Singularity Theory for Extended Cobordism Categories and an Application to Graph Theory

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Summer 2022

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Abstract

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In low-dimensional topology one often decomposes a space into pieces that can be individually understood then put back together, yielding a combinatorial description, and such descriptions can be used for diagrammatic reasoning, computations of algebraic invariants, constructions of new objects, and so on. A powerful tool for this is singularity theory, where a classification of the types of singularities that generically might be present (for example, Morse critical points) leads to systematic approaches to decompositions, and in the ideal case the classification reduces to analyzing singularities of polynomial functions. We develop singularity theory relevant to "n-categorical" decompositions of smooth manifolds, in which pieces each have a recursive decomposition of their boundaries. In particular, we study smooth functions from manifolds to flag-foliated $\mathbb{R}^n$, which serves as a model for the composition laws of an n-category. We use a refinement of the jet transversality techniques for the Thom–Boardman singularity types to define dense sets of functions that are suitably generic with respect to the foliations. We show how to compute codimensions of the submanifolds that correspond to a singularity type. For a few situations, for instance surfaces with embedded curves, we carry out a classification of the germs that appear in this dense set up to structure-preserving diffeomorphism. To this end, we prove stability results that aid in proving equivalence of germs and in finding polynomial representatives.

We explain how to use this classification of singularities to give a presentation of the symmetric monoidal 2-category of curves with embedded curves. As an application, we give a way to compute the Krushkal polynomial, which is a combinatorial invariant of graphs in surfaces, using an extended TQFT on the symmetric monoidal 2-category of surfaces that are decomposed into black and white regions.

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Acknowledgments

I don't know exactly how or when Ian became my advisor, though it always made sense to me that he was. I do know that I hung around him long enough, chatting about knots and graphs, that eventually he introduced me as his student at a conference. I have many things to thank Ian for, such as his generosity in giving his time whenever I'd drop by his office, his wide interests and knowledge to draw from, and the freedom he gave me to explore. (I bet you never envisioned that the questions you gave me about flow polynomials would eventually lead to singularity theory.) I appreciate very much the risks he let me take, or, if that wasn't it, his perhaps-misplaced trust in me that I knew what I was doing — whatever was the case may be, thanks.

I thank Slava Krushkal, who helped me through the years by showing keen interest in whatever tidbits about graph polynomials I'd come up with, and there have been a number of ideas I might have dropped (or, more often, never had) if it weren't for him. I thank Vera Serganova for all the algebra I learned from her. I also have to thank Chris Schommer-Pries, with whom I've only spoken little, but to whom this present work owes a great debt. I would like to make a special mention of Henry Cohn, who long ago helped me get into academia, and I have fond memories of our discussions at MSR.

I thank my fellow grad students for being there and for having been a joy to talk to. At the certain risk of missing most names from the list, to name just a few: Calvin McPhail-Snyder, Alex Sherman, Nic Brody, Morgan Weiler, and Kevin Donaghue. I also thank my friend Scott Kovach, who has long been patient hearing me attempt to explain things I'm trying to understand, and who's been there all these years.

My wife, Colleen Josephson, has been essential to me, and I have too much to thank her for. Lastly, I thank my family and friends for the support through the years, and may I see you again now that this work is done.

Introduction

There are a number of important tools in the study of manifolds that depend in some way on the study of singularities. To name some basic examples:

- Morse theory uses Morse functions, and a careful analysis of critical points leads to the existence of handle decompositions of manifolds. Furthermore, Cerf theory, which is a special kind of path of Morse functions, gives moves that let one go between any two handle decompositions.

- Knot diagrams depend on the existence of projections with transverse double points, and these double point singularities, the crossings, have long been used to describe the combinatorial structure of a knot. Every isotopy of knots can be perturbed and decomposed into a small set of moves, the Reidemeister moves, and these arise as different kinds of singularities: the moments where the knot temporarily fails to have a knot diagram in general position.

- General position arguments in 3-manifold topology require the existence of a collection of objects (such as curves or surfaces) being transverse to something else. Transverse points of intersection are a kind of singularity.

In this thesis, we are interested in developing theory to work with generic functions $M \to \mathbb{R}^n$ with $\mathbb{R}^n$ foliated with the flag foliation. This way we can decompose the manifold, then decompose the boundary of that manifold, and so on, which yields data that is amenable to being used for generators or relations for n-categories.

One of the concrete goals of this thesis is to work out in full detail a presentation for a 2D cobordism category with defects for the purpose of showing that one can extend classical graph invariants to extended TQFTs (functors from this cobordism 2-category to other 2-categories). Some of these, for closed surfaces, coincide with pre-existing invariants such as the Krushkal polynomial.

While in these low dimensions we could use ad hoc methods to derive a presentation, as done in [Abr96] and [Koc04] for the 2D cobordism category, instead we will follow the lead of Schommer-Pries in [SP09] and use jet transversality techniques from classical singularity theory to develop higher Morse theory, using suitably generic smooth functions $f : M \to [0, 1]^n$ to decompose a compact smooth m-dimensional manifold M in an "n-categorical" way.

