CSP problems 1

Reading:

- *Architectural geometry* chapter 6 (in the references folder) with chapters 5 and 2 for earlier reference if needed.
- Optional: Stillwell's *Four pillars of geometry*

Points, lines, planes review. (Reference if needed: *Four pillars*, chapter 3, beginning of ch. 4)
$\mathbb{R}^2$ is the plane, with points specified in coordinates (x, y); $\mathbb{R}^3$ is three-dimensional space with points of the form (x, y, z). The point $(0, 0)$ in $\mathbb{R}^2$ or $(0, 0, 0)$ in $\mathbb{R}^3$ is called the *origin*.
A straight line in $\mathbb{R}^2$ is the set of points of the form $(a, b)+t(u, v)$; where a, b, u, v are fixed numbers, at least one of u and v is nonzero (why?), and t is allowed to vary. Similarly, the set of points of the form $(a, b, c) + t(u, v, w)$, varying t, defines a line in $\mathbb{R}^3$ provided that (u, v, w) is not the origin.

1. a) Every two distinct points are contained in a unique line. Find the equation of the line (i.e. the values of $a, b, u,$ and v) in $\mathbb{R}^2$ that passes through the points $(1, 2)$ and $(13, 7)$.
   b) Conversely, almost every two lines in $\mathbb{R}^2$ intersect at a unique point. (Projective geometry will soon take away that annoying "almost"). Find the intersection of the line from part a) with $(0, 1) + t(-2, 3)$.
2. Find the equation of the line in $\mathbb{R}^3$ that passes through the points $(1, 1, 1)$ and $(1, 1, -6)$.
3. A plane in $\mathbb{R}^3$ is defined similarly, but with two degrees of freedom. Fix three points (a, b, c), (u, v, w) and $(u'v'w')$. What restrictions do you need on (u, v, w) and $(u'v'w')$ in order for
   $$(a, b, c) + t(u, v, w) + s(u', v', w')$$
   (varying s and t) to define a plane?
4. Thinking of the z direction as vertical, what does the equation of a horizontal plane look like?
5. a) How many points are needed to determine a plane? Does every collection of that many points determine a unique plane?
   b) Pick some points and write the equation of the plane through those points.
6. Planes can also be defined by equations of the form $ax + by + cz + d = 0$. How do you go back and forth between this and the previous notation?

Linear algebra review.

1. Show that the transformation $T(x, y) = (2x + y + 1, -7x + 3y)$ sends every straight line to a straight line. Or, if you prefer, do this for the general case $T(x, y) = (ax + by + c, dx + ey + f)$. ** but see the next question.
2. Putting $a = b = c = d = e = f = 0$ in the general case above gives $T(x, y) = (0, 0)$ which sends all of the plane to a single point. What else can go wrong?
3. $T(x, y) = (ax+by+c, dx+ey+f)$ is called an *affine transformation*, in the special case $c = f = 0$ it is called a *linear transformation*. Linear transformations can be written *in matrix form*:
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$
   One of the many advantages to this is that it is easy to compose transformations by multiplying matrices.
   If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $S = \begin{pmatrix} u & v \\ w & x \end{pmatrix}$ then T followed by S is the matrix $ST = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & v \\ w & x \end{pmatrix} = \begin{pmatrix} au + bv & aw + bx \\ cw + dv & cw + dx \end{pmatrix}$.

Practice multiplying some matrices (if you need to). Look up how this generalizes to three dimensions, and practice matrix multiplication there.

