and $b^{-nq(b)}a(S^1 \smallsetminus V^-) \subset V^+$, so we may find $2k$ disjoint attracting and repelling intervals for $d^{-nq(d)}c$ and $b^{-nq(b)}a$ as in the definition of k-Schottky. Now there exists a bending deformation that replaces c with $d^{-nq(d)}c$ and a with $b^{-nq(b)}a$, and it follows from Observation 2.25 that this deformation is conjugate to the original action. Thus, a and c are k-Schottky. ☐

Proposition 3.17. *Let a, c be two non separating curves with $i(a, c) = \pm 1$. Then $\rho(a)$ and $\rho(c)$ are in k-Schottky position.*

Proof. Choose b and d so that (b, a, c, d) is a 4-chain. Now follow the proof above. ☐

From Proposition 3.16 we deduce an enhanced version of Lemma 3.12.

Proposition 3.18. *Let (a, b, c) be a completable directed 3-chain. Then, up to reversing the orientation of the circle, the periodic points of a, b and c are in cyclic order $\langle\!\langle a^-, b^-, a^+, c^-, b^+, c^+ \rangle\!\rangle_k$.*

Proof. Following Proposition 3.14, we need only discard the possibility that the order is $\langle\!\langle a^-, b^-, a^+, c^+, b^+, c^- \rangle\!\rangle_k$. Suppose for contradiction that this order does hold. By Proposition 3.16, we know that a and c each have $2k$ intervals as in Definition 3.4, with pairwise disjoint closures. As $|\mathrm{Per}(a)| = |\mathrm{Per}(c)| = 2k$, each of these intervals contains exactly one periodic point, so their cyclic order is specified by the order of periodic points given above.

Note that ca is non-separating, as the 3-chain (a, b, c) is completable. Also, $\rho(ca)$ is k-savage, and we may take $I^-(ca) \subset I^-(a)$ and $I^+(ca) \subset I^+(c)$. With the same argument as above, $\rho(ca)$ has exactly one repelling periodic point in each interval of $I^-(ca)$, and one attracting periodic point in each interval of $I^+(ca)$.

If $\mathrm{Per}(b)$ is disjoint from $I^-(a) \cup I^+(c)$, then this is enough to imply that the periodic points of ca and b alternate, contradicting Lemma 3.12, since $i(ca, b) = 0$. Thus, it only remains to prove that $\mathrm{Per}(b)$ can be made disjoint from $I^-(a) \cup I^+(c)$ to finish the proof. This can be done in the same manner as that of Proposition 3.16. First, complete (a, b, c) into a directed 5-chain $(\alpha, a, b, c, \gamma)$. Then, consider a bending deformation of ρ, where b is unchanged but the action of a is replaced by that of $a\alpha^{Nq(\alpha)}$ and the action of c by $\gamma^{Nq(\gamma)}c$ for N large. By Observation 2.25 this new action is conjugate to ρ. Now, provided N is large enough, we can choose our Schottky intervals to be as narrow as we want, around the points α^-, $a(\alpha^+)$, γ^+ and $c^{-1}(\gamma^-)$ which, using Lemma 3.9, are disjoint from $\mathrm{Per}(b)$. □

3.3. **Euler number.** As a consequence of the work in the previous section, we can now show that the Euler number of ρ agrees with a geometric representation.

Theorem 3.19. *Let ρ be path-rigid, minimal and satisfy S_k. Then $|\mathrm{eu}(\rho)| = \frac{2g-2}{k}$.*

In fact, we will show the following stronger statement, which implies Theorem 3.19 by additivity of the Euler number on subsurfaces.

Theorem 3.20. *Up to changing the orientation of the circle, for every pair-of-pants subsurface $P \subset \Sigma_g$, the relative Euler class of ρ on P is $\frac{-1}{k}$.*

This is true in particular for one-holed torus subsurfaces, which will be of interest in the next subsection.

Definition 3.21. Let $i(a, b) = 1$. We say that the ordered pair (a, b) is of type $+$ if the periodic points of a and b are in the cyclic order $(\!(a^-, b^-, a^+, b^+)\!)_k$. Otherwise, we say that (a, b) is of type $-$.

As a consequence of Proposition 3.18, for every oriented, completable directed 3-chain (a, b, c), the pairs (a, b) and (b, c) have the same type. Thus, Lemma 2.11 implies that all one-holed tori have the same type. Thus, up to conjugating ρ by an orientation-reversing homeomorphism, we may suppose the type is always $+$.

Proof of Theorem 3.20. We begin by proving the claim for a pair of pants P, such that at least two boundary components of P are non-separating. Denote by a^{-1}, c^{-1}, and ac the three boundary components of P, with the convention of Figure 3, and suppose that a and c are non-separating. With these choices of orientations, the Euler number of ρ on P will be equal to

$\widetilde{\mathrm{rot}}(\widehat{a}\widehat{c}) - \widetilde{\mathrm{rot}}(\widehat{a}) - \widetilde{\mathrm{rot}}(\widehat{c})$, and there exists a curve b such that (a, b, c) is an oriented, completable, directed 3-chain (the end of the proof of Observation 2.9 justifies the existence such a curve b).

Since (a, b) is of type $+$, it follows from Proposition 3.18 that the periodic points of a and c are in cyclic order $(\!(a^-, a^+, c^-, c^+)\!)_k$; and by Proposition 3.16, they are in k-Schottky position, with Schottky intervals $I_j^\pm(a)$ and $I_j^\pm(c)$. Lift these to intervals $\tilde{I}_j^\pm(a)$ and $\tilde{I}_j^\pm(c) \subset \mathbb{R}$, indexed by integers, and in order

$$...\tilde{I}_j^-(a), \tilde{I}_j^+(a), \tilde{I}_j^-(c), \tilde{I}_j^+(c), \tilde{I}_{j+1}^-(a), ...$$

such that the projection to S^1 is given by taking indices mod k. It follows easily from the definition of Savage (see also Observation 3.5) that $\widehat{a}(\tilde{I}_j^+(a)) \subset \tilde{I}_{j+\ell}^+(a)$ for some ℓ (which depends of course on a) and in this case $\ell/k = \widetilde{\mathrm{rot}}(\widehat{a})$. An analogous statement holds also for c; let m/k denote its translation number.

Since a and c are in k-Schottky position, their product ac is k-savage, and we can take $I^-(ac) = I^-(c)$ and $I^+(ac) \subset I^+(a)$. Note that each of the k intervals of $I^+(ac)$ is contained in a different interval of $I^+(a)$. We now track images of intervals to compare translation numbers. Set the indexing of the intervals $\tilde{I}^\pm(ac)$ so that $\tilde{I}_1^+(a) = \tilde{I}_1^+(ac)$. This lies between $\tilde{I}_0^+(c)$ and $\tilde{I}_1^-(c)$, so we have

$$c(\tilde{I}_1^+(ac)) \subset \tilde{I}_m^+(c)$$

and similarly, since $\tilde{I}_m^+(c)$ lies between $\tilde{I}_m^+(a)$ and $\tilde{I}_{m+1}^-(a)$, we have

$$ac(\tilde{I}_1^+(ac)) \subset a(\tilde{I}_m^+(c)) \subset \tilde{I}_{m+\ell}^+(a) = \tilde{I}_{m+\ell}^+(ac).$$

Thus, $k \cdot \widetilde{\mathrm{rot}}(\widehat{a}\widehat{c}) = m + \ell - 1 = k \cdot \widetilde{\mathrm{rot}}(\widehat{a}) + k \cdot \widetilde{\mathrm{rot}}(\widehat{c}) - 1$ and hence $k(\widetilde{\mathrm{rot}}(\widehat{a}\widehat{c}) - \widetilde{\mathrm{rot}}(\widehat{a}) - \widetilde{\mathrm{rot}}(\widehat{c})) = -1$, as desired.

This implies Theorem 3.19, as we can cut the surface Σ_g into pairs of pants whose boundary components are all non-separating.

Now, if P is a pair of pants with possibly more than one separating boundary component, then $\Sigma_g \smallsetminus P$ admits a pants decomposition whose pants all have at most one separating boundary component. The fact that the contribution of P to the Euler class of ρ is $\frac{-1}{k}$ is then a consequence of Theorem 3.19 and the additivity of the Euler class. □

In fact, 3.17 implies an even stronger statement:

Proposition 3.22. *Let $i(a, b) = \pm 1$. The restriction of ρ to $\langle a, b\rangle$ is semi-conjugate to the restriction of a geometric representation in* $\mathrm{PSL}^k(2, \mathbb{R})$.

Proof. The fact that ρ is k-Schottky allows one to apply the classical ping-pong lemma to the action of a and b. A careful reading of the classical proof of the ping-pong lemma not only shows that some point $x \in S^1$ has a free orbit under a and b, but that the cyclic order of x is determined by the cyclic order of the domains $I \pm (a)$ and $I \pm (b)$, and their images under a and b; this determines the action up to semi-conjugacy. A detailed proof is written out in Lemma 4.2 in [16]. Alternatively, this can be proved by Matsumoto's theory of Basic Partitions, as in Example 3.31 and the comments following Example 3.32 below. □

3.4. **Surfaces bounded by separating curves.** Fix a standard basis $\{a_i, b_i\}$ for Γ_g, and let $c_i = [a_i, b_i]$. By considering the one-holed tori $T(a_i, b_i)$ in Theorem 3.20, we know that, up to reversing the orientation of the circle, $\mathrm{rot}(c_i) = -1/k$ for all i. We fix this orientation, for the rest of this section. We now discuss the cyclic order of their periodic points, and the images $c_i(\mathrm{Per}(c_j))$. This, combined with the fact that a_i and b_i are k-Schottky, will furnish enough combinatorial data to conclude (in the next subsection) that ρ is geometric.

Recall from the previous section that, up to reindexing the basis, we have the cyclic order of periodic points

$$\left(\!\left(a_1^-, b_1^+, a_1^+, b_1^-, a_2^-, b_2^+, a_2^+, b_2^-, \ldots\right)\!\right)_k$$

and that, for each i, the restriction of ρ to $\langle a_i, b_i\rangle$ is semi-conjugate to a $\mathrm{PSL}^k(2,\mathbb{R})$ geometric representation. Let c_i^+ denote points in the set of *closest points* of $\mathrm{Per}(c_i)$ to $\mathrm{Per}^-(b_i)$, and let c_i^- denote points in the set of closest points of $\mathrm{Per}(c_i)$ to $\mathrm{Per}^-(a_i)$. Both of these two "closest points" sets have cardinality k. We are slightly abusing notation here, because the points c_i^+ and c_i^- need not be attracting or repelling points. But this abuse will be justified shortly.

Using our notation for cyclic order of points, the fact that the restriction of ρ to $\langle a_i, b_i\rangle$ is semi-conjugate to a geometric representation implies that these points are in the cyclic order $\left(\!\left(c_i^-, a_i^-, b_i^+, a_i^+, b_i^-, c_i^+\right)\!\right)_k$ for each i. We now make a stronger claim.

Lemma 3.23. *The periodic points of a_i, b_i and c_i, with the notation above, are in the cyclic order*

$$\left(\!\left(c_1^-, a_1^-, b_1^+, a_1^+, b_1^-, c_1^+, c_2^- a_2^-, b_2^+, a_2^+, b_2^-, c_2^+, \ldots\right)\!\right)_k,$$

with the possibility that $c_i^+ = c_{i+1}^-$ (taking indices mod k) for some of the points c_i^+ and c_{i+1}^- (not necessarily respecting the repetition of the pattern k times).

Proof. Since c_i^+ and c_i^- are in $\partial\mathrm{Per}(c_i)$, it suffices to prove the following.

Claim. Let I be an interval of type (c_i^-, c_i^+), and let $j \neq i$. Then $\partial\mathrm{Per}(c_j) \cap I = \varnothing$.

Suppose for contradiction that this claim is false, and let $x \in \partial\mathrm{Per}(c_j) \cap I$. Let c_t be a positive one parameter family commuting with c_i and supported on I. Apply a bending deformation in the separating simple closed curve c_i, so that $\rho_t(a_i) = c_t\rho(a_i)c_t^{-1}$. Since this deformation is a conjugacy on $\langle a_i, b_i\rangle$, we have $|\mathrm{Per}\rho_t(a_i)| = 2k$ for all t, and there is exactly one point of $\mathrm{Per}^+(\rho_t(a_i))$ in each interval of type $(\rho_t(b_i)^+, \rho_t(b_i)^-)$, which moves continuously through the interval I. Now for some t, we have $x \in \mathrm{Per}^+(\rho_t(a_i)) \cap \mathrm{Per}(\rho_t(c_j))$. Since ρ was assumed minimal, the cardinality of $\mathrm{Per}^+(\rho_t(a_i)) \smallsetminus \mathrm{Per}(\rho_t(c_j))$ can only increase, hence must have been constant. Since $\{\rho_t(x) \mid t \in \mathbb{R}\} = I$, this implies that $I \subset \mathrm{Per}(\rho_t(c_j)) = \mathrm{Per}(\rho(c_j))$, a contradiction. This proves the claim, and hence the lemma. $\square$

In order to eliminate the case that some $c_i^+ = c_{i+1}^-$ and for future work, we use a result of [14].

Lemma 3.24.
1. *Any two points of the form c_i^+ and c_j^- are distinct.*
2. *We have $\widetilde{\mathrm{rot}}(\check{c}_i ... \check{c}_2 \check{c}_1) = \frac{-(2i-1)}{k}$ for all $i = 1, ..., g-1$.*
3. *Each point $c_i(c_{i-1}^+)$ lies in an interval of type (c_i^+, c_i^-), and within this interval, we have $c_i^+ < c_i(c_{i-1}^+) < c_{i+1}^+ < c_i^-$.*

In the statement above, the lifts $\check{c}_i$ to $\mathrm{Homeo}_{\mathbb{Z}}^+(\mathbb{R})$ are the "canonical lifts" of c_i, in the sense that regardless of the choices of lifts $\overline{a_i}$, $\overline{b_i}$ we have $[\overline{a_i}, \overline{b_i}] = \check{c}_i$; this holds because the participation of $T(a_i, b_i)$ to the Euler class of ρ is $\frac{-1}{k}$. Note that this is different than the canonical lifts defined in Notation 2.16.

The proof of Lemma 3.24 is accomplished by tracking the orbit of points under initial strings of the word $(\check{c}_i \cdots \check{c}_2 \check{c}_1)^k$. In essence, the constraint on the Euler number of ρ (i.e. the translation number of $\check{c}_g \cdots \check{c}_2 \check{c}_1$) combined with the cyclic order of the periodic points of the c_i places strong constraints on the rotation numbers and images of points under initial subwords of $\check{c}_g \cdots \check{c}_2 \check{c}_1$. However, since this computation is already carried out exactly in previous work of one of the authors, we will save time here by just quoting the relevant results. A very similar argument is given, with complete details, in the proof of Lemma 3.26 later on.

Proof of Lemma 3.24. Note that, for each i, the points c_i^+ are all in a single orbit, because of Proposition 3.22. The same is true for the c_i^-. Since we know the Euler number of ρ is $\frac{-(2g-2)}{k}$, and this is equal to $\widetilde{\mathrm{rot}}(\bar{c}_{g-1} \bar{c}_g ... \bar{c}_1) - 1/k$, Proposition 4.6 in [14] applied to the sets of lifts of periodic points $\tilde{c}_i^+$ gives the second and third part of the statement – the fact that $c_i(c_{i-1}^+)$ lies in an interval of type (c_i^+, c_i^-) is immediate, and the rest of (2) is a direct translation of Proposition 4.6 into our notation. With regards to notation, we warn the reader that, although Proposition 4.6 also uses c_i to denote homeomorphisms, rotation numbers there are assumed *positive.* However, the indexing and composition works out by using $\tilde{c}_i$ there to stand for our $(\hat{c}_i)^{-1}$.)

The first part now follows immediately from Corollary 4.8 of [14]. □

Given the lemma above, one can use a further result of [14] to understand the location of all periodic points of the c_i.

Corollary 3.25 (Cor. 4.8 of [14])**.** *Keeping the notation above, if x_i denotes* any *periodic orbit of c_i, then these points are in cyclic order*

$$\langle\!\langle x_1, x_2, ..., x_g \rangle\!\rangle_k .$$

In particular, $\mathrm{Per}(c_i) \cap \mathrm{Per}(c_j) = \varnothing$ *for all $i \neq j$; and these periodic sets satisfy the property that each connected component of $S^1 \smallsetminus \mathrm{Per}(c_i)$ either contains a point of the form c_j^+, or does not intersect* $\mathrm{Per}(c_j)$.

Proof. The statement that periodic points of $c_1, c_2, ..., c_{g-1}$ are in cyclic order

$$\langle\!\langle x_1, x_2, ..., x_{g-1} \rangle\!\rangle_k$$

is exactly the statement of Cor. 4.8 of [14]. Applying this again to $c_3, ...c_g, c_1$ instead gives the statement above. □

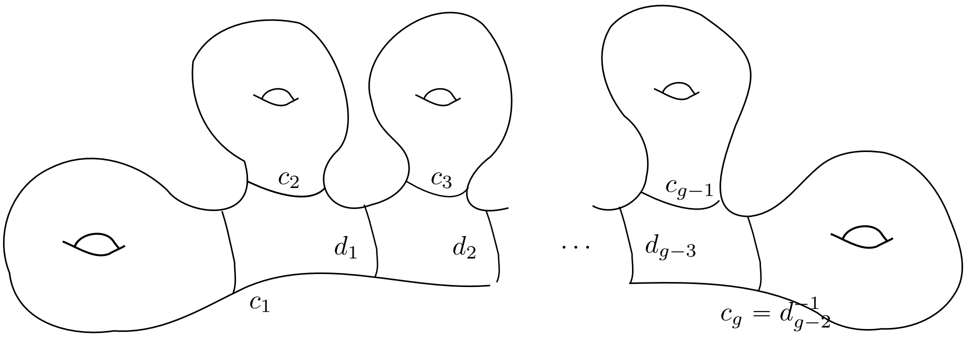

FIGURE 4. A flute decomposition of Σ_g in pairs of pants and tori

These lemmas (mostly) justify our use of notation $c_i^{\pm}$, as it is easy to compute from the location of periodic points of a_i and b_i that $c_i^{q(c_i)}$ is increasing on the intervals of type (c_i^-, c_i^+).