Before explaining how this is meant to work, it is instructive reviewing how Morse theory gives "1-categorical" decompositions of manifolds, and in particular how it gives a set of generators for the m-dimensional cobordism category Cob_m, which has as objects closed $(m-1)$-manifolds and has as morphisms diffeomorphism classes of m-dimensional cobordisms between them (where diffeomorphisms are relative to the boundaries). Given a cobordism M, there is a smooth map $f : M \to [0,1]$ transverse to $\{0,1\}$ such that $f^{-1}(0)$ and $f^{-1}(1)$ are respectively the source and target boundaries of M (and all such maps are homotopic). If $t \in [0,1]$ is a regular value, we may represent M as the composition of the cobordisms $M' = f^{-1}([0,t])$ and $M'' = f^{-1}([t,1])$, and by restricting f to these and reparameterizing codomains we get smooth maps $M' \to [0,1]$ and $M'' \to [0,1]$ that, in some sense, decompose f. Recall that f is a *Morse function* if (1) its critical points are nondegenerate and (2) the restriction of f to the critical points is injective — Morse functions are dense in $C^\infty(M,[0,1])$, so we may assume f is Morse. By decomposing M using regular values of f, it is possible to give M as a composition of cobordisms that each have at most one critical point of f. Reducing to the case of such a cobordism M:

- If M has no critical points, then it is induced by a diffeomorphism $\varphi : f^{-1}(0) \to f^{-1}(1)$ in the sense that it is diffeomorphic to $f^{-1}(1) \times [0,1]$ with φ defining the inclusion $f^{-1}(0) \to f^{-1}(1) \times \{0\}$ (such M are the isomorphisms in Cob_m).
- If M has a single critical point p, then p there is a coordinate chart $(x_1, \dots, x_m)$ such that f is locally of the form $f(x_1, \dots, x_m) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2$ for some $0 \le k \le m$. Analysis of this leads to a description of M as being diffeomorphic to $f^{-1}(0) \times [0,1]$ with a single k-handle $D^k \times D^{m-k} \cong D^m$ attached to $f^{-1}(0) \times \{1\}$ along its $(\partial D^k) \times D^{m-k}$ boundary (and smoothing corners).

Thus, morphisms of Cob_m are generated by diffeomorphisms and handle attachments, where the data of a handle attachment is an $(m-1)$-manifold along with an embedded $(\partial D^k) \times D^{m-k}$.

The second part to understanding the 1-categorical decomposition is finding a complete set of relations for these generating morphisms. An answer to this is Cerf theory [Cer70], which studies paths of smooth real-valued functions. We will revisit this from the point of view of jet transversality, but for now a *Cerf function* $f : M \times [0,1] \to [0,1]$ is a smooth map such that

1. the values of $t \in [0,1]$ for which f_t is not a Morse function are isolated and in $(0,1)$;
2. if $(p,u) \in M \times [0,1]$ is such that p is a degenerate critical point of f_u, then there is a coordinate chart $(x_1, \dots, x_n)$ at p and a coordinate chart (t) at u such that f is locally of the form

$$f(x_1, \dots, x_n, t) = f(p,u) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{m-1}^2 + x_m(\pm t + x_m^2);$$

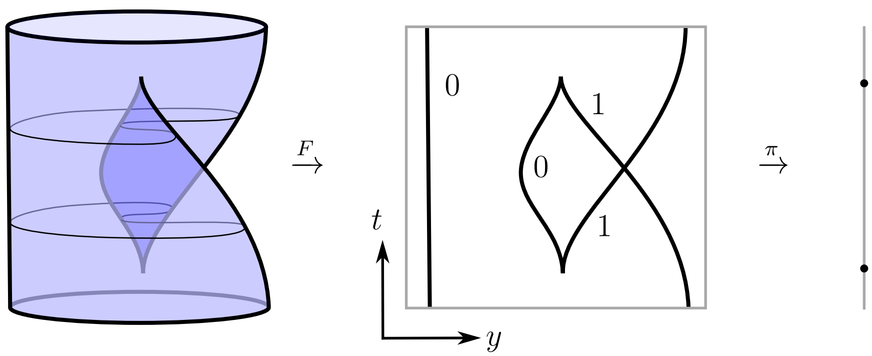


Figure 0.1: Graphics for Cerf function for $M = S^1$. *Left:* the space $S^1 \times [0,1]$ with the horizontal axis depicting the value of the Cerf function. *Middle:* the Cerf graphic, with two cusps, two index-0 curves, and two index-1 curves. *Right:* The projection onto the time axis including images of cusps.

3. if for $u \in [0,1]$ there are distinct critical points $p, p' \in M$ of f_u with the same image, then $df_{(p,u)}|_{T_{(p,u)}S} \neq df_{(p',u)}|_{T_{(p,u)}S}$, where S is the 1-dimensional submanifold

$$S = \{(x,t) \in M \times [0,1] \mid x \text{ is a nondegenerate critical point of } f_t\}.$$

Cerf functions are dense in $C^\infty(M \times [0,1],[0,1])$, so every pair of Morse functions for M is connected by a path of Cerf functions.

A key tool in the analysis of Cerf functions is the *Cerf graphic* $\Gamma \subset [0,1] \times [0,1]$, which is the image of all of the singularities through the map $F : M \times [0,1] \to [0,1] \times [0,1]$ defined by $F(x,t) = (f(x,t),t)$. (See Figure 0.1 for an illustration.) Let $S_2 \subseteq M \times [0,1]$ be the 0-dimensional submanifold consisting of the points from the second property of a Cerf function (the *cusps*) and the self-intersection points from the third property. Furthermore, let $S_1 \subseteq M \times [0,1]$ be the 1-dimensional submanifold consisting of the nondegenerate critical points of all those f_t that are Morse functions, and let $S_0 \subseteq M \times [0,1]$ be the codimension-0 submanifold consisting of everything not in S_1 and S_2. Letting $\pi : [0,1] \times [0,1] \to [0,1]$ be the projection $\pi(y,t) = t$, this decomposition has the property that both $F|_{S_i}$ and $(\pi \circ F)_{S_i}$ have constant-rank differentials for each i, and $F|_{S_i}$ maps onto a codimension-i immersed submanifold.