4. ** Suppose you have a function T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that sends every straight line to a straight line, and preserves intersections (so if two lines ℓ_1 and ℓ_2 intersect, so do $T(\ell_1)$ and $T(\ell_2)$). Can you show that T is affine? As a first step, assume $T(0,0) = (0,0)$ and try to show that T is linear. **this is an important problem, see how far you can get.
5. Find the matrix representatives for the following transformations:
   (a) Rotating $\mathbb{R}^2$ about the origin by 60 degrees counterclockwise.
   (b) Rotating $\mathbb{R}^3$ 60 degrees counterclockwise about the z-axis
   (c) The transformation of $\mathbb{R}^3$ that takes scales all distances by a factor of 3 and preserves directions
   (d) Make your own example (describe something in words, then write the matrix for it)
6. Which linear transformations of $\mathbb{R}^3$ send round spheres to round spheres?
7. What is the volume of the image of the unit square (vertices (0,0), (0,1), (1,0), (1,1)) after applying the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$? What does it mean when the image has volume zero?
8. Generalize the previous question to the unit cube in $\mathbb{R}^3$.

The projective plane. The projective plane is meant to capture the visual experience of a person with a 180° field of vision. Imagine an eye at the origin in $\mathbb{R}^3$ whose field of vision is exactly half of the space: say the set of points (x, y, z) with $z > 0$ together with those where $z = 0$ and $y > 0$. (Why is this exactly half the space? Why is just the points $z > 0$ not enough?)

A point (a, b, c) appears to the eye in exactly the same position as the point $(2a, 2b, 2c)$ and as $t(a, b, c)$ for any multiple $t > 0$. Thus we say that the ray $t(a, b, c)$ in the field of vision defines a *point* in the projective plane. Similarly, a line in the projective plane is given by a half-plane through the origin $t(a, b, c) + s(u, v, w)$ in the field of vision.

1. Convince yourself that this is what the eye sees as a point.
2. What does the eye see as a round circle? An ellipse?
3. Show that every two distinct lines in the projective plane meet at a point, and every two points are contained in a line.

Mathematicians usually simplify things by including all of $\mathbb{R}^3$ as the "field of vision" and saying that lines $t(a, b, c)$ (where t varies over all the real numbers now) are the points of the projective plane, and it's lines are defined by planes through the origin. This gives exactly the same space. *(What do circles and ellipses correspond to now?)*

Coordinates. Provided that $c \neq 0$, a line $t(a, b, c)$ always has a point on it where $c = 1$. Thus, most points in the projective plane can be written in the form $(x, y, 1)$. In this way, we identify (almost) all of the projective plane with $\mathbb{R}^2$. If you imagine a large canvas covering the plane $z = 1$, the points (x, y) on the canvas are exactly the points $(x, y, 1)$ seen by the eye.

1. What do parallel lines on the canvas correspond to in the projective plane?
2. What points in the projective plane are not on the canvas? (i.e. cannot be written in the form $(x, y, 1)$) What does this look like in the field of vision? (These are called the points "at infinity" or the "line at infinity")
3. Can you think of the "line at infinity" in some way as a horizon?
4. (open ended -ish) Of course, if $x \neq 0$ we can also write points as $(1, y, z)$; this is a canvas placed on the plane $x = 1$. If you have an image drawn on the $(x, y, 1)$ canvas and want to draw something on the $(1, y, z)$ canvas that looks the same to the eye, what do you need to do?

Homogeneous coordinates is the name of the notation $(a : b : c)$ to specify the point in projective space given by the line $t(a, b, c)$ in $\mathbb{R}^3$. In homogenous coordinates, $(1 : 2 : 3)$ is the same point as $(2 : 4 : 6)$.

Projective transformations. Read the section on projective transformations in *Architectural Geometry*. The first few sections of Chapter 5 in Stillwell may also be helpful.

1. What is a projective transformation?
2. ** They say "It turns out that we obtain all projective transformations between two planes (up to a congruence transformation of one plane) by perspective projection" Can you explain what they mean and then justify the statement? Why is it true?
3. ** In a previous problem, we characterized linear transformations of $\mathbb{R}^2$ as those that fix the origin, and send straight lines to straight lines, preserving intersection. Can you say something similar for projective transformations?
4. Work through the example on page 206.
5. What are the possible images of a square under a projective transformation? Perhaps start by working through the example on page 206 with various squares.