To summarize, Corollary 3.25 says that if one looks only at the periodic points of the curves a_i, b_i and c_i defined above, then ρ is indistinguishable from a representation semi-conjugate to a geometric one. Our next goal is to understand the dynamics of other separating curves. Keeping the indexing as above, we decompose Σ_g into the tori $T(a_i, b_i)$ and $g-2$ pants, by adding the curves $d_1 := c_2c_1$ and $d_j := c_{j+1}d_{j-1}$ for $i = 2, ..., g-2$. Note that $d_{g-2} = c_g^{-1}$, and for convenience we set the notation $d_0 := c_1$. These curves are illustrated in Figure 4 – for simplicity we have drawn unbased, unoriented curves in their free homotopy classes, but as always the basepoint and orientation is important.

Corollary 3.25 tells us the cyclic order of points $\mathrm{Per}(\check{c_j})$. In particular, $S^1 \smallsetminus \bigcup_{i\neq j} \mathrm{Per}(c_i)$ has exactly k connected components that contain points of $\mathrm{Per}(c_j)$. Enumerate these connected components in cyclic order and let $J_m(c_j)$ denote the closed interval between the leftmost and rightmost point of $\mathrm{Per}(c_j)$ in the m^{th} component, for $m = 1, 2, ..., k$. Note that the union of left endpoints of the $J_m(c_j)$, for fixed j, lie in a single orbit because of Corollary 3.25, and the same is true of the union of right endpoints. See Figure 5 for a schematic illustration of $J(c_1)$ and $J(c_2)$ in the case $k = 1$.

Abusing notation slightly, for $m \in \mathbb{Z}$ let $J_m(c_j)$ denote the lift of the previously defined intervals to $\mathbb{R}$, and reindex if needed so that these are in linear order

$$...J_m(c_1), J_m(c_2), ..., J_m(c_g), J_{m+1}(c_1), J_{m+1}(c_2), ...$$

With this notation, we have $\check{c_j}(J_m(c_j)) = J_{m-1}(c_j)$.

Lemma 3.26. *Let $x \in \mathbb{R}$ be greater than the leftmost endpoint of $J_i(c_1)$, and let $1 < n < g$. Then $(\check{c_n} \cdots \check{c_2}\check{c_1})^k(x)$ is greater than the rightmost endpoint of $J_{i-k(2n-1)}(c_n)$. In particular, for all x in the convex hull of $J_i(c_1) \cup J_i(c_n)$, we have $(\check{c_n} \cdots \check{c_2}\check{c_1})^k(x) > x - (2n-1)$.*

In this and later proofs, for sets $A, B \subset \mathbb{R}$, we use the notation $A > B$ to mean that $a > b$ holds for all $a \in A$ and $b \in B$. If $A = \{a\}$, then we write $a > B$.

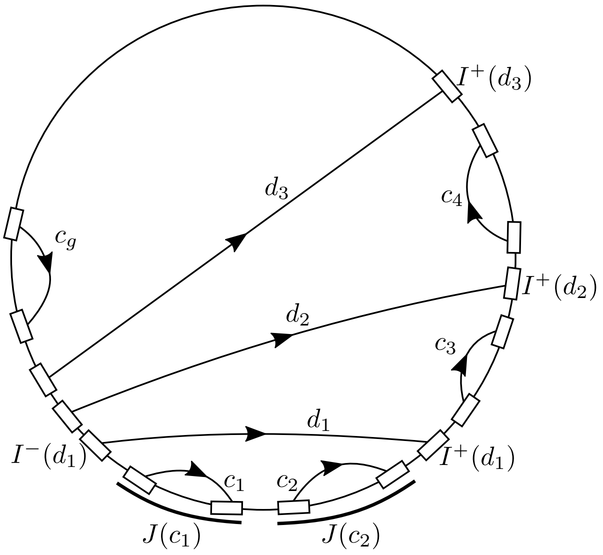

FIGURE 5. Sorting periodic points

Proof. Let x be greater than the leftmost endpoint of $J_i(c_1)$. Then $\check{c}_1(x)$ is greater than the leftmost endpoint of $J_{i-1}(c_1)$. Since $J_{i-1}(c_1) > J_{i-2}(c_2)$, we have $\check{c}_1(x) > J_{i-2}(c_2)$, hence $\check{c}_2\check{c}_1(x) > J_{i-3}(c_2)$. Repeating this argument, we have that $\check{c}_n \cdots \check{c}_2\check{c}_1(x) > J_{i-(2n-1)}(c_n) > J_{i-(2n-1)}(c_1)$. Now apply this k times, and conclude that $(\check{c}_n \cdots \check{c}_2\check{c}_1)^k(x) > J_{i-k(2n-1)}(c_n)$. ☐

From now on, we denote by $\overline{d_i} = \check{c}_i \cdots \check{c}_1$ the "prefered" lift of d_i to $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$.

Lemma 3.27. *For each $i = 1, 2, ...g-3$, all the periodic points of $\mathrm{Per}(d_i)$ are contained in $2k$ segments. Lifting these to $\mathbb{R}$, and denoting the segments by $I^-_m(d_i)$ and $I^+_m(d_i)$, they occur in the order*

$$J_{m-1}(c_g) < I^-_m(d_{g-3}) < \cdots < I^-_m(d_2) < \ldots < I^-_m(d_1) < J_m(c_1) \text{ and}$$
$$J_m(c_{i+1}) < I^+_m(d_i) < J_m(c_{i+2}) \text{ for all } i.$$

In brief, Lemma 3.27 says that the sets of periodic points of c_i, a_i, b_i and d_i are indistinguishable from those of a geometric representation. Figure 5 gives a cartoon of this in the case $k = 1$. Inside the boxes, the dynamics of c_i and d_i are not known. However, outside, the homeomorphisms behave like hyperbolic elements of $\mathrm{PSL}(2, \mathbb{R})$ with axes as indicated in the figure. The general case $k > 1$ is obtained by thinking of a lift of these dynamics to the k-fold cover of the circle, and composing with appropriate rotations.

Proof. We start with the base case of $d_1 = c_2c_1$. Lemma 3.26 above says that $\overline{d_1}^k(x) > x - 3$ on the convex hull of $J_m(c_1) \cup J_m(c_2)$, for all m. In particular, since $\widetilde{\mathrm{rot}}(\overline{d_1}) = \frac{-3}{k}$, it has no periodic points there. Now we perform a similar computation for the other half of the statement. Suppose that x is less than or equal to the right endpoint of $J_m(c_g)$. Then we have $\check{c}_2\check{c}_1(x) < J_{m-3}(c_3) < J_{m-3}(c_g)$. Hence $\overline{d_1}^k(x) < J_{m-3k}(c_3)$. This implies in particular that d_1 has no periodic points in the convex hull of $J_m(c_3) \cup J_m(c_g)$, for any m; this shows that the periodic set of d_1 has indeed the configuration as in Figure 5, and as stated in the Lemma.

We now repeat the argument of Lemma 3.26 using the collection of homeomorphisms $d_1, c_3, c_4, ...c_g$ instead of $c_1, c_2, c_3...c_g$. To do this, let $J_m(d_1)$ denote the intervals bounded by the leftmost and rightmost endpoints of $\mathrm{Per}(d_2)$ between consecutive points of $\mathrm{Per}(c_g)$ and $\mathrm{Per}(c_3)$. It follows from the dynamics of $d_1 = c_2c_1$ above that $\overline{d_1}(J_m(d_1)) = J_{m-3}(d_1)$. This can be fed it into the computation from Lemma 3.26 to conclude that $\overline{d_2}^k(x) > x-5$ holds on the convex hull of $J_m(d_1) \cup J_m(c_3)$ (hence d_m has no periodic points there) and the computation from the base case can be repeated to show $\breve{c}_3\overline{d_1}^k(x) = \overline{d_2}^k(x) < J_{m-3k}(c_4)$ for all x less than the right endpoint of $J_m(c_g)$. This process can be iterated; defining $J_m(d_i)$ and repeating first the calculation from Lemma 3.26, then the computation from the base case, for the homeomorphisms $d_i, c_{i+2}, ...c_g$ each time. The result is a description of periodic sets of the d_i in cyclic order exactly as in the statement of the Lemma. ☐

Finally, we discuss the images of periodic points of the d_i.

Lemma 3.28. *Keeping the notation from before, let $I^{\pm}_m$ denote the minimal intervals (possibly singletons) satisfying the conclusion of the Lemma above. Then we have, for all $m \in \mathbb{Z}$,*

$$I^+_{m-(2i+1)}(d_i) < \overline{d_i}(I^{\pm}_m(d_{i+1}) < J_{m-(2i+1)}(c_{i+2}).$$

Proof. The first inequality is immediate, given that $\widetilde{\mathrm{rot}}(\overline{d_i}) = \frac{-(2i+1)}{k}$. The second inequality can be deduced from the locations of periodic points, as follows. Assume for simplicity that $m = 0$. If instead there is some $x \in I^{\pm}_0(d_{i+1})$ and $y \in J_{-(2i+1)}(c_i)$ with $\overline{d_i}(x) > y$ then using the fact that $\widetilde{\mathrm{rot}}(\widetilde{c_{i+2}}) = -1/k$ and that $J(c_{i+2})$ is c_{i+2}-invariant, we have $\overline{d_{i+1}}(x) = \widetilde{c_{i+2}}\overline{d_i}(x) > J_{-2i-3}(c_i)$. Since $J_{-2i-3}(c_i) > I^-_{-2i-3}(d_{i+1})$, we conclude that $\overline{d_{i+1}}^k(x) > I^-_{k(-2i-3)}(d_{i+1})$. However, since $I^-_0(d_{i+1})$ was assumed minimal, and $I^+_0(d_{i+1}) < I^-_0(d_{i+1})$, there is some point $x' > x$ in $I^-_0(d_{i+1})$ that is periodic for d_{i+1}, i.e $\overline{d_{i+1}}^k(x') = x' + (-2i-3) \in I^-_{k(-2i-3)}(d_{i+1})$. But this contradicts the inequality above. ☐

3.5. **Basic partitions and combinations.** We keep the decomposition of the surface and the notation c_i and d_i from the previous section. Our next goal is to show that the cyclic order of periodic points and basic dynamics of the curves a_i, b_i, c_i and d_i established so far is sufficient information to determine the dynamics of ρ, namely, that it is geometric. For this, we apply tools developed by Matsumoto that allow one to recover a geometric representation from this kind of essentially combinatorial data. The first of these tools is a *Basic Partition.*

Definition 3.29 (Matsumoto [21])**.** Let G be a group generated by a finite symmetric set S, and let $\rho : G \to \mathrm{Homeo}^+(S^1)$. A *Basic Partition (BP)* for $\rho(G)$ is a collection P of disjoint closed intervals of S^1 satisfying

i) for each $I \in P$, there is a unique $s_I \in S$ such that $\rho(s_I)(I)$ is a union of $m = m(I)$ elements of P and $m - 1$ complementary intervals to P,

ii) for any $s \neq s_I$ in S, the image $\rho(s)(I)$ is a proper subset of an element of P, and
iii) for any complementary interval J to P and $s \in S$, either $\rho(s)(I)$ is contained in the interior of P, or is a complementary interval to P.

These properties of a BP (reminiscent of a Markov partition) imply that the cyclic order of the images of the endpoints of intervals in the BP under the elements $s \in S$ is determined by the cyclic order of the intervals I, and the intervals containing or comprising their images under the s_t. This observation, together with an inductive argument on word length, allows Matsumoto to prove the following.

Theorem 3.30 (4.7 in [21]). *Let* ρ_1 *and* $\rho_2 \in \mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$. *Suppose that* ρ_1 *has a basic partition* P. *If there exists* $\xi \in \mathrm{Homeo}^+(S^1)$ *such that* $\{\xi(I) : I \in P\}$ *is a BP for* ρ_2 *respecting the group action (i.e. with* $s_I = s_{\xi(I)}$ *and* $s_I(x) \in J \Leftrightarrow s_{\xi(I)}\xi(x) \in J$ *for any endpoint of a BP interval* x*), then* ρ_1 *and* ρ_2 *are semi-conjugate.*

The following is an essential and elementary example of a BP.

Example 3.31 (Basic Partition for one-holed torus groups). Let $G = \langle a, b\rangle$, and suppose that a and b are two hyperbolic elements of $\mathrm{PSL}(2,\mathbb{R})$ in 1-Schottky position, with fixed points in cyclic order a^-, b^+, a^+, b^-. Let x and y be the repelling and attracting fixed points o $[a,b]$, respectively. Then the points

$$y, x, b(y), b(x), ab(y), ba(x), a(y), a(x)$$

are in cyclic order, and the intervals

$$[x, b(y)],\ [b(x), ab(y)],\ [ba(x), a(y)],\ [a(x), y]$$

are a BP for $\rho(T)$. This is illustrated in Figure 6 (left).

Checking that the BP conditions hold in this example is easy and left to the reader. See [21, Ex 4.2].

Motivated by this, we call such a BP the *standard geometric BP* for a one-holed torus group. Similarly, we have

Example 3.32 (Basic Partition for pants subgroups). Let $G = \langle a, b\rangle$, and suppose that a and b are 1-Schottky, with domains in cyclic order $I^+(a)$, $I^-(b)$, $I^+(b)$, $I^-(a)$. Then there is a BP for a and b as illustrated in Figure 6.

It is a straightforward exercise to show that, if ρ is an action of G on S^1 that lifts to an action $\hat{\rho}$ on a k-fold cover of S^1, then the pre-image of a BP for ρ is a BP for $\hat{\rho}$ (see Lemma 4.8 in [21]). In particular, one can easily check that if a and b are k-Schottky with components of $I^{\pm}(a)$ and $I^{\pm}(b)$ alternating, then they have a BP which is a lift of the one in Example 3.31. Similarly, if instead, a and b are k-Schottky with cyclic order of intervals $(\!(I^+(a), I^-(b), I^+(b), I^-(a))\!)_k$, then they have a BP that is a lift of the partition in Example 3.32.

So far, our examples of BP's lend themselves to actions of free groups. To pass to amalgamated products, Matsumoto defines a *Basic Configuration* or *BC*. To define a *Basic Configuration (BC)*, one starts with a group G, a homomorphism $\rho : G \to \mathrm{Homeo}^+(S^1)$, and a decomposition of G into a

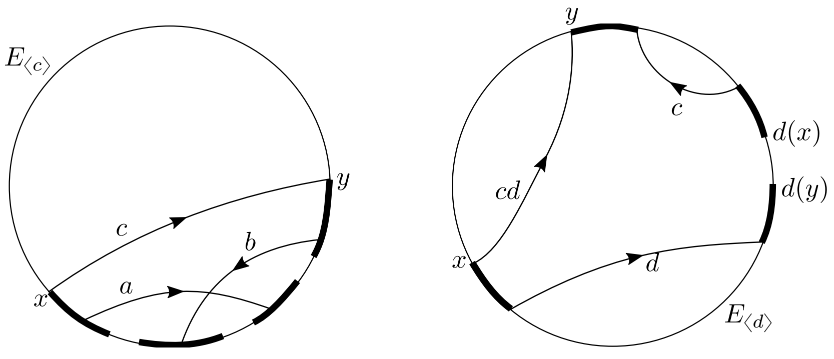

Figure 6. Two examples of Basic Partitions and "entrances" for cyclic subgroups

tree of groups such that the edge groups are all infinite cyclic. With this data, a BC consists of a set of Basic Partitions with disjoint interiors, one for each vertex group. For each edge group $G_e \cong \langle g\rangle$ adjacent to a vertex group G_v, the *entrance* $E_{\langle g\rangle}$ to the BP of v for e is defined to be the set of complementary intervals whose stabilizers are nontrivial subgroups of the cyclic group G_e. If v and v' are adjacent to e, then we require that their entrances for e have disjoint interiors, and union equal to S^1. This, together with a few other minor technical conditions (for brevity, we do not give the full list here, but refer the reader to [21, Assumption 6.3]), ensures that the vertex-group BP's satisfy a "combination theorem" reminiscent of the Klein–Maskit combination theorem, for any two vertex groups that share an edge.

The result is that the combinatorial data of a BC determines the cyclic order of the orbit of a point under $\rho(G)$, and hence determines the semi-conjugacy class of the group action. More precisely, given a decomposition of G as a tree of groups and ρ_1, $\rho_2 \in \mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$, Matsumoto shows the following.

Theorem 3.33 (Matsumoto). *Suppose that ρ_1 admits a BC, with P_V denoting the corresponding BP for each vertex group $V \subset G$. If there exists $\xi \in \mathrm{Homeo}^+(S^1)$ such that, for each V, the image $\xi(P_V)$ is a BP for $\rho_2(V)$ respecting the group action (as in Theorem 3.30), then ρ_1 and ρ_2 are semi-conjugate.*

The key example of a BC is that which comes from the decomposition of a surface group into a tree of groups corresponding to a decomposition of the surface into pants and one-holed-torus subsurfaces, exactly as in Figure 4. Using this decomposition, Matsumoto shows that the standard Fuchsian action of Γ_g, and hence any lift of this action to a finite cover of the circle admits a Basic Configuration. Following our previous terminology, we call these *standard geometric BCs*. Thus, to finish the proof of Theorem 3.3, we need to show that for our (path-rigid, minimal, S_k) action ρ admits a standard geometric BC.