These rank properties and local models for the S_0, S_1, and S_2 strata are sufficient to decompose the Cerf function "2-categorically" to obtain relations. The first direction of decomposition is the Cerf function as a path, where we use $\pi(F(S_2))$ as the singular values for the function $\pi \circ F$, and similar to before by choosing t outside this set we can give the Cerf function as a composition of the sub-paths on $[0,t]$ and $[t,1]$, thus giving a relation between two Morse functions as a pair of relations through a third. This lets us reduce to three cases for S_2:

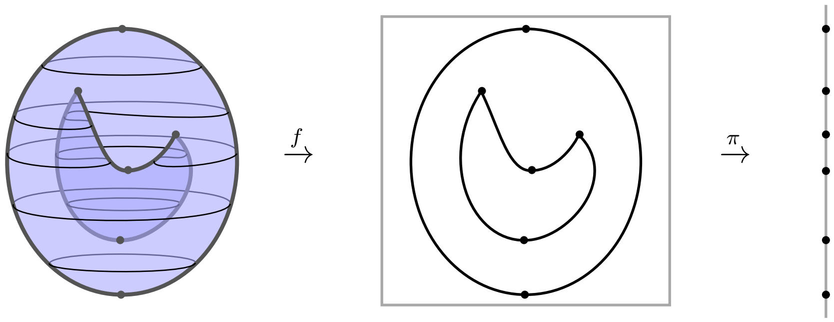


Figure 0.2: A generic function $f : S^2 \to [0,1]^2$ whose 0-dimensional singularities consist of two minima, a saddle, a maximum, and two cusps and whose 1-dimensional singularities consist of six folds connecting them. Level sets of regular values of $\pi \circ f$ horizontally have isolated minima and maxima along these folds.

- S_2 is empty. Then the Cerf function ends up being induced by isotopies of M and of the codomain $[0,1]$, and one can argue that this carries a decomposition for f_0 to a compatible decomposition for f_1 along with diffeomorphisms of each constituent cobordism.

- S_2 is a single cusp point. One may analyze the normal form from the second property and determine that it is either a birth or a death of a canceling pair of a $(k-1)$-handle and k-handle, where $1 \leq k \leq n$.

- S_2 is a pair of nondegenerate critical points with the same image. This corresponds to a pair of handles that can be attached in either order, and at the critical moment one may regard the handles as having disjoint attachment submanifolds.

Without the categorical language, this is the basis for Heegaard diagrams, Kirby calculus, thin position arguments of Gabai and Scharlemann–Thompson, and so on. For specifically categories, this analysis was carried out by [MS] to characterize the cobordism category for surfaces with boundary (for so-called “open-closed 2D TQFTs”).

Taking this “2-categorical” decomposition of Cerf functions seriously, Schommer-Pries generalizes the notion of a Cerf function to 2-dimensional Morse functions $f : \Sigma \to [0,1] \times [0,1]$, whose definition involves a stratification of Σ such that (1) f and $\pi \circ f$ are constant rank when restricted to each stratum and (2) each stratum has an associated local model for its points. (See Figure 0.2 for an illustration.) For Cerf functions, $\pi \circ F$ was trivially a Morse function for $M \times [0,1]$, but for 2-dimensional Morse functions there are additional singularities corresponding to $\pi \circ f$ being a Morse function.

Decomposing 2-dimensional Morse functions in a similar way to Cerf functions yields the “2-categorical” decomposition of Σ. The $\pi \circ f$ decomposition yields a vertical composition of 2-morphisms that, away from the codimension-2 singularities, the f decomposition of level sets yields a horizontal composition of 1-morphisms that, away from the codimension-1 singularities, yields the 0-morphisms (i.e., the objects, disjoint unions of points) of the 2-category; each generating k-morphism is diffeomorphic to a k-disk. For relations, there is a Cerf-like theory of paths between 2-dimensional Morse functions, whose graphics now lay within the cube $[0,1]^3$. These are decomposed "3-categorically" in the same sort of manner.

There is one issue glossed over here, which comes from a desire to have the presentation reflect the symmetric monoidal structure from disjoint unions — this additional structure is essential to the 2-category's finite presentation. The function f gives Σ as a "singular" covering space over $[0,1]^2$, with the cardinality of a fiber changing in a controlled way when crossing points of Γ. This covering space can have non-trivial monodromy, so it is not possible to identify each fiber with a finite set in a locally constant way. To solve this, Schommer-Pries introduces chambering graphs and foams, which respectively partition the 2D and 3D graphics into codimension-0 chambers inside of which such a fiber identification is possible. Along codimension-1 walls of these chambers there is a bijection to describe how sheets are glued, and around codimension-2 cells there is a cocycle condition for these bijections. To carry out this program, there is also a collection of moves for chambering graphs and foams that are proved to be complete.

An alternative to the useful but ad hoc chambering graphs and foams is to appeal to the fact that a monoidal 2-category is equivalently a 3-category with one object. This suggests considering 3-dimensional Morse functions $f : \Sigma \to [0,1]^3$ to decompose Σ. The 0-morphisms from the decomposition correspond to empty sets, which are the fibers outside the image of f, and in this region the covering space trivially has trivial monodromy. A benefit to this approach is that the pasting diagram for Σ may be regarded as being inside $[0,1]^3$ itself, like with the "manifold diagrams" of Dorn [Dor18] or the "surface diagrams" of McIntyre and Trimble [Tri10]. Using Roseman's work [Ros98], Carter–Rieger–Saito gave a 1-categorical description of knotted surfaces in $\mathbb{R}^4$ [CRS97], which Baez and Langford extended into a 2-category [BL98]. In principle, one should be able to extract a description of the 2-dimensional cobordism 2-category by looking at shadows of knots, however it would be beneficial having a self-contained calculation using higher Morse theory. We will pursue this idea for 1-categorical descriptions of graphs, using maps $f : \Gamma \to [0,1]^2$, but for surfaces we will defer the computation for future work since it strains the scope of this thesis.

