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Berstein Seminar in Topology - Fall 2021
Topic: Anosov flows on 3-manifolds

An Anosov flow is a dynamical system, but the study of these on 3-dimensional manifolds leads to some fascinating interactions with topology and geometry. This course will focus on geometric and topological questions related to these flows (such as building examples and classifying foliations).

Topics and Readings. As typical with Berstein seminar, the content will be in part directed by the students in the class, as will the pacing and the level of difficulty, to some extent. Below is a list of introductory material that should take at least the first third of the semester (perhaps the first half or more, depending on how much detail we explore), after which we can discuss further directions.

A. Introductory material.

1. **General overview, for reference.**

   T. Barthelmé's lecture notes, sections 1 and 2, give a good broad perspective on the subject and sketches of some of the results we will cover.

2. **Fundamental examples**

   a) The suspension of a linear Anosov map

   b) Geodesic flow on hyperbolic surfaces

   References: General reference on hyperbolic geometry, PSL(2,R), and geodesic flow: M. Einsiedler, T. Ward, *Ergodic Theory*, Springer GTM 259 https://link.springer.com/content/pdf/10.1007%2F978-0-85729-021-2_9.pdf

   -Katok's lectures, section 16 of https://www.ma.imperial.ac.uk/~santonak/m2r/S.KatokGeodesicFlowFuchsianGroups.pdf

   Plan: We will describe these two examples, prove they are Anosov, and, following Katok's notes (section 17) establish two important dynamical properties: topological transitivity, and the Anosov closing lemma.

   Optional: if someone is keen on more dynamics, we can learn Hopf's ergodicity argument form Einsiedler–Ward.

3. **Existence questions.** What manifolds admit Anosov flows? What are topological obstructions to admitting a flow?

   a) (optional) The manifold must be irreducible, i.e. every sphere bounds a ball
   This requires a nice excursion into foliation theory. Haefliger's theorem about foliations is presented in Chapter 5 of this book https://www.worldscientific.com/doi/epdf/10.1142/10366. It uses some basic general position / Morse theory style arguments.

   Given Haefliger's theorem, the rest of the proof is outlined in Barthelmé's notes. It uses a theorem of Palmeira from https://www.jstor.org/stable/pdf/1971256.pdf to conclude that the universal cover of the manifold is $\mathbb{R}^3$. Maybe someone can come up with an easier argument for this theorem in the 3-dimensional case (a simply connected 3 dimensional manifold foliated by 2-dimensional planes must be homeomorphic to $\mathbb{R}^3$), or another workaround.

b) Exponential growth of fundamental group Plante-Thurston's and/or Margulis' argument. An expository version is given in notes of Potrie: https://arxiv.org/pdf/2005.10889.pdf Plante and Thurston's paper is also short and quite readable: https://core.ac.uk/download/pdf/82322314.pdf

4. **Dehn surgery constructions** This is the classic method to produce new examples out of old ones (see also Handel–Thurston below for an alternative approach.)
   (a) Martelli section 10.1.1 and 10.1.2 on what is Dehn Filling
   (b) (optional) Thurston's hyperbolic Dehn surgery theorem. (not about flows, but a beautiful piece of geometric topology, and the justification for why we get hyperbolic examples).
   (c) Goodman's construction and Fried's construction. Proving in detail that the resulting flows are Anosov is tricky, but it is easy to understand the construction and believe that it works.

B. Possible further topics:

1. Ghys' complete classification of Anosov flows on Seifert fibered 3-manifolds
2. Pseudo-Anosov and quasigeodesic flows, work of Calegari and Frankel
3. Mosher's work on relationshipt between pseudo-Anosov flows and the Thurston norm (possibly too ambitious!)
4. Handel–Thurston's construction of examples on graph manifolds, and the program to generalize outlined in this (quite readable) recent paper of Ba and Clay: https://arxiv.org/pdf/2006.09101.pdf
5. Franks–Williams and Bonatti–Beguin–Yu gluing constructions
6. Barbot and Fenley's theory of $\mathbb{R}$-covered flows
7. ?? other, depending on interest