End of Proof of Theorem 3.3. First consider the restriction of ρ to a vertex subgroup of the form $\langle a_i, b_i\rangle$. By proposition 3.17, these are k-Schottky, hence semi-conjugate to the $\mathrm{PSL}^k(2,\mathbb{R})$ geometric action of a one-holed torus subgroup. Thus, they admit a BP that agrees with a k-fold lift of the standard geometric BP for a one-holed torus group. If these are chosen as in Example 3.31, then Lemma 3.27 implies that the BP's for distinct vertex subgroups are pairwise disjoint.

For the vertex subgroups corresponding to pants subsurfaces, Lemmas 3.27 and 3.28 imply that these admit a BP that agrees with a k-fold lift of the standard geometric BP for a pair of pants, which can be chosen as in Example 3.32. Indeed, this was the purpose of proving both of these lemmas in the previous section! The configuration of periodic points in Lemma 3.27 also implies that these can be chosen pairwise disjoint, and disjoint from the partitions for the one-holed torus vertex groups.

Since all of these BP's agree with k-fold lifts of the standard geometric BPs, they satisfy Matsumoto's additional hypothesis on stabilizers making them "pure" BPs. Moreover, if V_1 and V_2 are the vertex groups for some edge with edge group c_i or d_j, then Lemma 3.27 implies that the BP's can be chosen so that the entrances for this edge group have disjoint interiors, and union equal to S^1. There is actually some flexibility in choice of intervals in the BP here since the "blocks" containing periodic points of the edge group generators – the white boxes in Figure 5 – can be included either in BP intervals or in entrances. However, many possible choices satisfy Matsumoto's combination conditions, for instance, one can choose the complements of $J(c_1)$ and $J(c_2)$ to be the entrances for c_1 and c_2 in the BPs for $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ respectively; this determines the choice of intervals in the BP for the pants subgroup bordering $T(a_1, b_1)$ and $T(a_2, b_2)$, and one can proceed iteratively from there. Were the white boxes all single points, the choice would be canonical.

Chosen this way, these BPs form a *combinable pair* as in [21, Def 5.2], with the same combinatorial data as a k-fold lift of a Fuchsian representation. With this input, all the conditions of being a BC are satisfied, and Theorem 3.33 now says that ρ is semi-conjugate to a k-fold lift of a Fuchsian representation. □

4. Periodic considerations

As explained in the introduction, the content of this section is the proof of the following two statements.

Proposition 4.1. *If a representation* $\pi_1\Sigma_g \to G$ *is path-rigid then all non separating simple closed curves have rational rotation number.*

Theorem 4.2. *Suppose* ρ *is path-rigid and minimal. Then, for all* a, b *with* $i(a,b) = \pm 1$*, we have the implication*

$$\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing \Rightarrow S_k(a,b) \text{ for some } k.$$

4.1. **No irrational simple closed curve.**

We begin with the proof of Proposition 4.1. For $f \in \mathrm{Homeo}^+(S^1)$ with $\mathrm{rot}(f) \notin \mathbb{Q}$, we denote its minimal set by $K(f)$.

Lemma 4.3. *Let* $\rho\colon \langle a, b\rangle \to \mathrm{Homeo}^+(S^1)$ *be a representation such that* $\mathrm{rot}(\rho(a)) \notin \mathbb{Q}$*. Then at least one of these holds:*

(1) $K(\rho(a))$ *is* $\rho(b)$*-invariant*

(2) there exists a continuous deformation ρ_t of ρ, with $\rho_t([a,b]) = \rho([a,b])$ for all t, and $\gamma \in \langle a, b \rangle$ such that $\mathrm{rot}(\rho_t(\gamma))$ *is non-constant.*

Proof. If $\rho(a)$ is conjugate to a rotation (i.e. if $K(\rho(a)) = S^1$), then it lies in a one-parameter group of rotations, a_t, and in this case we may perform a bending deformation replacing $\rho(b)$ with $a_t\rho(b)$, which has nonconstant rotation number.

Thus, we now suppose that $K = K(\rho(a))$ is a Cantor set, and suppose that $\rho(b)(K) \neq K$. We will treat the case that $\rho(b)(K) \not\subset K$, and leave the case $\rho(b)^{-1}(K) \not\subset K$, symmetric, to the reader. In this case, we will first find $N \in \mathbb{Z}$ such that $\mathrm{rot}(\rho(a^N b)) = 0$, and then define the deformation of ρ.

Let $K' \subset K$ be the set of two-sided accumulation points of K. Since $\overline{K'} = K$, there exists $x \in K'$ such that $\rho(b)(x) \notin K$. Let I be the connected component of $S^1 \smallsetminus K$ containing $\rho(b)(x)$. Then $\rho(b)^{-1}(I)$ is a neighborhood of x, and since K is the minimal closed set invariant by $\rho(a)$, there exists a power $N \in \mathbb{Z}$ such that $\rho(a)^N(I) \subset \rho(b)^{-1}(I)$. It follows that $\rho(a)^N(I)$ is an attracting interval for $\rho(a^N b)$, hence $\mathrm{rot}(\rho(a^N b)) = 0$.

Now let β_t be a positive one-parameter family commuting with $\rho(a^N b)$. We claim that a bending deformation along $a^N b$ will change the rotation number of a. To see this, let J be a connected component of $S^1 \smallsetminus \rho(a^N b)$. Since $\rho(a^N b)$ does not preserve K, we can choose such a J that intersects K', and find m such that $\rho(a)^m(J) \cap J \neq \varnothing$. Let $\tilde{x} \in \mathbb{R}$ be a lift of a point in $\rho(a)^m(J) \cap J$.

Adapting the notation from Section 2.2.3 we have

$$\Delta(\tilde{x}, t_1, \dots, t_M) = \widehat{\beta_{t_M}}\widehat{\rho(a)} \circ \cdots \circ \widehat{\beta_{t_1}}\widehat{\rho(a)}(\tilde{x}) - \tilde{x} - k,$$

and $\delta(\tilde{x}, t) = \Delta(\tilde{x}, t, \dots, t)$. Up to changing orientation, we can suppose that $\delta(\tilde{x}, 0) > 0$. Since the open interval $\tilde{J}$ contains both $\tilde{x}$ and $\widehat{\rho(a)}^M(\tilde{x})$, there exists $t < 0$ such that $\Delta(\tilde{x}, 0, \dots, 0, t) < 0$, hence $\delta(\tilde{x}, t) < 0$. Thus, there exists t_0 such that $\delta(\tilde{x}, t_0) = 0$, hence $\mathrm{rot}(\rho_{t_0}(a)) = \frac{k}{M} \in \mathbb{Q}$. □

Now we can prove the proposition.

Proof of Proposition 4.1. By contradiction, suppose that there exists some non-separating simple curve a with $\rho(a) \notin \mathbb{Q}$. After semi-conjugacy, we may assume that ρ is minimal. By path-rigidity and Lemma 4.3, $\rho(a)$ cannot be a rotation, so $K(\rho(a))$ is a genuine Cantor set. Also, for all b with $|i(a,b)| = 1$, the set $K(\rho(a))$ is $\rho(b)$-invariant. But $\pi_1\Sigma_g$ is generated by such curves b (indeed, if $(a_1, b_1, \dots, a_g, b_g)$ is a standard generating set, and if $a = a_1$, consider the new generating set $(a_1b_1, b_1, a_2b_1^{-1}, b_1b_2, \dots, a_gb_1^{-1}, b_1b_g))$. It follows that $K(\rho(a))$ is invariant by the whole action, contradicting minimality. □

4.2. **Proof of Theorem 4.2.** We assume now that ρ is path-rigid, and $i(a,b) = \pm 1$. We will first establish some properties that do not use minimality, thus are robust under deformations of ρ. We will assume that ρ is minimal only in the end of the proof. By the Proposition 4.1, $\mathrm{Per}(a)$ and $\mathrm{Per}(b)$ are nonempty, and we assume for the rest of this section that $\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing$. As before, for simplicity, we drop the notation ρ.

The strategy of the proof is to first show that a and b *look like* they are semi-conjugate to elements satisfying $S_k(a,b)$, namely, if one takes each interval in $S^1 \smallsetminus \mathrm{Per}(a)$ that does not contain a point of $\mathrm{Per}(b)$ and collapses its closure to a point (and then repeats with the roles of a and b reversed), the remaining periodic points are attracting and repelling and alternate around the circle. We will then show that this collapse can be realized by a deformation of ρ, and we attain $S_k(a,b)$ if ρ is minimal. Much of this work will depend on the observation that the hypothesis that $\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing$ is invariant under deformations of the action.

We begin by discussing some basic combinatorics of periodic sets. Borrowing notation from the previous section, say that a connected component of $S^1 \smallsetminus (\mathrm{Per}(a) \cup \mathrm{Per}(b))$ is of type (x,y) if, with its natural orientation from the circle, it is bounded to the left by a point of $\mathrm{Per}(x)$ and to the right by a point of $\mathrm{Per}(y)$, for $x, y \in \{a,b\}$. Note that each (a,b) interval is followed by a collection (perhaps empty) of (b,b) intervals, and then a (b,a) interval. As the sets $\mathrm{Per}(a)$ and $\mathrm{Per}(b)$ are closed and disjoint, this implies that there exists an integer $m = m(\rho) \geqslant 1$ such that S^1 contains exactly m intervals of type (a,b) and m intervals of type (b,a), which alternate around the circle. By Remark 2.31, the integer m depends only on the semi-conjugacy class of ρ.

Definition 4.4. Let $X_a := \{I \text{ connected component of } S^1 \smallsetminus \mathrm{Per}(a) : I \cap \mathrm{Per}(b) \neq \varnothing\}$, and say that an interval I in X_a is *positive* if $a^{q(a)}$ is increasing on I, and *negative* otherwise.

Likewise, we define X_b, and its positive and negative intervals, by reversing the roles of a and b in the definition. Since the leftmost points of intervals in X_a are exactly the leftmost points of the (a,b) intervals, we have $|X_a| = m$.

Lemma 4.5. *The set X_a is $\rho(a)$-invariant, and the subset of positive (respectively, negative) intervals in X_a is also $\rho(a)$-invariant.*

Proof. Let $I \in X_a$ be a positive interval; we will use path-rigidity show that its image under a is another positive interval in X_a. The negative case is completely analogous.

For all $n \in \mathbb{Z}$, $a^n(I)$ is an interval between two consecutive periodic points of a, and on which $a^{q(a)}$ is increasing. Thus, for all n, $a^n(I)$ cannot be one of the negative intervals in X_a, and we need only show that $a(I) \in X_a$.

Suppose for contradiction that $a(I) \notin X_a$, i.e. that $a(I) \cap \mathrm{Per}(b) = \varnothing$. Then $a(\overline{I}) \subset J$ for some $J \in X_b$, by definition of X_a and X_b. We will use a bending deformation in b to produce a common periodic point for a and b, giving a contradiction.

Let b_t be a positive one-parameter family commuting with b, let $x \in I \cap \mathrm{Per}(b)$, and take lifts $\tilde{x} \in \tilde{I}$ of x and I to $\mathbb{R}$. We have $\delta_{b,a}(\tilde{x}, 0) > 0$, since I is a positive interval in X_a. If $t < 0$ is negative enough that $b_t(I) \cap I = \varnothing$, then we have $\widehat{b_t}(\hat{a}(\tilde{x})) < \hat{a}(\tilde{I})$ and hence $\hat{a}^{q(a)-1}(\widehat{b_t}(\hat{a}(\tilde{x}))) < \tilde{I} + k$. It follows that $\Delta_{b,a}(\tilde{x}, t, 0, \ldots, 0) < 0$, so $\delta_{b,a}(\tilde{x}, t) < 0$. In particular, there exists $t_0 \in \mathbb{R}$ such that $\delta_{b,a}(x, t_0) = 0$, i.e. $x \in \mathrm{Per}(b_{t_0}a) \cap \mathrm{Per}(b)$; this contradicts path-rigidity via Lemma 2.30. ☐

Obviously, reversing the roles of a and b above shows the positive and negative intervals of X_b are b-invariant.

While Remark 2.31 (and path-rigidity of ρ) showed that the cardinality of $S(a)$ and $S(b)$ are constant under deformations, the next lemma shows that the sets themselves are constant under particular bending deformations.

Lemma 4.6. *Let b_t be a positive one-parameter family commuting with b. For all $t \in \mathbb{R}$, let $X_b(t)$ denote the set of connected components I of $S^1 \smallsetminus \mathrm{Per}(b)$ such that $I \cap \mathrm{Per}(b_t a) \neq \varnothing$. Then $X_b(t) = X_b(0)$ for all t.*

Proof. Let $X_b(t)$ be as in the statement of the lemma, and let $X_a(t)$ denote the set of connected components of $S^1 \smallsetminus \mathrm{Per}(b_t a)$ containing points of $\mathrm{Per}(b)$, for $t \in \mathbb{R}$.

Let $K_a = \{(x,t) \in S^1 \times \mathbb{R} \mid x \in \mathrm{Per}(b_t a)\}$, and $K_b = \mathrm{Per}(b) \times \mathbb{R}$. These are closed, disjoint sets, and their intersections with each horizontal slice $S^1 \times \{t\}$ give the periodic sets for $b_t a$ and b.

For each connected component $I \subset S^1 \smallsetminus \mathrm{Per}(b)$, we set

$$T_I = \{t \in \mathbb{R} \mid I \in X_b(t)\} = \{t \in \mathbb{R} \mid I \cap \mathrm{Per}(b_t a) \neq \varnothing\}.$$

This is a closed set. Indeed, $\overline{I}$ being compact, the second projection, on the product space $\overline{I} \times \mathbb{R}$, is a closed map, and T_I is the image of K_a by this map. We claim T_I is also open. To see this, let $t_0 \in T_I$, and let $I_2, \dots, I_m$ be the other components of $S^1 \smallsetminus \mathrm{Per}(b)$ such that $t_0 \in T_{I_j}$. If $d > 0$ is the distance (for the product metric) between the disjoint compact sets $(S^1 \times [t_0 - 1, t_0 + 1]) \cap K_a$ and $(S^1 \times [t_0 - 1, t_0 + 1]) \cap K_b$, let $I_{m+1}, \dots, I_N$ be the remaining connected components of $S^1 \smallsetminus \mathrm{Per}(b)$ of length $\geqslant d$. Any component J of shorter length obviously satisfies $T_J \cap [t_0 - 1, t_0 + 1] = \varnothing$. Since the sets T_{I_j} are closed, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \cap T_{I_j} = \varnothing$ for all $j \geqslant m + 1$. It follows that $(-\varepsilon, \varepsilon)$ is contained in T_I, for otherwise the number m would fail to be constant. This proves that T_I is open, hence equal to $\varnothing$ or $\mathbb{R}$, and the intervals in $X_b(t)$ do actually not depend on t. ☐

The next lemma states that a and b look (on the level of periodic sets) as if they are semi-conjugate to hyperbolic elements of $\mathrm{PSL}^k(2, \mathbb{R})$, satisfying hypothesis $S_k(a, b)$.

Lemma 4.7. *Every two consecutive intervals of X_a (for the natural cyclic order from S^1) have opposite sign. In other words, $m = 2k$ for some $k \geqslant 1$, and the positive and negative intervals of X_a alternate.*

Proof. Let b_t be a positive one-parameter family of homeomorphisms commuting with $\rho(b)$. Suppose for contradiction that X_a has two successive positive intervals I_1 and I_2 (the negative case is analogous). Let $I \in X_b$ be the interval such that $I_1 \cap I \neq \varnothing$ and $I_2 \cap I \neq \varnothing$. Take $x \in I_1 \smallsetminus I$ such that $a^{q(a)}(x) \in I$. For t large enough, $b_t a^{q(a)}(x)$ can be taken arbitrarily close to the right endpoint of I, hence $a^{q(a)} b_t a^{q(a)}(x) \in I_2 \smallsetminus I$. Since b_t has positive dynamics, it follows that $(b_t a^{q(a)})^2$ moves every point of I to the right, thus, $\Delta_{b,a}(y, 0, \dots, 0, t) > 0$ for all $y \in I$. It follows that $\mathrm{Per}(b_t a) \cap I = \varnothing$ for t large enough, but this contradicts Lemma 4.6. ☐

As a consequence of this lemma, the intervals of X_a and X_b can be labelled, in their natural cyclic order (the cyclic order in which their leftmost points appear on S^1), as $\big(\!\big(I_a^+, I_b^+, I_a^-, I_b^-\big)\!\big)_k$ or $\big(\!\big(I_a^+, I_b^-, I_a^-, I_b^+\big)\!\big)_k$, borrowing the notation from Secion 3.

Lemma 4.8. *Let $I \in X_b$ have left endpoint in a positive interval of X_a. Then $a(I) \subset J$ for some $J \in X_b$. If instead $I \in X_b$ has left endpoint in a negative interval of X_a, then $a^{-1}(I) \subset J$ for some $J \in X_b$.*

Note that, because periodic points alternate and positive and negative intervals alternate, in both cases, J is a positive interval of X_b if and only if I is. Of course, the statement of the lemma also holds with the roles of a and b exchanged.

Proof. Let $x_1, x_2, \dots x_6$ be points in cyclic order such that (x_1, x_3) and (x_4, x_6) are consecutive (positive and negative, respectively) intervals in X_a, and $I = (x_2, x_5) \in X_b$. Let $y_i = a(x_i)$ for $i = 1, 3, 4, 6$. Then (y_1, y_3) and $(y_4, y_6) \in X_a$ and both intersect some interval of X_b, say (y_2, y_5). The statement of the lemma is that $a(x_5) \leqslant y_5$ and $a(x_2) \geqslant y_2$.