Having given all this motivation, we will now explain the basic notion we will use for generic functions $f : M \to [0,1]^n$. In general, we develop a version of the Thom–Boardman classification of singularities for smooth maps between foliated manifolds, and in this case we give $[0,1]^n$ the standard flag foliation, whose k-dimensional leaves are the cosets of $[0,1]^k \subseteq [0,1]^n$ for all $0 \leq k \leq n$, or in other words the level sets of the projections $\pi_k : [0,1]^n \to [0,1]^n/[0,1]^k$. It is possible to recursively construct the following stratification of M for a dense subset of such f. We start with a single stratum, M itself, and then repeatedly apply the following to replace strata with finer stratifications:

1. Given a stratum S, for each $p \in S$ we consider the rank of $d(\pi_k \circ f|_S)_p$ for each $0 \leq k < n$, which measures how transverse $f|_S$ is to each leaf of the foliation. Each

possible sequence of ranks gives an associated submanifold of S, and together these stratify S. (This ensures that each $\pi_k \circ f|_S$ is a submersion onto its image.)

2. Given a stratum S, a point $p \in S$ might locally be in the closure of other strata whose union is locally a submanifold. We can carry out the previous rank decomposition to further stratify S.

3. More delicately, given a stratum S, a point $p \in S$, and a stratum S' whose closure contains p, we can consider the images of S and S' locally as submanifolds of $[0,1]^n$. There is a subspace of $T_{f(p)}[0,1]^n$ that may be regarded as the tangent space of the closure of S', and we stratify S based on the dimension of the intersection of this subspace with $T_{f(p)}f(S)$.

4. If the images of one or more strata intersect, we may further stratify them by how many points from these strata map to the same point. (To avoid this getting out of hand, this is not applied to strata for which the restriction of f is a submersion.)

In summary: the strategy is to make everything in sight be as generic as possible with respect to each other.

This process eventually stabilizes, and furthermore it is possible to compute the codimensions of each stratum obtained this way, which is useful for determining which of the strata might be nonempty. This is important because it leaves only a finite list of strata to characterize.

Locally the images of the strata stratify $[0,1]^n$, giving a graphic that is "progressive" in a similar sense to Dorn and McIntyre–Trimble, which is the property that allows us to produce an n-categorical decomposition of M. Strata with codimension-k images in $[0,1]^n$ give k-morphisms for the category.

To handle a graph G embedded in a surface Σ with this framework, for example, we can initialize the stratification with three strata: the vertices of G, the interiors of the edges of G, and the complement $\Sigma \setminus G$. For relations, we can take the product of these strata with $[0,1]$ in $\Sigma \times [0,1]$. We will carry this out for surfaces with embedded 1-manifolds. We will also consider foliations of the domain; for example in our analogue of Cerf theory we give $M \times [0,1]$ the codimension-1 product foliation.

It should be said this stratification is insufficient to classify singularities completely, in the sense that strata obtain in a particular way often consist of one of multiple types of singularities. Despite this, for small n we are still able to derive and enumerate normal forms for each stratum using only the way in which it was constructed. This is likely due to the fact that, forgetting the foliation, only Morin singularities appear when $n \leq 3$. These are relatively easy to analyze, and they are characterized by being those strata where we recursively only consider whether $d(f|_S)_p$ drops rank by 1. When $n = 4$, elliptic and hyperbolic umbilics can also appear, which would be necessary to analyze if we were to carry out a higher-Morse-theoretic 4-categorical decomposition of 4-manifolds

Overview

Chapter 1 goes over the necessary singularity theory for the analysis in Chapter 2. We review standard jet bundle material, but in we include general information about pro-objects and pro-open sets before introducing smooth manifold germs and maps between them in Section 1.1.3, which we use as a device to avoid needing to choose collar neighborhoods when gluing. We provide a parameterized version of the Hadamard Lemma in Lemma 1.1.69, and in the next section we give versions of the Malgrange Preparation Theorem, a deep theorem that lets us solve differential equations via algebra, and we give a parameterized version in Corollary 1.1.76. A key theorem is the Thom Transversality Theorem in Section 1.2, which lets us assume jet extensions are transverse to any countable collection jet submanifolds. Section 1.3 explains the types of foliations we will come across, and there is a description of them in terms of pseudogroups.

Section 1.4 discusses stability and proves some algebraic characterizations of stability of a germ with a foliated codomain. We use these later to prove singularity types are equivalent to one that is polynomially defined. After that is a description of the Thom–Boardman singularity types, with an interlude in Section 1.5.1 to calculate normal bundles to submanifolds of $\mathrm{Hom}(V, W)$ given by multiple rank conditions on multiple subspaces.

Then we enter Chapter 2, where we work out all the singularity types in a number of situations. We replicate from scratch the definition and existence of Morse and Cerf functions in Section 2.1 and then work out all the singularity types for curves in the plane in Section 2.3. Starting from Section 2.4 we rederive the singularity types for $\mathbb{R}^n \to \mathbb{R}^n$ with $n = 1, 2, 3$. This in some sense reproduces the work in [SP09], but we do it in a way that is much stricter, which is to only use the diffeomorphism groups for the domain and codomain. Finally for the chapter, starting from Section 2.7 we classify singularities for surfaces with embedded curves, and what we derive are just singularities running through embedded curve, rather than reproducing the whole calculation from the previous section.

Chapter 3 gives the presentations of the extended cobordism categories using the generators and relations we developed in the previous chapter. In Section 3.4 we show how to do gradient flows like in Morse theory, where the answer is to use some Riemannian metric for which the gradient is subordinate to all the singularities.

Chapter 4 is about some applications of the presentation. We define an extended TQFT with 1-dimensional defects such that the two regions on either side are colored opposite colors. We use this in Section 4.2 (and in particular Section 4.2.6 to give a the Krushkal polynomial, which is an invariant of graphs in surfaces, as a simple state sum over a TQFT. In Section 4.1.1 we give a characterization of a Frobenius algebra as a direct sum of a “radicular” element and an element in the Jacobson radical.

The presentations for surfaces with embbedded curves are given in Figures 3.2 to 3.4, and the black-white cobordism category presentation in Figure 4.2.

Chapter 1

Singularity theory for decompositions

1.1 Jet bundles

This section is a review of jets and jet bundles — experts of which should free to skip ahead. In essence, the k-jet at $x \in M$ of a smooth map $f : M \to N$ is the kth-order Taylor polynomial of f at x. This is defined in such a way that the k-jets assemble into a fiber bundle $J^k(M, N)$ over both M and N, and f defines a section $j^k f : M \to J^k(M, N)$ called the k-jet extension of f. For the special case of 1-jets, the jet bundle $J^1(M, N)$ can be identified with $\mathrm{Hom}(TM, TN)$ and $j^1 f$ with df. The k-jet of f at x depends on no more than the germ of f at x (see Section 1.1.4).

We will follow a combination of [GG73], [Hir76], [KMS93], and [AGZV12].

Definition 1.1.1. Let M and N be smooth manifolds and $p \in M$. Suppose f and g are smooth maps $M \to N$. We say f and g have *kth order contact at p* if for every smooth $\varphi : N \to \mathbb{R}$ and $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$, then $\varphi \circ f \circ \gamma$ and $\varphi \circ g \circ \gamma$ have the same kth order Taylor polynomial centered at 0. The kth order contact class of f at p is the *k-jet* of f at p, and it is denoted by $j^k f_p$.

$J^k_p(M, N)$ denotes the set of k-jets at $p \in M$ of smooth maps $M \to N$. The disjoint union of these for all $p \in M$ is the set $J^k(M, N)$, known as the *k-jet bundle*.

The *source map* $\alpha : J^k_p(M, N) \to M$ is defined by $\alpha(j^k f_p) = p$, and the *target map* $\beta : J^k_p(M, N) \to N$ is defined by $\beta(j^k f_p) = f(p)$. We denote the set of k-jets with source p and target q by $J^k_p(M, N)_q$.

For a smooth map $f : M \to N$, we write $j^k f : M \to J^k(M, N)$ for the *k-jet extension*, which is defined by $p \mapsto j^k f_p$. $\Diamond$

Lemma 1.1.2. *Let L, M, and N be smooth manifolds with $p \in L$ and $q \in M$, and let $f, f' : L \to M$ and $g, g' : M \to N$ be smooth maps with $q = f(p) = f'(p)$. If $j^k f_p = j^k f'_p$ and $j^k g_q = j^k g'_q$, then $j^k(g \circ f)_p = j^k(g' \circ f')_p$. Thus, we are justified in defining the composition $\circ : J^k_q(M, N) \times J^k_p(L, M)_q \to J^k_p(L, N)$ by $j^k g_q \circ j^k f_p = j^k(g \circ f)_p$.*

Proof. For all smooth $\varphi : N \to \mathbb{R}$ and $\gamma : \mathbb{R} \to L$ with $\gamma(0) = p$,

$$(\varphi \circ g) \circ f \circ \gamma = (\varphi \circ g) \circ f' \circ \gamma$$

hence $j^k(g \circ f)_p = j^k(g \circ f')_p$. Similarly, $j^k(g \circ f')_p = j^k(g' \circ f')_p$. □

The fibers of jet bundles are functorial on the category of pointed smooth manifolds, and in some cases the jet bundles themselves are functorial:

Definition 1.1.3. Let L, M, and N be smooth manifolds and $f : L \to M$ and $g : M \to N$ smooth maps. We define the following induced maps:

- $g_* : J^k(L, M) \to J^k(L, N)$ by $g_*(j^k h_q) = j^k(g \circ h)_q$ for $q \in M$,
- $f^* : J^k_{f(p)}(M, N) \to J^k_p(L, N)$ by $f^*(j^k h_{f(p)}) = j^k(h \circ f)_p$ for $p \in L$, and
- if f is a diffeomorphism, $f^* : J^k(M, N) \to J^k(L, N)$ by $f^*(j^k h_q) = j^k(h \circ g)_{f^{-1}(q)}$. ◊

For sake of fixing notation, we recall the definition of the kth order Taylor polynomial in multiple variables:

Definition 1.1.4. Let $U \subseteq \mathbb{R}^m$ be open and $f : U \to \mathbb{R}$ be smooth. The *kth order Taylor polynomial of f at $p \in U$* is

$$T^k f_p = \sum_{|I| \leq k} \frac{\partial_I f(0)(x-p)^I}{I!}$$

where $I \in \mathbb{N}^m$ is a *multiindex*, and

$$\begin{aligned} |I| &= I_1 + \cdots + I_m \\ I! &= I_1! \cdots I_m! \\ \partial_I f &= \frac{\partial^{|I|} f}{\partial^{I_1} x_1 \ldots \partial^{I_m} x_m} \\ (x-p)^I &= (x_1 - p_1)^{I_1} \cdots (x_m - p_m)^{I_m}. \end{aligned}$$