Similar to the Proof of Lemma 4.5, we assume the contrary and find a deformation with a common periodic point for a and b. Suppose that $a(x_5) > y_5$ (the proof of the other inequality is symmetric), and choose a positive one-parameter family b_t commuting with b. Since $a^{-1}(y_5) \in (x_2, x_5)$, there exists $t \in \mathbb{R}$ such that $b_t a^{-1}(y_5) \in (x_1, x_3)$. As (y_1, y_3) is $a^{q(a)}$-invariant, it follows that $a^{-q(a)+1} b_t a^{-1}(y_5) < y_5$, ie, $\Delta_{b,a}(y_5, 0, \dots, 0, t, 0) > 0$. On the other hand, as (y_4, y_6) is a negative interval of X_a, we have $\delta_{b,a}(y_5, 0) < 0$. It follows that for a suitable $t_0 \in \mathbb{R}$, y_5 is a periodic point of $b_{t_0} a$. But it is a periodic point of b as well: together with Lemma 2.30, this contradicts the path-rigidity of ρ. The statement concerning $\rho(a)^{-1}$ is symmetric, and proved in the same manner. ☐

Now we state a lemma of purely technical nature, that will allow us to compress the periodic sets, in each interval of $X_a \cup X_b$, to singletons. In the statement and proof, we use $\tau_t \colon \mathbb{R} \to \mathbb{R}$ to denote the translation $x \mapsto x + t$.

Lemma 4.9. *Let $n \geqslant 1$, and for all $i = 1, \dots, n$, let f_i be an increasing homeomorphism from $\mathbb{R}$ to some interval $(a_i, b_i) \subset \mathbb{R}$. Assume that $a_i > -\infty$ for at least one i, and that $b_j < +\infty$ for at least one j. For all $t \in \mathbb{R}$, we set $F_t = \tau_t \circ f_n \circ \dots \circ \tau_t \circ f_1$. Then, there exists a subset $N \subset \mathbb{R}$ of finite Lebesgue measure and consisting of a countable union of segments, such that for all $t \notin N$, the map F_t admits a unique fixed point in $\mathbb{R}$.*

Let us postpone the proof of this lemma to the next paragraph, and use it now to finish the proof of Theorem 4.2.

Proof of Theorem 4.2. Assume now that ρ is minimal. Let b_t be a positive one-parameter family commuting with the action of b. We will first prove that for some t, $b_t a$ has exactly $2k$ periodic points; the conclusion will then follow easily.

Let X_a^+ denote the set of positive intervals of X_a. As observed in Lemma 4.5, $\rho(a)$ induces a permutation of X_a^+; which has say, ℓ orbits, all of cardinality $n = k/\ell$. Fix an interval $I_0 \in X_b$ whose left endpoint lies in an element of X_a^+. By successive applications of Lemma 4.8, for $m = 1, 2, \dots n-1$ we have $\rho(a)^m(I_0) \subset I_m$ for some $I_m \in X_b$, and $\rho(a)^n(I_0) \subset I_0$ because $\rho(a)^n$

fixes X_a^+. Note that, there must exist some m such that $\rho(a)(I_{m-1}) \subset I_m$ is a strict inclusion at the left endpoint (and similarly, another for the right endpoint) as otherwise a and b would have an endpoint of I_0 as a common periodic point.

For each j, let $\phi_j \colon I_j \to \mathbb{R}$ be a homeomorphism such that $\phi_j \circ b_t \circ \phi_j^{-1} = \tau_t$, and for $j \in \{1, \dots, n\}$, set $f_j = \phi_{j+1} \circ a \circ \phi_j^{-1}$, using cyclic notation for the last index. Then, Lemma 4.9 applies, and provides a set $N_{I_0} \subset \mathbb{R}$ of finite Lebesgue measure, such that for all $t \notin N_{I_0}$, $(b_t a)^n = \phi_1^{-1} \circ F_t \circ \phi_1$ has a unique fixed point in I_0.

We repeat this procedure for each element I of X_b, using a^{-1}, instead of a, for the intervals of X_b whose left endpoint lies in a negative interval of X_a. The resulting, finitely many, sets N_I, each of finite Lebesgue measure, cannot cover $\mathbb{R}$, hence there exists $t \in \mathbb{R}$ such that each element of X_b contains a unique periodic point of $b_t a$. By Lemma 4.6, $b_t a$ does not have any other periodic points, hence $b_t a$ admits exactly $2k$ periodic points. As $b_t a$ is obtained by a bending deformation that does not change a, by Lemma 4.7 these $2k$ periodic points have alternating attracting and repelling dynamics. One may now repeat the same procedure reversing the roles of a and b, to obtain a further deformation where b has exactly $2k$ periodic points, that are alternately attracting and repelling. If ρ is assumed to be minimal, by Observation 2.24, this was originally true of ρ. ☐

4.3. **Proving Lemma 4.9.**

The statement of Lemma 4.9 came from our attempt to understand the argument in the first four lines of page 644 in [5]. The case $n = 1$ of Lemma 4.9 gives an alternative end of the proof of [5, Lemma 2.7].

To give an outline of the major ideas, we sketch the proof in the (easier) case $n = 1$. Suppose, for all $t, x \in \mathbb{R}$ we have $F_t(x) = f(x) + t$, where f is an increasing homeomorphism from $\mathbb{R}$ to a bounded interval (a, b). Define $T(x) := x - f(x)$; so $T(x)$ is the unique number such that x is a fixed point of $F_{T(x)}$. We want to prove that most t are realized as $T(x)$ by a unique x. If $T(x_1) = T(x_2)$ for some $x_1 < x_2$, then we have $f(x_2) - f(x_1) = x_2 - x_1$, ie, the intervals $[f(x_1), f(x_2)]$ and $[x_1, x_2]$ have the same length, say ℓ. Since T cannot increase faster than the identity map, all elements $x \in [x_1, x_2]$ have $|T(x) - T(x_1)| \leqslant \ell$. In summary, although we have found a segment $[T(x_1) - \ell, T(x_1) + \ell]$ of length 2ℓ that contains t's which may have several preimages under T, it has cost us the length ℓ in the image of f, and this reservoir is finite.

Now let us turn this into a precise proof, in the general case $n \geqslant 1$.

Proof of Lemma 4.9. We will show that there exists a countable union of segments, $N_+ \subset \mathbb{R}_+$, of finite Lebesgue measure, such that for all $t \in \mathbb{R}_+ \setminus N_+$, F_t has a unique fixed point. The case for $t < 0$ is symmetric and left to the reader.

Let j be an index such that $b_j < +\infty$. Let $A_t = \tau_t \circ f_j \circ \cdots \circ \tau_t \circ f_1$, and $B_t = \tau_t \circ f_n \circ \cdots \circ \tau_t \circ f_{j+1}$. For fixed t, both maps A_t and B_t are homeomorphisms to their images so $F_t = B_t \circ A_t$ has a unique fixed point x if and only if $A_t \circ B_t$ has a unique fixed point (in which case it is $B_t(x)$). In other words, we may suppose without loss of generality that $j = n$. (For the $t < 0$ case, one supposes instead that $a_n > -\infty$.)

Let $G(t, x) = f_n \circ \tau_t \circ f_{n-1} \circ \cdots \circ \tau_t \circ f_1(x) = F_t(x) - t$. This map G is strictly increasing in the variable x, and increasing (strictly, if $n \geqslant 2$) in t. The monotonicity of G, and the assumptions $\sup(a_j) > -\infty$ and $b_n < +\infty$, imply that the range of the map $G\colon \mathbb{R}_{\geqslant 0} \times \mathbb{R} \to \mathbb{R}$ is a bounded interval. Let (a_0, b_0) denote this interval, with $b_0 = b_n$.

If $x \geqslant b_0$, the map $t \mapsto F_t(x)$ is a homeomorphism between $\mathbb{R}_{\geqslant 0}$ and $[F_0(x), +\infty)$, and $F_0(x) = G(x, 0) < b_0$. Hence, there is a unique $t = T(x)$ such that $F_t(x) = x$. This defines a function $T\colon [b_0, +\infty) \to (0, +\infty)$.

Sublemma 4.10. *The map T satisfies the following inequalities.*

(T1) *For every $x \in [b_0, +\infty)$, we have $a_0 < x - T(x) < b_0$.*

(T2) *For all $x_1, x_2 \in [b_0, +\infty)$ such that $x_1 < x_2$, we have*

$$f_1(x_1) - f_1(x_2) < T(x_2) - T(x_1) < x_2 - x_1.$$

In particular, T is continuous, at bounded distance from the identity, and its rate of increase is bounded above by 1.

Proof. The inequality (T1) follows directly from the fact that the range of G is (a_0, b_0), and from the defining identity $F_{T(x)}(x) = x$ for all $x \geqslant b_0$. Let us turn to (T2).

Suppose $x_1 < x_2$. If $T(x_2) \geqslant T(x_1)$ the first inequality holds trivially, so suppose $T(x_2) < T(x_1)$. By definition we have $F_{T(x_1)}(x_1) < F_{T(x_2)}(x_2)$, so $G(x_1, T(x_1)) < G(x_2, T(x_2))$, ie

$$f_n \circ \tau_{T(x_1)} \circ \ldots \circ f_2(f_1(x_1) + T(x_1)) < f_n \circ \tau_{T(x_2)} \circ \ldots \circ f_2(f_1(x_2) + T(x_2)).$$

As $t \mapsto f_n \circ \tau_t \circ \ldots \tau_t \circ f_2$ is increasing in t, we have

$$f_n \circ \tau_{T(x_2)} \circ \ldots \circ f_2(f_1(x_1) + T(x_1)) < f_n \circ \tau_{T(x_2)} \circ \ldots \circ f_2(f_1(x_2) + T(x_2)),$$

hence $f_1(x_1) + T(x_1) < f_1(x_2) + T(x_2)$ and $f_1(x_1) - f_1(x_2) < T(x_2) - T(x_1)$.

For the second inequality, if $T(x_2) \leqslant T(x_1)$ it is automatically satisfied, as $x_2 - x_1 > 0$, so suppose $T(x_2) > T(x_1)$. Since G is increasing in t and strictly increasing in x, we have $F_{T(x_2)}(x_2) - T(x_2) > F_{T(x_1)}(x_1) - T(x_1)$, ie, $T(x_2) - T(x_1) < x_2 - x_1$. □

Now, we define a map $H\colon \mathbb{R}_{\geqslant b_0} \to \mathbb{R}$ by $H(x) = \sup\{T(x'),\ x' \leqslant x\}$.

Lemma 4.11. *The map $H\colon \mathbb{R}_{\geqslant b_0} \to [T(b_0), +\infty)$ is continuous, surjective, and for all $A \geqslant T(b_0)$, the set $H^{-1}(A)$ is a segment, $[a, b]$, with possibly $a = b$, and $T(a) = T(b) = A$.*

This is an easy exercise in undergraduate analysis; we leave the details to the reader.

For the last step of the proof, let $W \subset [T(b_0), +\infty)$ be the set of all elements $w \in [T(b_0), +\infty)$ such that $H^{-1}(w)$ is not a singleton, and for all $w \in W$, write $H^{-1}(w) = [a_w, b_w]$.

Sublemma 4.12. *The set W is countable, and $\sum_{w \in W} b_w - a_w \leqslant b_0 - a_0$.*

Proof. First, since the segments $[a_w, b_w]$ all have positive length and are disjoint in $[b_0, +\infty)$, there can only be countably many of them. Now, for any $w \in W$, we have $F_w(a_w) = a_w$ and $F_w(b_w) = b_w$, by Lemma 4.11. In other words, $G(w,a_w)+w=a_w$ and $G(w,b_w)+w=b_w$. Thus, the segment $[G(w,a_w),G(w,b_w)]$ has the same length b_w-a_w.

Since the image of G is an interval of length b_0-a_0, it suffices to prove that the segments $[G(w,a_w),G(w,b_w)]$ are pairwise disjoint. To do this, take $w_1,w_2\in W$, with $w_1<w_2$. Since H is increasing, we have $b_{w_1}<a_{w_2}$. By monotonicity of G, we have $G(w_1,b_{w_1})<G(w_2,a_{w_2})$ and these segments are disjoint indeed. $\square$

Now, for all $w\in W$, define $N_w:=[w-(b_w-a_w),w]$, and define

$$N_+=[0,b_0-a_0]\cup\bigcup_{w\in W}N_w.$$

This may not be a disjoint union, but, by Lemma 4.12, this countable union of segments has finite Lebesgue measure. Hence, the proof of Lemma 4.9 boils down to the following observation.

Sublemma 4.13. *For all $t\in\mathbb{R}_{\geqslant 0}\smallsetminus N_+$, the map F_t admits a* unique *fixed point.*

Proof. Let $t>b_0-a_0$ be such that F_t has at least two distinct fixed points, say x_1,x_2 with $x_1<x_2$. We want to prove that $t\in N_w$ for some $w\in W$. By definition, the x_i satisfy $G(t,x_i)+t=x_i$. Since $G(t,x)>a_0$ for all x, and $t>b_0-a_0$, this implies $x_1,x_2\in[b_0,+\infty)$. By definition of the map T, we have $T(x_1)=T(x_2)=t$. Let $x_0=\min\{x\leqslant x_2\,|\,T(x)=H(x_2)\}$. Then $x_0<x_2$. Indeed, if $H(x_2)=t$ then $x_0\leqslant x_1$, and if $H(x_2)>t$ then the maximum $H(x_2)$ is reached at some point to the left of x_2. Thus, $x_0=a_w$ for some $w\in W$, and we also have $b_w\geqslant x_2$.

We claim now that $t\in N_w$. Since $x_2\leqslant b_w$, by definition of H we have $w=H(b_w)\geqslant t=T(x_2)$. Applying inequality (T2) to x_2 and b_w now gives $w-t\leqslant b_w-x_2$, so $w-t\leqslant b_w-a_w$, hence $t\geqslant w-(b_w-a_w)$. Thus we indeed have $t\in N_w$. $\square$

This concludes the proof of Lemma 4.9. $\square$

5. Proof of Theorem 1.6

In this section we finish the proof of the main result for path-rigid representations, showing that a path-rigid representation ρ of $\pi_1\Sigma_g$ is either geometric, or (as an unlikely case) must have Euler class zero and a genus $g-1$ subsurface whose fundamental has finite orbit under ρ.

Definition 5.1. Let $\rho:\Gamma_g\to\mathrm{Homeo}^+(S^1)$, and let $T\subset\Sigma_g$ be a one-holed torus. We say T is a *good torus* if T contains a nonseparating simple closed curve a with $\mathrm{rot}(\rho(a))=0$, and *bad* otherwise.

We say T is *very good* if $\rho(\pi_1(T))$ has a finite orbit in S^1.

Observation 5.2 (Very good implies good)**.** *Suppose that T is very good, and let a and b be free generators for $\pi_1(T)$, represented by simple closed curves on Σ_g with $i(a,b)=\pm$ (here orientation of curves is not important). Since T is very good,* rot *is a homomorphism to a finite subgroup of $\mathbb{R}/\mathbb{Z}$. If $0\neq|\mathrm{rot}(\rho(a))|\leqslant|\mathrm{rot}(\rho(b))|<1$, then there exists n such that $|\mathrm{rot}(\rho(a^nb))|<|\mathrm{rot}(\rho(a))|$. Since a,a^nb are again free generators represented by simple closed* *curves, one may repeat this procedure until the process terminates with a simple closed curve of rotation number zero.*

Assumption 5.3. For the remainder of this section, we assume that $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ is path-rigid.

As in Section 3, we will frequently drop the notation ρ when the context is clear, using a to denote $\rho(a)$.

5.1. **Bad tori.**

This subsection contains the proof of Proposition 1.11. Under our standing assumption that ρ is path-rigid, we will show that if Σ_g contains a bad torus T, then $\Sigma_g \backslash T$ contains only very good tori. We begin with some general preliminaries on rotation numbers in order to show that a bad torus T always contains simple closed curves with rotation number arbitrarily close to zero. This means in particular that there are points "almost fixed" by simple closed curves in T. We then study these "almost fixed" points for some specific sequences of generators for $\pi_1(T)$, and leverage properties of these sets to show that there cannot exist two disjoint bad tori.

Definition 5.4. If $f, g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, we say that g *dominates* f, and we write $f < g$, if we have $f(x) < g(x)$ for all $x \in \mathbb{R}$.

It is immediate that $<$ is a left- and right-invariant partial order on $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Furthermore, it satisfies the following obvious properties.

(1) $\forall f, g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}), f > g \Leftrightarrow f^{-1} < g^{-1}$;
(2) $\forall f \in \mathrm{Homeo}^+(S^1),\ \widehat{f} > \mathrm{Id} \Leftrightarrow \mathrm{rot}(f) \neq 0$;
(3) $\forall f, g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}), \begin{cases} f < g \Rightarrow \widetilde{\mathrm{rot}}(f) \leqslant \widetilde{\mathrm{rot}}(g), \text{ and} \\ (f < g \text{ or } g < f) \Leftrightarrow \widetilde{\mathrm{rot}}(f^{-1}g) \neq 0. \end{cases}$

Property (2) uses the notation $\widehat{f}$ from Notation 2.16, which is also adopted throughout this section. The following easy observation will be handy.