For smooth $f : U \to \mathbb{R}^n$, then the *kth order Taylor polynomial of f at $p \in \mathbb{R}^m$* is

$$T^k f_p = (T^k(f_1)_p, \ldots, T^k(f_n)_p),$$

where f_i is the ith component of f. ◊

Lemma 1.1.5. *Let M and N be smooth manifolds and $p \in M$, and suppose $f, f' : M \to N$ are smooth maps with $f(p) = f'(p)$. Using the notation for local homeomorphisms (see Definition 1.3.1), let $\varphi : M \hookrightarrow \mathbb{R}^m$ be a chart at p with $\varphi(p) = 0$ and let $\psi : N \hookrightarrow \mathbb{R}^n$ be a chart at $f(p)$. Then $j^k f_p = j^k f'_p$ if and only if $T^k(\psi \circ f \circ \varphi^{-1})_0 = T^k(\psi \circ f' \circ \varphi^{-1})_0$.*

Proof. By Lemma 1.1.2, since φ and ψ are locally invertible, $j^k f_p = j^k f'_p$ if and only if $j^k(\psi \circ f \circ \varphi^{-1})_0 = j^k(\psi \circ f' \circ \varphi^{-1})_0$. Hence we may reduce to the case of smooth maps $f, f' : \mathbb{R}^m \to \mathbb{R}^n$ sending 0 to 0. That is, we want to show $j^k f_0 = j^k f'_0$ if and only if $T^k f_0 = T^k f'_0$.

First suppose $T^k f_0 = T^k f'_0$. Whenever $\omega : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}^m$ are smooth with $\gamma(0) = 0$, then $\omega \circ f \circ \gamma$ and $\omega \circ f' \circ \gamma$ have the same kth order Taylor series at 0 by the chain rule. Hence, $j^k f_0 = j^k f'_0$.

Conversely, suppose $j^k f_0 = j^k f'_0$. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the standard projection for $1 \le i \le n$, and for arbitrary constants $c_1, \ldots, c_m$ consider maps $\gamma(t) = (c_1 t, \ldots, c_m t)$. Since $\pi_i \circ f \circ \gamma$ and $\pi_i \circ f' \circ \gamma$ have the same kth order Taylor series, we have

$$\sum_{|I| \le k} \frac{\partial_I f_i(0) c^I}{I!} = \sum_{|I| \le k} \frac{\partial_I f'_i(0) c^I}{I!}$$

where $c^I = c_1^{I_1} \cdots c_m^{I_m}$ for each multiindex I. This is an equality between symmetric polynomials in $c_1, \ldots, c_n$, hence we can deduce $\partial_I f_i(0) = \partial_I f'_i(0)$ for all $|I| \le k$ and i. $\square$

We have so far only defined $J^k(M, N)$ as a set, but it has the structure of a fiber bundle over both M and N with respective projections α and β. We will first describe $J^k(M, N)$ when both M and N are open subsets of Euclidean spaces. Let P^k_m denote the vector space of polynomials in m variables of degree at most k. We give P^k_m a smooth structure by using the basis of monomials to identify it with a Euclidean space of dimension $\frac{(k+m)!}{k!m!}$. Given open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ we define the function

$$\begin{aligned} T^k_{U,V} : J^k(U, V) &\to U \times (P^k_m)^n \\ \sigma &\mapsto (\alpha(\sigma), T^k \sigma_{\alpha(\sigma)}). \end{aligned}$$

The Taylor polynomial $T^k \sigma_{\alpha(\sigma)}$ makes sense since the choice of smooth representative for σ does not matter due to Lemma 1.1.5. By the same lemma, we see $T^k_{U,V}$ is injective. We define smooth maps $\alpha' : U \times (P^k_m)^n \to U$ and $\beta' : U \times (P^k_m)^n \to V$ by $\alpha'(x, p) = x$ and $\beta'(x, p) = p(x)$, and we see these and the other functions form a commutative diagram:

$$\begin{tikzcd} J^k(U,V) \arrow[rr, "T^k_{U,V}"] \arrow[dr, "\alpha\times\beta"'] & & U \times (P^k_m)^n \arrow[dl, "\alpha'\times\beta'"] \\ & U \times V & \end{tikzcd}$$

We use this to define a smooth structure on $J^k(U, V)$. In fact, $T^k_{U,V}$ maps homeomorphically onto the smooth submanifold $(\beta')^{-1}(V) = \{(x, p) \mid p(x) \in V\}$. The following lemma implies that $J^k(M, N)$ is a smooth fiber bundle over $M \times N$, where given charts $\varphi : M \mathrel{\circ\!\!\rightarrow} \mathbb{R}^m$ and $\psi : N \mathrel{\circ\!\!\rightarrow} \mathbb{R}^n$, then $J^k(\operatorname{im}\varphi, \operatorname{im}\psi)$ is used as a local trivialization of $J^k(M, N)$ over $\operatorname{dom}\varphi \times \operatorname{dom}\psi$.

Lemma 1.1.6 ([GG73, Lemma II.2.6])**.** *With the above notation, suppose $U' \subseteq \mathbb{R}^m$ and $V' \subseteq \mathbb{R}^n$ are open, $g : U \to U'$ is a diffeomorphism, and $h : V \to V'$ is a smooth map. Then*

$$T^k_{U',V'} \circ (g^{-1})^* \circ h_* \circ (T^k_{U,V})^{-1} : (\beta')^{-1}(V) \to U' \times (P^k_m)^n$$

is smooth.