Observation 5.5. *Let $f, g \in \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Suppose that $\widetilde{\mathrm{rot}}(f) < \widetilde{\mathrm{rot}}(g)$ and $\widetilde{\mathrm{rot}}(g^{-1}f) \neq 0$. Then $f < g$.*

Proof. If not, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \geqslant g(x_0)$, hence $\widetilde{\mathrm{rot}}(g^{-1}f) \geqslant 0$. Thus, $\widetilde{\mathrm{rot}}(g^{-1}f) > 0$, and this implies $g^{-1}f > \mathrm{Id}$, hence $f > g$ and $\widetilde{\mathrm{rot}}(f) \geqslant \widetilde{\mathrm{rot}}(g)$, a contradiction. ☐

Building on this observation, we show that the infimum of rotation numbers of non-separating simple curves in a bad torus is necessarily zero. More precisely, we have the following.

Lemma 5.6. *Let (a, b) be standard generators of a bad torus T. Then, there exist integers m, n, unique and well-defined modulo $q(a)$, with $(n - m)p(a) = 1 \bmod q(a)$, and such that for all j not divisible by $q(a)$, we have $\widehat{a^n b} < \widehat{a^j}$, and $\check{a^j} < \overbrace{a^m b}$. Moreover, if $p(a) = 1$, then we have $\widehat{a^n b}^{\,2} < \widehat{a}$, or $\overbrace{a^{n-1} b}^{\,-2} < \widehat{a}$, or both.*

Of course, the same statement holds with the roles of a and b exchanged. In this lemma, the standing assumption that ρ path-rigid is used only to guarantee that all non-separating simple closed curves are mapped to homeomorphisms of rational rotation number (see Proposition 4.1).

Proof. Let F be a finite orbit of a. If there exists some point $x \in F \cap b^{-1}(F)$, then there exists $N > 0$ such that $\rho(a)^N \rho(b)(x) = x$, thus $\mathrm{rot}(a^N b) = 0$, contradicting the fact that T was bad. Thus, $F \cap b^{-1}(F) = \varnothing$.

Now we show these sets alternate in the circle. Suppose for contradiction that some connected component $I = (x_1, x_2)$ of $S^1 \setminus F$ contains at least two points of $b^{-1}(F)$. Let $y_1 \in b^{-1}(F)$ be the leftmost point of $b^{-1}(F)$ in I, and y_2 be the second leftmost such point. Then there exists $N > 0$ such that $a^N b(y_1) = x_1$. It follows that $a^N b(y_2) = x_2$ and $(a^N b)^{-1}(I) = (y_1, y_2) \subset I$. In particular, $\mathrm{rot}(a^N b) = 0$, a contradiction.

Now that we know these sets alternate, choose $x \in b^{-1}(F)$, and let y_r, y_ℓ, be the next point of F to the right and to the left of x, respectively. Then there exists a unique pair $(n, m) \in \{0, \ldots, q(a) - 1\}^2$ such that $a^n b(x) = y_r$ and $a^m b(x) = y_\ell$. In particular, $(n - m)p(a) = 1 \bmod q(a)$. These m, n are obviously the only candidates, modulo $q(a)$, for the dominations $\widehat{a^n b} < \widehat{a^j}$ and $\widecheck{a^m b} > \widecheck{a^{-j}}$, for an integer j such that $a^k(y_\ell) = y_r$. (This shows m and n do not depend on F). We claim that this pair (n, m) satisfies the statement of the lemma.

To see this, lift the points of F to $\mathbb{R}$ and let $x_1 < x_2 < \ldots < x_{q(a)}$ denote $q(a)$ consecutive points of the lift. Then, for all i, $\widehat{a^n b}(x_i) \leqslant x_{i+1}$; hence $\widetilde{\mathrm{rot}}(\widehat{a^n b}^{q(a)}) \leqslant 1$, Hence, $\widetilde{\mathrm{rot}}(\widehat{a^n b}) \leqslant \frac{1}{q(a)}$, and for every integer j not divisible by $q(a)$ we have $\widetilde{\mathrm{rot}}(\widehat{a^n b}) \leqslant \widetilde{\mathrm{rot}}(\widehat{a^j})$. So we cannot have $\widehat{a^n b} > \widehat{a^j}$. But $\widehat{a^j}^{-1} \widehat{a^n b}$ cannot have translation number zero, for otherwise $a^{n-j} b$ would have rotation number zero and the torus would be good. Thus, $\widehat{a^n b} < \widehat{a^j}$. An essentially identical argument shows that $\widecheck{a^m b} > \widecheck{a^j}$.

It remains only to prove the statement regarding the case $p(a) = 1$. As we have seen, $\widehat{a} > \widehat{a^n b}$, and $\widehat{a} > \widehat{b^{-1} a^{1-n}} = \widecheck{a^{n-1} b}^{-1}$, and this immediately implies $\widehat{a} = \widehat{a^n b} \cdot \widehat{b^{-1} a^{1-n}}$. As $(a, a^n b)$ and hence $(b^{-1} a^{1-n}, a^n b)$ are also standard generating sets of $\pi_1(T)$ (obtained by Dehn twists) and T is bad, we must either have $\widehat{b^{-1} a^{1-n}} > \widehat{a^n b}$, or $\widehat{b^{-1} a^{1-n}} < \widehat{a^n b}$, otherwise the non-separating simple closed curve $a^{n-1} b a^n b$ would have rotation number zero. The statement follows. $\square$

As a consequence, we have the following.

Proposition 5.7. *Let (a, b) be a standard generating set for a bad torus. Let $(a_k, b_k)_{k \geqslant 0}$ be the sequence of standard generating sets, defined inductively as follows.*

- *Define $(a_0, b_0) = (a, b)$.*
- *If k is even, let $a_{k+1} = a_k$ and $b_{k+1} = a_k^{n(k)} b_k$, where $0 \leqslant n(k) \leqslant q(a_k) - 1$ is the integer given by Lemma 5.6 applied to the generators (a_k, b_k).*
- *If k is odd, let $b_{k+1} = b_k$ and $a_{k+1} = b_k^{n(k)} a_k$, where $0 \leqslant n(k) \leqslant q(a_k) - 1$ is obtained, similarly, by inputting (b_k, a_k) into Lemma 5.6.*

Then for all $k \geqslant 0$ even we have $\widehat{a_{k+1}} > \widehat{b_{k+1}}$, and for $k \geqslant 0$ odd we have $\widehat{a_{k+1}} < \widehat{b_{k+1}}$.

Moreover, for all $k \geqslant 0$, we have $\widehat{a_k} > \widehat{a_{k+2}}^2$, and $\widehat{b_k} > \widehat{b_{k+2}}^2$. In particular, both sequences $(\mathrm{rot}(a_k))_{k \geqslant 0}$ and $(\mathrm{rot}(b_k))_{k \geqslant 0}$ converge to zero.

Note that the sequence (a_k, b_k) is built so that both $\mathrm{rot}(a_k)$ and $\mathrm{rot}(b_k)$ converge to zero from above. This choice is arbitrary.

Proof. The first consideration follows immediately from the first statement of Lemma 5.6. Let us prove the second. Let $k \geqslant 0$ be even. If $p(a_k) \geqslant 2$, let $n = n(k) \geqslant 0$ be such that $np(a_k) = 1 \bmod q(a_k)$, as in Lemma 5.6. Then $\mathrm{rot}(a_k^n) = \frac{1}{q(a_k)}$, and $\widehat{a_k^n}^{p(a_k)} = \widehat{a_k}$. By lemma 5.6 we have $\widehat{b_{k+1}} < \widehat{a_k^n}$, hence $\widehat{b_{k+1}}^{p(a_k)} < \widehat{a_k}$, and $\widehat{a_{k+2}}^2 < \widehat{a_k}$.

Otherwise, $p(a_k) = 1$, and again we take $n(k)$ as in Lemma 5.6. If $\widehat{a_k^{n(k)} b_k}^2 < \widehat{a_k}$ then we may conclude as above. Otherwise, $\widehat{b_k^{-1} a_k^{1-n}}^2 < \widehat{a_k}$, ie, $\widehat{b_{k+1}^{-1} a_{k+1}}^2 < \widehat{a_k}$. Thus either $n(k+1)$ equals -1 modulo $q(b_{k+1})$, or not, in which case $\widetilde{\mathrm{rot}}(\widehat{b_{k+1}^{n(k+1)} a_{k+1}}) < \widetilde{\mathrm{rot}}(\widehat{b_{k+1}^{-1} a_{k+1}})$, and then $\widehat{b_{k+1}^{n'} a_{k+1}} < \widehat{b_{k+1}^{-1} a_{k+1}}$. In either case we conclude that $\widehat{a_{k+2}}^2 < \widehat{a_k}$.

For k odd, and for b_k instead of a_k, things are symmetric. Note that $\widehat{a_{k+2}}^2 < \widehat{a_k}$ implies in particular that $0 < \widetilde{\mathrm{rot}}(\widehat{a_{k+2}}) < \frac{1}{2}\widetilde{\mathrm{rot}}(\widehat{a_k})$, hence the sequences $(\widetilde{\mathrm{rot}}(\widehat{a_k}))$ and $(\widetilde{\mathrm{rot}}(\widehat{b_k}))$ converge to zero from above. ☐

Let $T = T(a, b)$ be a bad torus, and let (a_k, b_k) be the sequence furnished by Proposition 5.7. Let $x \in S^1$, and let $\tilde{x} \in \mathbb{R}$ be a lift of x. Then, by Proposition 5.7, the sequence $(\widehat{a_k}(\tilde{x}))_k$ is decreasing, bounded below by $\tilde{x}$, hence it converges to some real number that we denote by $\tilde{x} + j_T(x)$. Note that $j_T(x)$ does not depend on the choice of the lift of x. We define

$$F_T := \{x \in S^1,\ j_T(x) = 0\}.$$

The reader should interpret this as the set of points that are moved arbitrarily small distances by elements $\{a_k\}$ (or that are *almost fixed* by elements towards the tail end of this sequence). Although the notation (a, b) is suppressed, F_T as defined is dependent on the generating set we started with. (But see Step 1 of the proof of Proposition 5.9 below). As usual, we let $\widetilde{F_T}$ denote the preimage of F_T in $\mathbb{R}$.

Proposition 5.8 (Properties of F_T). *(1) F_T is a non-empty, proper subset of S^1, and it has no isolated points (in particular, it is infinite).*

(2) For every $x \in S^1$, we have $\min\{\widetilde{F_T} \cap [\tilde{x}, \infty)\} = \tilde{x} + j_T(x)$. In particular, $x + j_T(x) \in F_T$ for all x.

(3) The commutator $[a, b]$ fixes F_T pointwise (in particular, it has rotation number zero).

Proof. Let $x \in \mathbb{R}$. For all $k \geqslant 0$ we have $\widehat{a_k}(x) > x + j_T(x)$, hence, $\widehat{a_k}^2(x) > x + j_T(x) + j_T(x + j_T(x))$. But $\widehat{a_{k-2}}(x) > \widehat{a_k}^2(x)$, and, by definition, $\widehat{a_{k-2}}(x)$ converges to $x + j_T(x)$. This proves that $x + j_T(x) \in F_T$, which, thus, is non-empty. Further, if the open interval $(x, x + j_T(x))$ contained a point $y \in \widetilde{F_T}$, then for large k we would have $x + j_T(x) > \widehat{a_k}(y) > y > x$, contradicting that a_k preserves orientation. This proves property (2).

To prove property (3), let $x \in \widetilde{F_T}$ and observe, just as above, that the sequence $\widehat{a_k}^4(x)$ also converges to x. Fix $\varepsilon > 0$, and let k be even, and large enough so that $x_1 = x$, $x_2 = \widehat{a_k}(x)$, $x_3 = \widehat{a_k}^2(x)$ and $x_4 = \widehat{a_k}^3(x)$ all lie in the interval $[x, x+\varepsilon]$. By Lemma 5.6, $a_{k+1} = a_k$ and $\widehat{b_{k+1}}$ is dominated by $\widehat{a_{k+1}}$. Thus, $\widehat{b_{k+1}}(x_3) \in (x_3, x_4)$, and $\widehat{b_{k+1}}^{-1}(x_2, x_3) \subset (x_1, x_3)$. It follows that $[a_{k+1}, b_{k+1}] = [a, b]$ maps the point x_2 into the interval (x_1, x_3), hence, for all $\varepsilon > 0$, $[a, b]$ maps a point of $[x, x+\varepsilon]$ in $[x, x+\varepsilon]$, whence $[a, b](x) = x$.

It remains to prove that $F_T \neq S^1$, and F_T has no isolated point. If F_T was equal to S^1, then $[a, b]$ would be the identity of S^1. Thus, the restriction of ρ to $\langle a, b\rangle$ would have abelian image and rotation number would be a homomorphism. In particular, the proof of Observation 5.2 shows that T is not bad.

Finally if x were an isolated point of F_T, we could take $x_0 \in S^1$ such that $[x_0, x) \cap F_T = \varnothing$. Let x_1 be the next point of F_T to the right of x. Then $x_0 + j_T(x_0) = x$, and $x + j_T(x) = x_1$. It follows that for all $k \geqslant 0$, $\widehat{a_k}^2(x_0) \geqslant x_1$, hence $x_0 + j_T(x_0) \geqslant x_1$, a contradiction. ☐

We remark that the complement of the set F_T is a union of half-open intervals, so F_T is not closed. Using F_T and j_T, we now prove the following major step towards Proposition 1.11

Proposition 5.9. *There cannot exist two disjoint bad tori in* Σ_g.

Proof. By contradiction, let $T = T(a, b)$ and $T' = T(a', b')$ be two disjoint bad tori. Up to re-indexing and reversing some of these curves, we may suppose that (a, b, a', b') is the beginning of a standard basis of $\pi_1\Sigma_g$.
Step 1: We have $j_T = j_{T'}$.

We proceed by contradiction. Suppose these functions are non-equal and let $x_0 \in S^1$ be such that $j_T(x_0) \neq j_{T'}(x_0)$, without loss of generality assume $j_T(x_0) < j_{T'}(x_0)$. Let $(a_k, b_k)_{k\geqslant 0}$ and $(a'_k, b'_k)_{k\geqslant 0}$ be the sequences of generators of T and T' furnished by Proposition 5.7. For k large enough, we have $\widehat{a_k}(x_0) < x_0 + j_{T'}(x_0)$. Let m be as in Lemma 5.6 applied to (a_k, b_k), and put $\alpha = a_k$, and $\beta = a_k^m b_k$. Then (α, β) is a standard generating set for T, and $\widehat{\alpha} > \widehat{\beta^{-1}}$. Since $\mathrm{rot}(b'_\ell) \to 0$, for $\ell \geqslant 0$ large enough we have $\widetilde{\mathrm{rot}}(\widehat{b'_\ell}) < \widetilde{\mathrm{rot}}(\widehat{\beta^{-1}})$. But $\widehat{b'_\ell}(x_0) > x_0 + j_{T'}(x_0)$ (indeed, $\widehat{b'_\ell}$ dominates $\widehat{a'_{\ell+1}}$, by construction of the sequences in Proposition 5.7), hence $\widehat{a_k}$ does not dominate $\widehat{b'_\ell}$. We now prove a sub-lemma to derive a contradiction, this will conclude the proof of Step 1.

Lemma 5.10. *Let* $T = T(a, b)$ *be a bad torus, and let* b' *be a non separating simple curve outside* $T(a, b)$ *such that* $b'^{-1}a$ *and* bb' *are simple. Suppose that* $\widehat{a} > \widehat{b^{-1}}$ *and* $\widetilde{\mathrm{rot}}(\widehat{b^{-1}}) > \widetilde{\mathrm{rot}}(\widehat{b'})$. *Then* $\widehat{a}$ *dominates* $\widehat{b'}$.

Proof. Suppose that $\widehat{a}$ does not dominate $\widehat{b'}$. Then $\widehat{b^{-1}}$ does not dominate $\widehat{b'}$, either. Observation 5.5 then asserts that both $b'^{-1}a$ and bb' have rotation numbers zero. Now the two curves $b'^{-1}a$ and bb' are standard generators of a torus $T(b'^{-1}a, bb')$, and since $\mathrm{rot}(b'^{-1}a) = 0$, this homeomorphism lies in a one-parameter family, so as in Observation 2.25, there is a path-deformation of ρ replacing the action of bb' with $b'^{-1}a \cdot bb'$. Hence, $\mathrm{rot}(bb') = 0 =$

$\mathrm{rot}(b'^{-1}a \cdot bb') = \mathrm{rot}(ab)$. However, we assumed that $T(a,b)$ was bad, so $\mathrm{rot}(ab) \neq 0$. This gives the desired contradiction. ☐

Step 2: We can deform the representation so that $j_T \neq j_{T'}$.

As shown in the proof of Proposition 5.8, $[a,b] \neq \mathrm{id}$, but $F_T \subset \mathrm{Fix}([a,b])$. Let $x \notin \mathrm{Fix}([a,b])$, so then $j_T(x) > 0$. Let $y = x + j_T(x)$, let I be the connected component of $S^1 \smallsetminus \mathrm{Fix}([a,b])$ containing x, and let c_t be a one-parameter family of homeomorphisms commuting with $[a,b]$, and with support equal to $\overline{I}$.

Then, regardless of whether $y \in I$, the distance between $c_t(x)$ and $c_t(y)$ varies, in a nonconstant way, with t: it goes to zero as $t \to \infty$ if $y \in I$, and simply changes if $y \notin I$. Now, consider a bending deformation of our representation ρ, by setting $\rho_t(\gamma) = \rho(\gamma)$ for all curves outside T, and $\rho_t(\gamma) = c_t\rho(\gamma)c_{-t}$ for $\gamma \in \langle a, b\rangle$. This deformation changes the value of $j_T(x)$, without changing the value of $j_{T'}(x)$. In particular, after this path-deformation, Step 1 no longer holds! This gives a contradiction. ☐

Supposing again that $T(a,b)$ is a bad torus, it remains to show that any torus in $\Sigma_g \smallsetminus T(a,b)$ is not only good, but *very good.*

Lemma 5.11. *Let $T = T(a,b)$ be a bad torus, and let γ be a non-separating simple closed curve outside of T, with* $\mathrm{rot}(\gamma) = 0$. *Then $F_T \subset \mathrm{Fix}(\gamma)$.*

Proof. Let $(a_k, b_k)_{k \geqslant 0}$ be the sequence given by Proposition 5.7, and orient γ so that $\gamma^{-1}a_k$ is also a (non-separating) simple curve. Fix $k \geqslant 0$, and let $\alpha = a_k$, and $\beta = a_k^m b_k$, as in Lemma 5.6. Then, as a consequence of Lemma 5.10, we have $\widehat{a_k} > \widehat{\gamma}$. This holds for all $k \geqslant 0$, hence, for all $x \in \mathbb{R}$ we have $\widehat{\gamma}(x) \leqslant x + j_T(x)$. In particular, if $x \in \widetilde{F_T}$, we have $\widehat{\gamma}(x) \leqslant x$.

For the reverse inequality, first observe that, as in Lemma 5.10, the conditions $\breve{\alpha} < \breve{b^{-1}}$ and $\widetilde{\mathrm{rot}}(\breve{b^{-1}}) < \widetilde{\mathrm{rot}}(\breve{\gamma})$ imply the domination $\breve{\alpha} < \breve{\gamma}$ (this is exactly the statement of Lemma 5.10 upon reversing the orientation of $\mathbb{R}$). And $\breve{\gamma} = \widehat{\gamma}$, since $\mathrm{rot}(\gamma) = 0$. Thus, fix $x \in \widetilde{F_T}$; we want to prove that $\breve{\gamma}(x) \geqslant x$. Fix $\varepsilon > 0$. For k large enough, the sequence (a_k, b_k) from Proposition 5.7 satisfies $\widehat{a_k}(x) < x + \varepsilon$. Let $(a', b') = (a_k, b_k)$ for such a large k, and now define $b'' = b'$ and $a'' = b'^m a'$ and then $\alpha = a''$ and $\beta = a''^n b''$, where m, and then n, are given by Lemma 5.6 with these two successive pairs. Then, we have $\widetilde{\mathrm{rot}}(\breve{\alpha}) < \widetilde{\mathrm{rot}}(\breve{\beta^{-1}}) < \widetilde{\mathrm{rot}}(\breve{\gamma})$, hence, $\breve{\alpha} < \breve{\gamma}$, ie, $\widehat{\alpha^{-1}}$ dominates $\widehat{\gamma}^{-1}$. It follows that $\widehat{\gamma}(x) \leqslant x + \varepsilon$. ☐

End of the proof of Proposition 1.11. Suppose that $T = T(a,b)$ is a bad torus, and let T' be a torus disjoint from T. Since T' is good, by Lemma 5.10 we may take $T' = T(a', b')$ where $\mathrm{rot}(a') = 0$. Then we have $\mathrm{Fix}(a') \supset F_T$ by Lemma 5.11. This is also true after replacing a' with a deformation $b'_t a'$, so $\mathrm{Per}(b') \supset F_T$ or equivalently $\mathrm{Fix}(b'^{q(b')}) \supset F_T$. Since this is also true after replacing b' with any deformation $a'_t b'$, it must be that $F_T \subset P(a', b')$. By Lemma 2.19 (1), this means that $\langle a', b'\rangle$ has a finite orbit in S^1. ☐

5.2. **Good tori.** In this section, we prove Proposition 1.12. Recall this was the statement that if ρ is path-rigid and non-geometric, then there cannot exist two disjoint good tori which are both not very good.

In the course of the proof, we will develop some tools that will be used again in Section 6 for the proof of Theorem 1.5. The proof proceeds by showing that any path-rigid, minimal representation that fails the hypothesis above on tori necessarily satisfies hypothesis S_k. Our main tool is the movement of periodic sets by bending deformations, as introduced in Paragraph 2.2.3.

To motivate the first step, observe that if ρ has two disjoint good tori $T(a,b)$ and $T(d,e)$ with $\operatorname{rot}(a) = \operatorname{rot}(e) = 0$, and if neither of these tori are very good, then $P(a,b) = P(e,d) = \varnothing$. We can also find c so that (a,b,c,d,e) is a 5-chain. This is the set-up of the next Proposition.

Proposition 5.12. *Let ρ be path-rigid minimal and let (a,b,c,d,e) be a 5-chain. Suppose that both $P(a,b)$ and $P(e,d)$ are empty. Then we have $S_k(b,c)$, for some $k \geqslant 1$.*

Proof of Proposition 5.12. After changing orientations of these curves, we may suppose that (a,b,c,d,e) is a directed 5-chain. By Theorem 4.2, it suffices to show that $\operatorname{Per}(b) \cap \operatorname{Per}(c) = \varnothing$. Since $P(a,b) = \varnothing$, Lemma 2.21 says that $\partial N(a,b)$ is finite. Choose a positive one-parameter family $(e_t)_{t\in\mathbb{R}}$, commuting with $\rho(e)$. Since $P(e,d) = \varnothing$, we have $\operatorname{Per}(e_t d) \subset U(e,d)$ for all t, so the sets $\operatorname{Per}(e_t d)$, for varying t, are pairwise disjoint. Thus, we can choose t_0 so that $\operatorname{Per}(e_{t_0} d) \cap \partial N(a,b) = \varnothing$. Abusing notation, we now replace d with $e_{t_0} d$ (we will not further use e). With this change in notation, we now have $\partial N(a,b) \cap P(d,c) = \varnothing$. The remaining step will be a useful tool later in Section 6, so we split it off to a separate statement (Lemma 5.13), proved below. ☐

Lemma 5.13. *Let ρ be path-rigid, and let (a,b,c,d) be a 4-chain. Suppose that $P(a,b) = \varnothing$ and $\partial N(a,b) \cap P(d,c) = \varnothing$. Then* $\operatorname{Per}(b) \cap \operatorname{Per}(c) = \varnothing$*.*

Proof. Let a_t and d_t be positive one-parameter families commuting with a and d respectively. By Lemma 2.30, it suffices to find t and s such that $\operatorname{Per}(a_t b) \cap \operatorname{Per}(d_s c) = \varnothing$. Let $F_0 = \partial N(a,b) \cap \partial N(d,c)$. As $P(a,b)$ is empty, the set $\partial N(a,b)$ is finite, by Lemma 2.21. Hence, F_0 is finite. Let $F_1 = \partial N(a,b) \smallsetminus F_0$ and $F_2 = (P(d,c) \cup \partial N(d,c)) \smallsetminus F_0$. By construction, the F_i are disjoint closed sets, so we can take $\varepsilon > 0$ smaller than the minimum distance between any two of them. Fix t large, so that (by Lemma 2.19), $\operatorname{Per}(a_t b)$ is contained in the ε-neighborhood of $F_0 \cup F_1$, hence disjoint from F_2. Since $F_0 \subset N(a,b)$, it is also disjoint form $\operatorname{Per}(a_t b)$, i.e. $\operatorname{Per}(a_t b) \cap (F_0 \cup F_2) = \varnothing$. Now let $\eta > 0$ be smaller than the distance between $F_0 \cup F_2$ and $\operatorname{Per}(a_t b)$. By Lemma 2.19 again, for s large enough, the set $\operatorname{Per}(d_s c)$ is in the η-neighborhood of $F_0 \cup F_2$. Hence, $\operatorname{Per}(a_t b)$ and $\operatorname{Per}(d_s c)$ are disjoint, as desired. ☐

Our next goal is to propagate $S_k(\cdot,\cdot)$ to other curves. For this, we define two stronger properties.

Definition 5.14 (Strengthenings of S_k)**.** We say that two curves a and b satisfy $S_k^+(a,b)$ if they satisfy $S_k(a,b)$ and if additionally, we have $a(\operatorname{Per}(b)) \cap \operatorname{Per}(b) = \varnothing$. We say that a and b satisfy $S_k^{++}(a,b)$ if they satisfy both $S_k^+(a,b)$ and $S_k^+(b,a)$.

Property $S_k^+(\cdot,\cdot)$ is what allows one to move families of periodic points continuously by twist deformations, as described in the following lemma.

Lemma 5.15 (Periodic points move continuously)**.** *Let* a *and* b *be any curves with* $i(a,b) = -1$ *satisfying* $S_k^+(a,b)$*. Then there exists a continuous family* a_t *commuting with* a *such that* $\mathrm{Per}(a_t b) \cap \mathrm{Per}(a_s b) = \varnothing$ *for all* $s \neq t$*, and* $|\mathrm{Per}(a_t b)| = 2k$ *for all* t*.*

Since property $S_k(a,b)$ immediately implies that $\mathrm{Per}(b) \subset U(a,b)$, the nontrivial part of this lemma is controlling the cardinality of $\mathrm{Per}(a_t b)$ at all times t. This requires a special choice of one-parameter family a_t, which we construct by hand.

Proof of Lemma 5.15. Together with Lemma 4.8, the asumption $a\mathrm{Per}(b) \cap \mathrm{Per}(b) = \varnothing$ completely prescribes the cyclic order on the set $\bigcup_n a^n(\mathrm{Per}(b))$; it follows that we may choose a neighborhood V of $\mathrm{Per}(b)$, consisting of $2k$ open intervals, such that $a^n(V) \cap a^m(V) = \varnothing$ for all $n, m \in \mathbb{Z}$.

We now construct a continuous family of homeomorphisms a_t commuting with a, supported on $\bigcup_{n\in\mathbb{Z}} a^n V$. A slight variation on this construction would give a *positive* family of homeomorphisms, but this is not required by the lemma.

Choose one point in each of the periodic orbits of b; let $x_1, x_2, \ldots, x_m$ denote these points. We may parametrize S^1 so that, for each x_i, b agrees with a rigid rotation by $p(b)/q(b)$ on a small neighborhood of $b^k(x_i)$ for $k = 0, 1, \ldots, q(b) - 2$ and so that b maps a neighborhood of $b^{q(b)-1}(x_i)$ to a neighborhood of $x_i = b^{q(b)}(x_i)$ by the map $x \mapsto 2x$ or $x \mapsto x/2$, in coordinates, depending on whether the orbit of x_i is repelling or attracting.

Let $V_{i,k}$ denote the connected component of V containing $b^k(x_i)$. Note that, by construction, these sets partition V. Define a_t to be the identity on $V_{i,k}$ for $k = 0, 1, \ldots, q(b) - 2$ and all i. To define a_t on $V_{i,q(b)-1}$ we proceed as follows. Let U_i be a neighborhood of $b^{q(b)-1}(x_i)$ whose closure is contained in $V_{i,q(b)-1}$. Using the local coordinates in which b is linear, define a_t to agree with the translation $x \mapsto x+t$ on U_i. Since U_i has closure contained in $V_{i,q-1}$, there exists $\varepsilon > 0$ so that for all $t \leqslant \varepsilon$, this map extends to a homeomorphism of $V_{i,q(b)-1}$ fixing the endpoints, and we may take these extensions to vary continuously in t. From now on, we restrict to such $t \leqslant \varepsilon$. Finally, we extend a_t to a family of homeomorphisms of S^1 that is equivariant with respect to a on $\bigcup_{n\in\mathbb{Z}} a^n U$, where $U = \bigcup_i U_i$, and agrees with the identity elsewhere.

The property that a_t agrees with $x \mapsto x + t$ on U_i implies that $(a_t b)^{q(b)}$ has a unique fixed point in each set $b^{-n} U_i$ for $n = 0, 1, 2, \ldots, q(b) - 1$. Since the union of such sets open $b^{-n} U_i$ covers $\mathrm{Per}(b)$, there is some $\delta > 0$ such that $|b^{q(b)}(x) - x| \geqslant \delta$ for all $x \in S^1 \smallsetminus U$, hence for all t sufficiently small, we will have $\mathrm{Per}(a_t b) \subset U$. ☐

Another function of property S_k^+ is given by the following lemma.

Lemma 5.16. *Let* (a, b, c) *be a completable* 3*-chain. Then* $S_k^+(a,b)$ *implies* $S_k(b,c)$*.*

As hinted by this statement, the stronger Property S_k^{++} can actually be propagated along chains, as follows.

Proposition 5.17. *Let (a, b, c) be a completable 3-chain. Suppose that $S_k^{++}(a, b)$ holds. Then $S_k^{++}(b, c)$ holds as well.*

To prove these two statements, we will need a quick sub-lemma.

Lemma 5.18 (Per has empty interior)**.** *Let a and b be any curves with $i(a, b) = \pm 1$, and let b_t be a positive one-parameter family commuting with b. Then, for all but countably many t, the set $\mathrm{Per}(b_t a)$ has empty interior.*

Proof. Let $X = S^1 \smallsetminus P(b, a)$. Then for $t \neq s$, we have $\mathrm{Per}(b_t a) \cap \mathrm{Per}(b_s a) \cap X = \varnothing$. In particular, the set

$$T = \{t : \mathrm{Per}(b_t a) \cap X \text{ contains a nonempty open set}\}$$

is countable. Note also that if $\mathrm{Per}(b_t a)$ contains a nonempty open set U, then $U \cap X = U \smallsetminus P(b, a)$ is open and nonempty, since $P(b, a)$ is closed with empty interior, hence $t \in T$. It follows that for all $t \notin T$, $\mathrm{Per}(b_t a)$ has empty interior. □

Proof of Lemma 5.16. Complete (a, b, c) to a 4-chain (a, b, c, d), and let $(d_t)_{t\in\mathbb{R}}$ be a positive one-parameter family commuting with d. By Lemma 5.18, $\mathrm{Per}(d_{t_0} c)$ has empty interior for some $t_0 \in \mathbb{R}$. Now, by Lemma 5.15, there exists a one-parameter group $(a_s)_{s\in\mathbb{R}}$, an interval $I \subset \mathbb{R}$ and $2k$ maps, $\phi_j : I \to S^1$, each homeomorphism to its image, such that the $2k$ periodic points of $\mathrm{Per}(a_s b)$ are precisely $\phi_1(s), \ldots, \phi_{2k}(s)$, for all $s \in I$. The set $\bigcap \phi_j^{-1}(\mathrm{Per}(d_{t_0} c))$ then has empty interior in I, hence there exists $s_0 \in I$ such that $\mathrm{Per}(a_{s_0} b) \cap \mathrm{Per}(d_{t_0} c) = \varnothing$, and we conclude that $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$ by Lemma 2.30. We conclude by using Theorem 4.2. □

Proof of Proposition 5.17. Complete the 3-chain into a 5-chain, (e, a, b, c, d), and apply Lemma 5.16 to the 3-chains (a, b, c) and (e, a, b) to conclude $S_k(b, c)$ and $S_k(a, e)$. By Lemma 3.8, we may then use a bending deformation of a along e to move the periodic set of a disjoint from any finite set, so in particular $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. Now take a positive one-parameter family a_t commuting with a. Since $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$ the points $a_{-t}\mathrm{Per}(c)$ move continuously in t, so there is some t such that $b\mathrm{Per}(c) \cap a_{-t}\mathrm{Per}(c) = \varnothing$. Thus, $a_t b\mathrm{Per}(c) \cap \mathrm{Per}(c) = \varnothing$ hence by Lemma 2.30 $b\mathrm{Per}(c) \cap \mathrm{Per}(c) = \varnothing$. Thus, we conclude that $S_k^+(b, c)$ holds. By Lemma 5.16, this implies that $S_k(c, d)$ holds as well. In particular, $\mathrm{Per}(d)$ is finite. We can now apply Lemma 3.8 and use a bending deformation so that $\mathrm{Per}(a_t b) \cap \mathrm{Per}(d) = \varnothing$, which implies that $\mathrm{Per}(b) \cap \mathrm{Per}(d) = \varnothing$, and repeat the argument above (with d and c playing the roles of a and b) to conclude $S_k^+(c, b)$ holds as well. □

Proposition 5.17, Theorem 3.3, and the connectedness of the graph in Lemma 2.11 immediately gives the following.

Corollary 5.19. *Let ρ be a path-rigid, minimal representation, and suppose there exists (a, b) such that $S_k^{++}(a, b)$ holds. Then ρ is geometric.*

This consequence is strong enough to imply the main result of the companion article [17]. We explain this now, as it will be used again in Section 6.

Corollary 5.20. *Let ρ be a path-rigid, minimal representation, and suppose that there is some torus $T(a,b)$ such that the relative Euler number of $T(a,b)$ is ± 1. Then ρ is semi-conjugate to a Fuchsian representation.*

Proof. Since $T(a,b)$ has Euler number 1, it follows form [21] that the restriction of ρ to $\langle a,b\rangle$ is semi-conjugate to a geometric representation in $\mathrm{PSL}(2,\mathbb{R})$. (This is not difficult: that $\widetilde{\mathrm{rot}}([\widehat{\rho(a)},\widehat{\rho(b)}] = \pm 1$ easily implies that $\rho(a)$ and $\rho(b)$ are 1-Schottky, hence as in Proposition 3.22 are semi-conjugate to a geometric representation in $\mathrm{PSL}(2,\mathbb{R})$. See the beginning of §3 in [21].) In particular, property $S_1^{++}(a,b)$ holds. It follows from Corollary 5.19 that ρ is geometric. $\square$

Given Corollary 5.19, to achieve our main goal of this section, we need only show the following.