Proof. This map is defined by $(x, p) \mapsto (g(x), T^k(h \circ p \circ g^{-1})_{g(x)})$, where $x \in U$ and $p \in (P^k_m)^n$ is such that $p(x) \in V$. The first component is smooth since g is smooth. The second component is smooth since by the chain rule the coefficients of the kth Taylor polynomial are smooth functions of p and $g(x)$. □

Theorem 1.1.7 ([GG73, Theorem II.2.7], [KMS93, 12.4])**.** *If M and N are smooth manifolds, then*

- *$J^k(M, N)$ is a smooth manifold of dimension $m + n\frac{(k+m)!}{k!m!}$.*
- *$\alpha : J^k(M, N) \to M$, $\beta : J^k(M, N) \to N$, and $\alpha \times \beta : J^k(M, N) \to M \times N$ are smooth submersions, each giving $J^k(M, N)$ the structure of a fiber bundle.*
- *If $h : N \to N'$ is a smooth map, then $h_* : J^k(M, N) \to J^k(M, N')$ is smooth.*
- *If $h : M' \to M$ is a diffeomorphism, then $h^* : J^k(M, N) \to J^k(M', N)$ is a diffeomorphism.*
- *If $f : M \to N$ is smooth, then the k-jet extension $j^k f : M \to J^k(M, N)$ is a smooth section of α.*

Remark 1.1.8. One can identify $J^1_0(\mathbb{R}, M)$ with TM thought of as velocity vectors, since for $\sigma \in J^1_0(\mathbb{R}, M)$ and $\varphi : M \to \mathbb{R}$, the value of $\frac{\partial}{\partial t}|_{t=0}(\varphi \circ \sigma)$ does not depend on the representative for σ. This gives $J^1_0(\mathbb{R}, M)$ the structure of a vector bundle. There is also a vector bundle structure for $J^1(M, \mathbb{R})_0$ from the fact that $\mathbb{R}$ is a vector space, and $J^1(M, \mathbb{R})_0$ can be identified with T^*M since the map $J^1_p(M, \mathbb{R})_0 \otimes J^1_0(\mathbb{R}, M)_p \to \mathbb{R}$ given by $\tau \otimes \sigma \mapsto \frac{\partial}{\partial t}|_{t=0}(\tau \circ \sigma)$ is a nondegenerate pairing.

For $\sigma \in J^1_p(M, N)_q$, the induced map $\sigma_* : J^1_0(\mathbb{R}, M)_p \to J^1_0(\mathbb{R}, N)_q$ is linear, and in fact we can view $J^1(M, N)$ as a vector bundle over $M \times N$, giving an isomorphism of vector bundles

$$J^1(M, N) \cong \mathrm{Hom}(TM, TN),$$

where $\mathrm{Hom}(TM, TN)_{(p,q)} = \mathrm{Hom}(T_pM, T_qN)$. From this point of view, $j^1 f$ can be identified with $df : M \to \mathrm{Hom}(TM, f^*TN)$, which may be regarded as a section over the graph of f.

However, $J^k(M, N)$ is *not* a vector bundle for general k. It at least has a canonical section (over $M \times N$) of jets of constant functions. See Section 1.1.4 for the well-known characterization of $J^k_p(M, N)_q$ as ring homomorphisms. ■

Lemma 1.1.9. *Let M and N be smooth manifolds and $k, \ell \in \mathbb{N}$. The map*

$$\begin{aligned} J^{k+\ell}(M,N) &\to J^k(M, J^\ell(M,N)) \\ j^{k+\ell} f_p &\mapsto j^k(j^\ell f)_p \end{aligned}$$

is well-defined and smooth. In particular, we get a smooth map of bundles over M:

$$\begin{aligned} J^{\ell+1}(M,N) &\to \mathrm{Hom}(TM, TJ^\ell(M,N)) \\ j^{\ell+1} f_p &\mapsto d(j^\ell f)_p. \end{aligned}$$

Proof. See, for example, the proof of [GG73, II.3.4]. The idea is that if $f \in C^\infty(M,N)$, then $j^\ell f : M \to J^\ell(M,N)$ is smooth by Theorem 1.1.7. Hence, this defines an element of $J^k(M, J^\ell(M,N))$. The k-jet of j^ℓ depends smoothly on only the partial derivatives of f up to order $k+\ell$, so this map $C^\infty(M,N) \to J^k(M, J^\ell(M,N))$ factors through $J^{k+\ell}(M,N)$. The second statement follows from Remark 1.1.8. □

Remark 1.1.10. The sequence of jet bundles assemble into an inverse system

$$J^0(M,N) \leftarrow J^1(M,N) \leftarrow J^2(M,N) \leftarrow \cdots$$

by maps $j^{k+1} f_p \mapsto j^k f_p$, which is, in coordinates, Taylor polynomial truncation. We will only be using this inverse system to lift submanifolds of jet bundles to a common higher jet bundle, but the inverse limit of this system, the *infinite jet bundle* $J(M,N)$, can be thought of as a bundle of infinite Taylor series. By a classic theorem of Borel, every element of $J(M,N)$ is in fact the infinite Taylor series of a smooth map $M \to N$. ■

1.1.1 Generalities about pro-objects

We will be using various notions of germs, and to make these precise this section collects background on the theory of pro-objects of a category. Pro-objects are, informally, formal cofiltered limits of objects. In the Bourbaki formulation of topology [Bou98], pro-objects appear extensively in the form of filters on a topological space, where they may be regarded as "generalized sets."

An example of how pro-objects are "generalized spaces" is in Stone duality. The category FinSet of finite sets may be regarded as the category of finite discrete topological spaces. Using the idea from algebraic geometry that the opposite category of a category of spaces is its category of algebras, then it's significant that $\mathrm{FinSet}^{\mathrm{op}}$ is equivalent to the category FinBool of finite boolean algebras. General boolean algebras are filtered colimits of finite boolean algebras (which is to say Bool is equivalent to Ind(FinBool)), and so the category $\mathrm{Bool}^{\mathrm{op}}$ of spaces is equivalent to Pro(FinSet), the category of profinite sets. Profinite sets have concrete descriptions as inverse limits of finite sets in Set.