Proposition 5.21. *Let (a,b,c,d,e) be a 5-chain, and suppose that $P(a,b) = P(e,d) = \varnothing$. Then we have $S_k^{++}(b,c)$.*

Proof. Suppose $P(a,b) = P(e,d) = \varnothing$. By Proposition 5.12, we have $S_k(b,c)$ and $S_k(c,d)$ for some $k \geqslant 1$. Since $P(e,d) = \varnothing$ and $\mathrm{Per}(b)$ is finite, we have a bending deformation $e_t d$ such that $\mathrm{Per}(b) \cap \mathrm{Per}(e_t d) = \varnothing$, hence $\mathrm{Per}(b) \cap \mathrm{Per}(d) = \varnothing$. Hence, $\mathrm{Per}(b) \cap d_t c\mathrm{Per}(b) = \varnothing$ for some t, so we have $\mathrm{Per}(b) \cap c\mathrm{Per}(b) = \varnothing$, ie, $S_k^+(c,b)$ holds. By Lemma 5.16, this gives $S_k(a,b)$. In particular, $\mathrm{Per}(a)$ is finite, and so there exists a bending deformation replacing c with $d_t c$ such that $\mathrm{Per}(a) \cap \mathrm{Per}(d_t c) = \varnothing$, and hence $\mathrm{Per}(a) \cap \mathrm{Per}(c) = \varnothing$. Repeating the argument above, we conclude $S_k^+(b,c)$ holds. $\square$

The main result of this section is now a quick corollary. We restate it here for convenience and to summarize our work.

Corollary 5.22. *Let ρ be a path-rigid, minimal representation. Suppose ρ admits two disjoint good tori that are not very good. Then ρ is geometric.*

Proof. Let $T(a,b)$ and $T(d,e)$ be these two disjoint good tori. Since they are good, we may suppose $\mathrm{rot}(a) = \mathrm{rot}(e) = 0$. Since they are not very good, we have $P(a,b) = \varnothing$ and $P(e,d) = \varnothing$. We may find a curve c such that (a,b,c,d,e) is a 5-chain, and then Proposition 5.21 and Corollary 5.19 imply that ρ is geometric. $\square$

5.3. **Finite orbits.**

The goal of this section is the proof of the following proposition.

Proposition 5.23. *Let $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be a path-rigid representation, and let $\Sigma = \Sigma_{g-1,1}$ be a subsurface containing only very good tori. Then $\rho_{|\pi_1\Sigma}$ has a finite orbit.*

If $T(a,b)$ is very good, then a and b act with a finite orbit, so $\mathrm{rot}(ab) = \mathrm{rot}(a) + \mathrm{rot}(b)$. Thus, in a subsurface where all tori are very good, rotation number is additive on any pair of curves with intersection number ± 1. This motivates the following proposition, which gives our first step.

Proposition 5.24. *Let Σ be a one-holed surface of genus $\geqslant 2$. We suppose that $\pi_1\Sigma$ acts on the circle in such a way that all nonseparating simple curves* *have rational rotation number, and that for all* γ_1, γ_2 *with* $i(\gamma_1, \gamma_2) = \pm 1$, *we have* $\mathrm{rot}(\gamma_1\gamma_2) = \mathrm{rot}(\gamma_1) + \mathrm{rot}(\gamma_2)$.

Then, there exist two curves γ_1, γ_2 *with* $i(\gamma_1, \gamma_2) = \pm 1$ *and* $\mathrm{rot}(\gamma_1) = \mathrm{rot}(\gamma_2) = 0$.

Proof. Let $(a_1, \dots, b_g)$ be a standard generating set of $\pi_1\Sigma$, ie, where the loop $[a_1, b_1] \cdots [a_g, b_g]$ is homotopic to the boundary, and consider the non-completable directed 5-chain $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (a_1^{-1}b_1a_1, a_1, \delta_1, a_2, b_2^{-1})$, with the notation of Section 2.1. Our proof proceeds by iteratively applying Dehn twists in this system of curves.

Recall from Convention 2.15 that the effect of a Dehn twist along γ_i replaces γ_{i-1} (if $i \geqslant 2$) with $\gamma_i^{-1}\gamma_{i-1}$, and replaces γ_{i+1} with $\gamma_{i+1}\gamma_i$ (if $i \leqslant 4$), while leaving the other curves unchanged. By the asumption of the proposition, and since $i(\gamma_i, \gamma_{i+1}) = 1$ for each i, the Dehn twist τ_i along γ_i changes the vector $(r_1, r_2, r_3, r_4, r_5)$ of rotation numbers of the γ_i, by replacing r_{i-1} with $r_{i-1} - r_i$ and replacing r_{i+1} with $r_{i+1} + r_i$. This Dehn twist also changes the standard generating system $(a_1, \dots, b_g)$ into a new generating system (with a new chain $(\gamma_1, \dots, \gamma_5)$ with rotation numbers changed as above), in which we may perform new Dehn twists and iterate this process.

In other words, Proposition 5.24 amounts to proving that the operations $\tau_i \colon (r_1, \dots, r_5) \mapsto (r'_1, \dots, r'_5)$ with $r'_{i-1} = r_{i-1} - r_i$ and $r'_{i+1} = r_{i+1} + r_i$, and $r'_j = r_j$ otherwise, can be iterated to transform any vector in $(\mathbb{Q}/\mathbb{Z})^5$ to a vector of the form $(0, 0, r_3, r_4, r_5)$. This should seem well known to any reader familiar with the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$, but we give the details anyway.

Start with a vector $(r_1, \dots, r_5)$, and lift it (without change in notation) to $\mathbb{Q}$. Since the τ_i are linear operations, we may suppose instead that r_1, r_2, r_3, r_4, r_5 are integers with greatest common divisor 1. We use the notation $x \wedge y$ to denote the greatest common divisor of x and y.

Using only τ_4 and τ_5 and their inverses, we can apply the Euclidean algorithm to r_4 and r_5, changing $(r_1, r_2, r_3, r_4, r_5)$ into $(r_1, r_2, r_3 + N, r_4 \wedge r_5, 0)$, where N is some multiple of $r_4 \wedge r_5$. By applying again a power of τ_4, we get to $(r_1, r_2, r_3, r_4 \wedge r_5, N)$. Similarly, by applying a sequence of τ_3 and τ_4, and then some power of τ_3 to correct the effect on r_2, we can arrive at a vector of the form $(r_1, r_2, r_3 \wedge r_4 \wedge r_5, \star, \star)$. We then do the same operation on the second and third entries of the vector using τ_3 and τ_2, and again finally on the first and second via τ_2 and τ_1. After these operations, the new vector is of the form $(1, r'_2, r'_3, r'_4, r'_5)$, and after using a power of τ_1, we get the vector $(1, 1, r'_3, r'_4, r'_5)$.

Applying τ_3 maps this vector to $(1, 1 - r'_3, r'_3, r'_4 + r'_3, r'_5)$. We can then apply $\tau_1^{r'_3}$, to get $(1, 1, r'_3, r'_4 + r'_3, r'_5)$. Let us call τ'_3 this combination. Then, the operations τ'_3 and τ_4 enable to run the Euclidean algorithm on r'_3 and r'_4, without changing first two entries of the vector. This allows us to arrive at a vector of the form $(1, 1, 0, n, m)$. Now we apply τ_2, getting $(0, 1, 1, n, m)$, and then τ_3, getting $(0, 0, 1, n + 1, m)$. □

Our interest in Proposition 5.24 is that it is much easier to keep track of the dynamics of two curves if their rotation numbers are zero. In this case, we do not need a condition as strong as $\mathrm{Per}(a) \cap \mathrm{Per}(b) = \varnothing$ in order to control the fixed points as in Lemma 5.15. More precisely, we have the following statement.

Proposition 5.25. *Suppose* $\mathrm{rot}(a) = \mathrm{rot}(b) = 0$. *Then for every* $\varepsilon > 0$, *there exists a one-parameter family* $(a_t)_{t \in \mathbb{R}}$ *commuting with* a, *an interval* $J \subset \mathbb{R}$, *and a finite collection of homeomorphisms* $\phi_i : J \to S^1$ *with disjoint images, such that for all* $t \in J$,

$$\mathrm{Fix}(a_t b) \cap \big(S^1 \smallsetminus V_\varepsilon(P(a,b))\big) = \{\phi_1(t), \cdots, \phi_n(t)\}.$$

In other words, for all $t \in J$, the fixed points of $a_t b$ at distance $\geqslant \varepsilon$ to $P(a,b)$ are finite in number and move continuously in t. Note that, in the statement above, we do not require a_t to be a *positive* family.

Proof. Fix a positive one-parameter family α_t commuting with a. We will modify α_t to obtain the desired family a_t.

When $\mathrm{rot}(a) = \mathrm{rot}(b) = 0$, we have $P(a,b) = \mathrm{Fix}(b) \cap \partial \mathrm{Fix}(a)$, and the set $U(a,b)$ has a very simple description: $x \in U(a,b)$ if and only if x and $b(x)$ are in the same connected component of $S^1 \smallsetminus \partial\mathrm{Fix}(a)$. Thus, $U(a,b) = \bigcup_I (I \cap b^{-1}(I))$, where I ranges over the connected components of $S^1 \smallsetminus \partial\mathrm{Fix}(a)$. As each connected component I is a-invariant, we may define a_t separately on each, affecting only $\mathrm{Fix}(a_t b) \cap I$.

For every connected component I of $S^1 \smallsetminus \partial\mathrm{Fix}(a)$, let $U(I)$ denote $I \cap b^{-1}(I)$. By definition, each endpoint of $U(I)$ lies in $\partial N(a,b) \cup P(a,b)$. Thus, by Lemma 2.19, all but finitely many intervals $U(I)$ lie in $V_\varepsilon(P(a,b))$. On all the corresponding connected components I of $S^1 \smallsetminus \partial\mathrm{Fix}(a)$, there is nothing to worry about, and we set $a_t = \alpha_t$.

Now we treat the remaining (finitely many) intervals I of $S^1 \smallsetminus \mathrm{Fix}(a)$ such that $U(I)$ is nonempty, considering the configuration of I and $b^{-1}(I)$. As a first case, suppose that I and $b^{-1}(I)$ share an endpoint, i.e. a point in $P(a,b)$. If this is the right endpoint, define $a_t = \alpha_t$ on I. If the left endpoint is shared, take instead $a_t = \alpha_{-t}$. If $I = b(I)$, either choice will work. In each case, for all s sufficiently large, we have

$$\mathrm{Fix}(a_s b) \cap I \subset V_\varepsilon(P(a,b)). \tag{5.1}$$

As a second case, suppose b shifts I. If the shift is to the right, i.e. $I = (x_1, x_3)$ and $b(I) = (x_2, x_4)$ with x_1, x_2, x_3, x_4 in cyclic order, define $a_t = \alpha_t$ on I, and if the shift is to the left, set $a_t = \alpha_{-t}$. In either case, for all s sufficiently large, we have

$$\mathrm{Fix}(a_s b) \cap I = \varnothing. \tag{5.2}$$

We are left with the case where either $b(\bar{I}) \subset I$ or $\bar{I} \subset b(I)$. Suppose the first holds, as the second can be dealt with by a symmetric argument. Note that (using α_t and b) we are in the case $n = 1$ of Lemma 4.9 of the preceding section. Thus, there exists $s \in \mathbb{R}$ such that $\alpha_s b$ has a unique fixed point in I. Moreover, $b(\bar{I}) \subset I$ implies that this unique fixed point is an attracting point, i.e. we may take local coordinates so that the map $\alpha_s b$ agrees with $x \mapsto x/2$ at the origin. After reparametrization of α_t on I, we may assume that this time s is sufficiently large to satisfy (5.1) and (5.2) above. Working in coordinates, let $(-\delta, \delta)$ be a neighborhood of 0 contained in a fundamental domain for a. Let τ_t be a smooth family of bump functions supported on

$(-\delta, \delta)$ and agreeing with $x \mapsto x + t$ on an even smaller (fixed) neighborhood of 0, for all $t < \delta' < \delta$. Extend τ_t a-equivariantly to a homeomorphism of I Now define a_t on I to agree with α_t for $t < s$, to agree with $\tau_{t-s}\alpha_s$ for $s \leqslant t \leqslant s + \delta'$, and arbitrarily (for example, constant in t) for $t \geqslant s + \delta'$. Varying t in $J := (s, s+\delta')$, the homeomorphism $a_t b$ has a unique fixed point in I that moves continuously with t, as desired. Of course, we can choose parameterizations of a_t on each of these (finitely many) intervals so that J does not depend on I. This proves the lemma. □

Using this tool, we can propagate finite orbits over chains.

Proposition 5.26. *Let* $a, \gamma_1, \gamma_2, \gamma_3, ...\gamma_k$ *be a chain. Suppose that* $\mathrm{Per}(a)$ *has empty interior,* $\mathrm{rot}(\gamma_i) = 0$ *for all* i*, the subgroup* $\langle a, \gamma_1\rangle$ *has a finite orbit and* $\langle \gamma_i, \gamma_{i+1}\rangle$ *has a global fixed point. Then* $\langle a, \gamma_i, ..., \gamma_k\rangle$ *has a finite orbit.*

Proof. Inductively, suppose the statement holds for chains of length k and take a chain of length $k+1$ of the form $a, \gamma_1, ..., \gamma_k$. By inductive hypothesis the group generated by the first k elements $\langle a, \gamma_1, ..., \gamma_{k-1}\rangle$ has a finite orbit, i.e. there is a periodic orbit of a contained in $\bigcap_{i=1}^{k-1} \mathrm{Fix}(\gamma_i)$.

Since $\mathrm{Per}(a)$ has empty interior, for any $n \in \mathbb{N}$, we can use Proposition 5.25 to produce a homeomorphism $c(n)$ lying in a one-parameter family commuting with γ_k such that $\mathrm{Fix}(c(n)\gamma_{k-1}) \cap \mathrm{Per}(a) \subset V_{1/n}(P(\gamma_{k-1}, \gamma_k))$. Indeed, with the notation of that proposition, there exists $t \in J$ such that $\phi_j(t) \notin \mathrm{Per}(a)$ for all j, because $\bigcap_j \phi_j^{-1}(\mathrm{Per}(a))$ has empty interior in J. Do this for each $n \in \mathbb{N}$; we do not require that the $c(n)$ all belong to a common one-parameter family, all that is important is that they are each obtainable by a bending deformation, hence give a semi-conjugate representation.

The result is a sequence of bending deformations $c(n)\gamma_{k-1}$ of γ_{k-1} such that

$$\mathrm{Fix}(c(n)\gamma_{k-1}) \cap \mathrm{Per}(a) \subset V_{1/n}(\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)).$$

Since $\langle a, \gamma_1, ..., \gamma_{k-1}\rangle$ has a finite orbit, and this property is stable under semi-conjugacy, it follows that, for every n, $\bigcap_{i=1}^{k-2} \mathrm{Fix}(\gamma_i) \cap \mathrm{Fix}(c(n)\gamma_{k-1})$ contains a full orbit of a. For each n, choose one such full orbit, and denote it by $\mathcal{O}_n$. After passing to a subsequence, the sets $\mathcal{O}_n$ converge pointwise to a finite subset of $\bigcap_{i=1}^{k-2} \mathrm{Fix}(\gamma_i) \cap \mathrm{Per}(a)$ that is invariant under a (as these are both closed conditions) so the limit is a full orbit. Moreover, this orbit is contained in every open neighborhood of $\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)$, so also lies in $\mathrm{Fix}(\gamma_{k-1}) \cap \mathrm{Fix}(\gamma_k)$. This gives a periodic orbit of a in $\bigcap_{i=1}^{k} \mathrm{Fix}(\gamma_i)$, as desired. □

We can now prove the main result advertised at the beginning of this section.

Proof of proposition 5.23. Let $\Sigma_{g,1}$ be a surface with one boundary component, in which all tori are very good. Recall that our goal is to show that ρ has a finite orbit. Since all tori are very good, we may use Proposition 5.24 to find a standard system of generators $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$ where $\mathrm{rot}(a_i) = \mathrm{rot}(b_i) = 0$ for all $i = 2, 3, \ldots, g-1$. Since $T(a_1, b_1)$ is good, we may also assume that $\mathrm{rot}(b_1) = 0$.

Let $\delta_i = a_{i+1}^{-1} b_{i+1} a_{i+1} b_i^{-1}$ as in Section 2.1, so that $(a_1, \delta_1, a_2, \delta_2, \dots \delta_{g-2}, a_{g-1}, b_{g-1})$ forms a chain. For each i, we can use Lemma 5.18 in order to assume without loss of generality that $\mathrm{Per}(\delta_i)$ has empty interior, and then apply Proposition 5.26 to the chain (δ_i, a_i, b_i). It follows that $\langle \delta_i, b_i \rangle$ has a finite orbit, hence

$$\mathrm{rot}(\delta_i) + \mathrm{rot}(b_i) = \mathrm{rot}(a_{i+1}^{-1} b_{i+1} a_{i+1}) = \mathrm{rot}(b_{i+1}).$$

Thus, $\mathrm{rot}(\delta_i) = 0$ for all i.

Lemma 5.18 implies that, after a deformation, we may assume that $\mathrm{Per}(a_1)$ has empty interior. Thus, we can apply Proposition 5.26 to the chain $(a_1, \delta_1, a_2, \delta_2, \dots \delta_{g-2}, a_{g-1}, b_{g-1})$ to conclude that the subgroup generated by these elements has a finite orbit. As this subgroup is equal to $\pi_1(\Sigma_{g-1,1})$, this proves the proposition. □

5.4. **Proof of Theorem 1.6.** Theorem 1.6 is now a quick consequence of Proposition 5.23 and Corollary 5.20.

Proof of Theorem 1.6. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Homeo}_+(S^1)$ be a path-rigid representation, and suppose that ρ is not geometric. If Σ contains a bad torus T, then by Proposition 1.12, $\Sigma \setminus T$ contains only very good tori. If Σ contains no bad torus, but some torus T' that is not very good, then Proposition 1.12 implies that $\Sigma \setminus T'$ contains only very good tori. In either case, there is a genus $g-1$ subsurface $\Sigma_{g-1,1}$ containing only very good tori, hence by Proposition 5.23 the restriction of ρ to $\Sigma_{g-1,1}$ has a finite orbit. In particular, the boundary curve of this subsurface has zero rotation number, and the restriction of ρ to this subsurface has relative Euler number zero.

It follows that the Euler number of the remaining (not very good) torus is either 0 or ±1. By Corollary 5.20, if it is ±1, then ρ is geometric. Thus, the remaining torus has Euler number 0, and by additivity the Euler number of ρ is zero. □

We conclude by noting that if Σ has only very good tori, then the proof of Proposition 5.23 actually shows that ρ has a finite orbit (hence automatically Euler number zero).

6. PROOF OF THEOREM 1.5 AND LAST COMMENTS

6.1. **Proof of Theorem 1.5.** Here is where we use the stronger hypothesis of rigidity, instead of path-rigidity. Our proof relies on the following observation, that was hinted to us by work in the recent article [1].

Lemma 6.1. *Let ρ be a rigid, minimal representation. Let $T = T(a,b)$ be a very good torus. Then only finitely many points of S^1 have a finite orbit under $\langle a, b \rangle$. In particular, if* $\mathrm{rot}(a) = 0$ *then $P(a,b)$ is a finite set.*

This observation is the *only* place where we use rigidity, rather than the weaker path-rigidity, in our proof.

Proof. Let $F(a,b)$ denote the set of points whose orbit under $\langle a, b \rangle$ is finite. To simplify the exposition of the proof, fix a metric on S^1 so that a and b act on $F(a,b)$ by rigid rotations. Given any $\varepsilon > 0$, let $J_1, J_2, \dots$ denote the (finitely many) connected components of $S^1 \setminus F(a,b)$ consisting of intervals of length greater than ε (by our choice of metric, this is a $\langle a, b \rangle$-invariant set).

If $F(a,b)$ is finite, and ε small enough, then $\bigcup_i \overline{J_i} = S^1$. Otherwise (even in the case where $\bigcup_i \overline{J_i} = \varnothing$), we may divide $S^1 \smallsetminus \bigcup_i \overline{J_i}$ into finitely many disjoint open intervals $I_1, I_2, \ldots$ each of length at most ε and with endpoints in $F(a,b)$, such that these intervals are permuted by $\langle a,b\rangle$, and such that $S^1 = (\bigcup_i \overline{J_i}) \cup (\bigcup_i \overline{I_i})$.

Since T is very good, we can suppose without loss of generality that $\mathrm{rot}(a) = 0$. We claim that there exist $a', b' \in \mathrm{Homeo}^+(S^1)$, agreeing with a and b on $S^1 \smallsetminus \bigcup_i I_i$, such that $[a',b'] = [a,b]$ holds globally, and such that $\mathrm{Per}(b') \cap \bigcup I_i = \varnothing$.

Let $c = [a,b]$. As $\bigcup_i I_i$ is a,b-invariant, constructing a' and b' amounts to solving the equation $b'c = a'^{-1}ba'$ on $\bigcup_i J_i$. That this can be solved is shown in [5, Lemma 2.7]; as their notation and context is slightly different, we explain the strategy. Take coordinates identifying each J_i with $\mathbb{R}$. If b' is defined on some J_i (with image in J_j) to increase sufficiently quickly (as a homomorphism $\mathbb{R} \to \mathbb{R}$), then $b'c$ will also be strictly increasing, hence conjugate to b'. One then defines a' to be this conjugacy.

Let ρ' be the representation obtained from ρ by replacing (a,b) by (a',b'). As $\varepsilon > 0$ is arbitrary, this ρ' can be taken arbitrarily close to ρ, in $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$. Since ρ is assumed rigid, for small enough ε, ρ' must be semi-conjugate to ρ. As ρ was supposed to be minimal, then there is a *continuous* semi-conjugacy $h : S^1 \to S^1$ such that $h \circ \rho' = \rho \circ h$. Let

$$F' := \{x \in S^1 \mid x \text{ has finite orbit under } \langle \rho'(a), \rho'(b)\rangle\}.$$

By construction of ρ', this set is finite. However, $h(F') = F(a,b)$. It follows that $F(a,b)$ was finite as well. □

To conclude the proof of Theorem 1.5, let ρ be a rigid, minimal representation, and assume for contradiction that ρ is non-geometric. As a consequence of Lemma 6.1, Proposition 5.8 and Lemma 5.11, ρ cannot have bad tori. In order to derive a contradiction, we will show that all good tori are actually very good. We pursue this in the spirit of Proposition 5.12.

Lemma 6.2. *Suppose* $P(a,b) = \varnothing$. *Then* $\partial N(a,b) \subset \partial\mathrm{Per}(a) \cup b^{-1}(\partial\mathrm{Per}(a))$.

Proof. Assume $P(a,b) = \varnothing$ and let $x \in \partial N(a,b)$. Since $P(a,b) = \varnothing$, the set $N(a,b)$ is closed, hence $x \in N(a,b) \cap \overline{U(a,b)}$.

Suppose that $x \notin (\partial\mathrm{Per}(a) \cup b^{-1}(\partial\mathrm{Per}(a))$. Then, there exists two intervals, I, J, neighborhoods of x, with $I \subset S^1 \smallsetminus \partial\mathrm{Per}(a)$ and $J \subset S^1 \smallsetminus b^{-1}(\partial\mathrm{Per}(a))$. As $x \in \overline{U(a,b)}$, there exists $u \in U(a,b) \cap I \cap J$. Let a_t be a positive one-parameter family commuting with a. Since $b(J)$ contains $b(x)$ and $b(u)$ and $b(J) \cap \partial\mathrm{Per}(a) = \varnothing$, there exists $t_0 \in \mathbb{R}$ such that $a_{t_0}b(x) = b(u)$. Similarly, there exists $t_1 \in \mathbb{R}$ such that $a_{t_1}(u) = x$. Thus, $\Delta_{a,b}(x, t_1 + T(u), T(u), \ldots, T(u), T(u) + t_0) = 0$, and it now follows easily that $x \in U(a,b)$. This proves the lemma. □

Lemma 6.3. *Suppose* $\mathrm{rot}(a) = 0$ *and suppose* $\langle a,b\rangle$ *has no finite orbit. Choose a positive one-parameter group* b_t *commuting with* b. *Then for all* $x \in S^1$, *there exist at most two values of* t *such that* $x \in \partial N(b_t a, b)$.

Proof. Since $\langle a,b\rangle$ has no finite orbit, $P(a,b) = \varnothing$ and hence $P(b_t a, b) = \varnothing$ for all t. Let $x \in S^1$; we will apply Lemma 6.2 to the pairs $(b_t a, b)$. If

$x \in \mathrm{Per}(b)$, then $x \notin N(b_t a, b)$, and in particular $x \notin \partial N(b_t a, b)$ for all $t \in \mathbb{R}$. Thus, suppose $x \notin \mathrm{Per}(b)$.

By Lemma 6.2, if $x \in \partial N(b_t a, b)$, then $x \in \partial \mathrm{Per}(b_t a) \cup b^{-1}(\partial \mathrm{Per}(b_t a))$. Note that x cannot be in $P(b, a)$, as $x \notin \mathrm{Per}(b)$. Hence, if there exists some $t \in \mathbb{R}$ such that $x \in \mathrm{Per}(b_t a)$, then $x \in U(b, a)$, and this t is unique. Similarly, if there exists some $t \in \mathbb{R}$ such that $b(x) \in \mathrm{Per}(b_t a)$, then $b(x) \in U(b, a)$, and this t is unique. This concludes the proof. ☐

Using these tools, we will now show that ρ (always assumed rigid and minimal) satisfies hypothesis S_k. We divide the first part of this proof into two lemmas.

Lemma 6.4. *Let (a, b, c, d) be a 4-chain, and suppose $\mathrm{rot}(a) = \mathrm{rot}(d) = 0$ holds. Suppose that $T(a, b)$ is good but not very good. Then we have $S_k(b, c)$.*

Proof. By Lemma 6.1, the set $P(d, c)$ is finite, and using Lemma 6.3, we can first deform a, to some $b_t a$, so that $\partial N(a, b)$ does not intersect $P(d, c)$. Then by Lemma 5.13, we have $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$. ☐

Lemma 6.5. *Let (a, b, c, d) be a 4-chain, and suppose $S_k(a, b)$ and $\mathrm{rot}(d) = 0$ hold. Then we have $S_k(b, c)$.*

Proof. By Lemma 6.1, the set $P(d, c)$ is finite. By Lemma 3.8 in the torus $T(a, b)$, the set $\mathrm{Per}(b)$ is disjoint from $P(d, c)$.

Hence, $\mathrm{Per}(b) \subset U(d, c) \cup N(d, c)$, and $\mathrm{Per}(b)$ is finite. Thus, for all but finitely many t, we have $\mathrm{Per}(b) \cap \mathrm{Per}(d_t c) = \varnothing$. Hence $\mathrm{Per}(b) \cap \mathrm{Per}(c) = \varnothing$ by Lemma 2.30. ☐

Now we can complete the proof of the Theorem.

Proof of Theorem 1.5. Let ρ be a rigid, minimal representation. As we said above, ρ does not admit any bad torus. If all tori are very good, then as in the proof of Theorem 1.6, we know that ρ admits a finite orbit, a contradiction.

Thus, ρ admits a good torus, $T(a, b)$, which is not very good. We may suppose $\mathrm{rot}(a) = 0$. As all tori are good, we may choose a curve d outside $T(a, b)$ with $\mathrm{rot}(d) = 0$, and we may form a 4-chain (a, b, c, d). By Lemma 6.4, we have $S_k(b, c)$ for some k.

Now rename (b, c) into (a, b), and forget about the other curves, remembering only that we have two curves a, b with $S_k(a, b)$. Since all tori are good, we may choose a curve d outside $T(a, b)$ such that $\mathrm{rot}(d) = 0$, and such that there exists a standard generating system beginning with (a, b, d, γ) Define $u = \gamma a^{-1} b^{-1} a$ and $v = \gamma a^{-1}$. Then (u, a, b, v), (d, u, a, b) and (a, b, v, d) are 4-chains (we encourage the reader to refer to Figure 1 and draw these curves u and v for him/herself). Apply Lemma 6.5 to the 4-chain (a, b, v, d). This proves that $S_k(b, v)$ holds. The same lemma applied to the 4 chain (d, u, a, b) implies $S_k(u, a)$. Hence, the 4-chain (u, a, b, v) satisfies $S_k(u, a)$, $S_k(a, b)$ and $S_k(b, v)$. We can deform a along u, thanks to Lemma 3.8, in such a way that $\mathrm{Per}(a) \cap \mathrm{Per}(v) = \varnothing$, hence we have $S_k^+(b, a)$, and we can deform b along v, in such a way that $\mathrm{Per}(b) \cap \mathrm{Per}(u) = \varnothing$, hence we have $S_k^+(a, b)$. Finally, this proves $S_k^{++}(a, b)$, and thus ρ is geometric by Corollary 5.19. ☐

6.2. **Comments and further questions.** We conclude this paper by discussing some natural questions and directions for further work.

6.2.1. *Path-rigidity.* Given Theorem 1.6, we expect that path-rigiditity should suffice to imply that a representation is geometric. The most obvious route to this result would be through an improvement of Lemma 6.1, as it is the only place where we use the stronger hypothesis of rigidity.

Question 6.6. *Does Lemma 6.1 hold when "rigid" is replaced by "path-rigid"?*

This question also arises naturally out of the work of Alonso–Brum–Rivas in [1], which served as our inspiration for Lemma 6.1. Their main result is the following.

Theorem 6.7 (Alonso–Brum–Rivas [1]). *Let* $\rho \in \mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ *or in* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(\mathbb{R}))$*. In any neighborhood* U *of* ρ*, there exists a representation* ρ' *without global fixed points.*

Since it is unknown whether these representation spaces are locally connected, their result does not imply that there is a *path-deformation* of ρ without global fixed points. Thus, the obvious problem arising out of their work is to upgrade this result to path-deformations. A first step in this direction would be to attempt to reprove [1, Lemma 3.9, 3.10]. These lemmas show that, in any neighborhood of ρ, there exists a representation ρ' whose fixed points are isolated and either attracting or repelling points. Can ρ' be attained by deforming along a path? If so, can this be generalized to finite orbits, rather than fixed points, for actions on S^1? This is essentially the content of Question 6.6 above.

6.2.2. *The commutator equation.* More general than Question 6.6 above, the following basic problem appears to be essential in understanding the topology of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

Probem 6.8. *For fixed* $h \in \mathrm{Homeo}^+(S^1)$*, describe the topology of the set*

$$\nu_h := \{f, g \in \mathrm{Homeo}^+(S^1) \times \mathrm{Homeo}^+(S^1) \mid [f,g] = h\}.$$

As it stands, remarkably little is known about this space. If $\mathrm{rot}(h) \in \mathbb{Q} \smallsetminus \{0\}$, then it is known that ν_h is not connected; however, we do not know the number of connected components, nor do we know in any circumstances whether ν_h is locally connected or not.

This problem is strongly related to Question 1.7 on classifying connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$ that we raised in the introduction. For instance, Goldman's classification of connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ given in [9] is built upon a complete understanding of this space for $\nu_h \cap \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. This is of course a much easier problem, as $\mathrm{PSL}(2,\mathbb{R})$ is a finite dimensional Lie group, and the commutator map is smooth. The result of the first author in [14] (that Euler number does not classify connected components of $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$, unlike the $\mathrm{PSL}(2,\mathbb{R})$ case) may also serve as warning that the topology of ν_h space should be more complicated than its intersection with $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$.

Throughout this paper, we navigated within ν_h by making bending deformations. This raises a few obvious questions, such as the following.

Question 6.9. *Let* $h \in \mathrm{Homeo}^+(S^1)$*, and let* $(f,g), (f',g')$ *in the same path-component of* ν_h*. Identifying* f, g *with the image of generators of a* *one-holed torus, Can we move from* (f, g) *to* (f', g') *by using bending deformations? More generally, given* ρ *and* ρ' *in the same path-component of* $\mathrm{Hom}(\Gamma_g, \mathrm{Homeo}^+(S^1))$*, is there a path from* ρ *to* ρ' *using bending deformations in simple closed curves on* Σ_g*?*

This question is reminiscent of Thurston's earthquake theorem for Teichmüller space. It also calls to mind work of Goldman–Xia [10], who use the analogous (positive) result for bending deformations in connected components of classical character varieties in order to studying the action of the mapping class group on these varieties. As well as justifying our use of bending deformations alone, a positive answer to Question 6.9 would give another analogy between classical character varieties and $\chi(\Gamma_g, \mathrm{Homeo}^+(S^1))$.

6.2.3. *Bad tori.* In Section 5, we needed a long series of lemmas in order to prove that a path-rigid representation cannot contain two disjoint bad tori. However, we do not know any example of a path-rigid representation with even one single bad torus. Besides being an interesting question in itself, the question of existence bad tori could provide an alternative route to prove that path-rigid representations are geometric even in Euler class zero. The strategy would be to show that a path-rigid representation of Γ_g cannot admit a bad torus, and then prove and enhanced version of Lemma 5.13.

However, we were somewhat surprised to be unable to tackle the following even more basic technical question.

Question 6.10. *Let* $T(a, b)$ *be a one-holed torus. Does there exist a representation* $\rho\colon \pi_1(T) \to \mathrm{Homeo}^+(S^1)$ *such that the rotation number of every nonseparating simple closed curve is rational, but nonzero?*

This is, obviously, related to understanding the mapping class group actions on character varieties, as we are insisting on finding a nonseparating simple closed curve.

By contrast, relaxing the condition that curves be simple gives a problem already solved by a classical (though not widely known) result of Antonov.

Theorem 6.11 (Antonov [2])**.** *Let* $\rho : \langle a, b\rangle \to \mathrm{Homeo}^+(S^1)$ *be a minimal action. Either* ρ *has abelian image, and is conjugate to an action by rotations, or (up to taking a quotient of* S^1 *by a finite order rotation* r*, in the case that* ρ *commutes with* r*), the probability that the rotation number of the image of a random word of length* N *in* $\{a, b, a^{-1}, b^{-1}\}$ *is zero tends to 1 as* N *tends to* ∞*.*

In the case where ρ commutes with a finite order rotation, say of order n, the rotation numbers of random words equidistribute in $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$. Thus, for any representation, *most* words have rational rotation number.

6.2.4. *Local versus global rigidity.* Thus far, we have discussed rigidity and path-rigidity of representations; rigidity being the natural notion to study from our interest in character spaces, and path deformations being easier to work with in practice. However, from a dynamical perspective, it is also interesting to study *local rigidity* or *stability* of actions. The following definition appears in [15], and is also discussed in [1].

Definition 6.12 (3.1 in [15])**.** A representation ρ is called *locally rigid* if it has a neighborhood *in the representation space* $\mathrm{Hom}(\pi_1\Sigma_g, \mathrm{Homeo}^+(S^1))$ containing only representations semi-conjugate to ρ.

In many circumstances, this condition is much easier to satisfy than rigidity or path-rigidity. For example, an element $g \in \mathrm{Homeo}^+(S^1)$ (thought of as a representation of $\mathbb{Z}$), with finitely many fixed points, all of which are attracting or repelling, is easily seen to be locally rigid, but it is semi-conjugate to (and in the same connected component of $\mathrm{Hom}(\mathbb{Z}, \mathrm{Homeo}^+(S^1))$ as the identity. We do not know if this phenomenon generalizes to representations of Γ_g.

Question 6.13. *Is there a representation* $\rho \in \mathrm{Hom}(\Gamma_g; \mathrm{Homeo}^+(S^1))$ *that is locally rigid, but not rigid?*

Again, a natural first step to this question could be to study the local topology of the sets ν_h defined above.

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