Without exploring any sort of formal duality, an example we will consider is the category Man of smooth manifolds and the functor $C^\infty : \mathsf{Man}^{\mathrm{op}} \to \mathbb{R}\text{-Alg}$ that takes a smooth manifold to its ring of functions. The category $\mathbb{R}$-Alg already has all filtered colimits, and so we can extend C^∞ to be a functor $\mathrm{Pro}(\mathsf{Man})^{\mathrm{op}} \to \mathbb{R}\text{-Alg}$, whose domain consists of "generalized smooth manifolds." The infinite jet bundle from Remark 1.1.10 is one such example of an object in Pro(Man). For decomposition purposes, we are interested in "generalized smooth submanifolds" that arise from formal intersections of open submanifolds. These have what Schommer-Pries in [SP09] calls a *halation* structure, which gives a tidy way to compose cobordisms without needing to choose arbitrary collar neighborhoods of the boundary.

For the following, we collect results from [Isa02], Lurie's account in [Lur18, Section 6.1], and properties of limits in the reference [Kel05, Section 3.4].

We start by working through facts about copresheaves for categories that admit finite limits to try, and the definition of the category of pro-objects will appear in Definition 1.1.24 along with basic properties starting in Theorem 1.1.25.

Definition 1.1.11. Suppose $\mathcal{C}$ is a category. It is *cofiltered* if every finite diagram in C has a cone, and it is *finitely complete* it admits all finite limits (evidentally, every finitely complete category is cofiltered). Dually, C is filtered if every finite diagram in C has a cocone.

A *cofiltered limit* in $\mathcal{C}$ is a limit $\lim_{i\in I} F(i)$ for $F : I \to \mathcal{C}$ a functor with I a cofiltered category. Dually, a *filtered colimit* in $\mathcal{C}$ is a colimit $\mathrm{colim}_{i\in I} F(i)$ for $F : I \to \mathcal{C}$ a functor with I a filtered category. $\Diamond$

Similarly to how finitely complete categories are characterized by having a terminal object and admitting binary products and equalizers, recall that a cofiltered category $\mathcal{C}$ is characterized by three properties:

1. There exists an object of $\mathcal{C}$.

2. For all objects $x, y \in \mathcal{C}$ there exists an object $z \in \mathcal{C}$ and morphisms $z \to x$ and $z \to y$.

3. For all objects $x, y \in \mathcal{C}$ and morphisms $f, g : x \to y$ there is an object $z \in \mathcal{C}$ and morphism $e : z \to x$ such that $e \circ f = e \circ g$.

From these three properties one may deduce that every finite diagram in $\mathcal{C}$ has a cone.

Example 1.1.12. Suppose P is a poset thought of as a category, where for $x, y \in P$ with $x \le y$ we have a morphism $x \to y$. A subset $F \subseteq P$ is an *order filter* if the following conditions hold:

1. F is nonempty.

2. F is downward directed (for every $x, y \in F$, there is some $z \in F$ such that $z \le x$ and $z \le y$).

3. F is upward closed (for every $x \in F$ and $y \in P$, if $x \le y$ then $y \in F$).

Since morphism sets in P are subsingletons, then the first two conditions of an order filter imply that P is a cofiltered category. The third condition is unnecessary for this.

The poset $\mathrm{Open}(X)$ of open subsets of a topological space X is finitely complete (and thus a cofiltered category) since it is a meet-semilattice with respect to intersections. $\Diamond$

Definition 1.1.13. Suppose $\mathcal{C}$ and $\mathcal{D}$ are finitely complete categories. We use $[\mathcal{C}, \mathcal{D}]$ or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ to denote the category of functors from $\mathcal{C}$ to $\mathcal{D}$. A functor $F : \mathcal{C} \to \mathrm{Set}$ is *left exact* if for every finite diagram $G : I \to \mathcal{C}$ then there is an isomorphism

$$F(\lim_{i\in I} G(i)) \approx \lim_{i\in I} F(G(i)).$$

We write $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the full subcategory of left exact functors. $\Diamond$

For a small category $\mathcal{C}$, we have the hom bifunctor $\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$, and from this we get the contravariant Yoneda embedding functor $Y : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ defined by $Y(c) = \mathcal{C}(c,-)$ on objects. Elements of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ are known as *copresheaves*, and functors of the form $\mathcal{C}(c,-)$ are known as *representable copresheaves*. The Yoneda lemma is that, for all $c \in \mathcal{C}$ and $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, we have a bijection $\mathrm{Nat}(\mathcal{C}(c,-), F) \approx F(c)$ defined by $\eta \mapsto \eta_c(\mathrm{id}_c)$ that is natural in both c and F.

Lemma 1.1.14. *Suppose $\mathcal{C}$ is a finitely complete small category. For all $c \in C$ we have that $\mathcal{C}(c,-) \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$.*

Proof. This is a simple matter of expanding definitions. Suppose $F : I \to \mathcal{C}$ is a finite diagram. Then since Set is complete and the hom functor is continuous,

$$\mathcal{C}(c,-)(\lim_{i\in I} F(i)) \approx \mathcal{C}(c, \lim_{i\in I} F(i)) \approx \lim_{i\in I} \mathcal{C}(c, F(i)) \approx \lim_{i\in I} \mathcal{C}(c,-)(F(i)). \qquad \square$$

We review the well-known fact that every copresheaf is a colimit of representable copresheaves, which will be stated in Lemma 1.1.15. Given a functor $F : \mathcal{C} \to \mathrm{Set}$, its *category of elements* $\mathrm{el}(F)$ has as objects pairs (c, x) for $c \in \mathcal{C}$ and $x \in F(c)$ and has as morphism sets

$$\mathrm{el}(F)((c,x),(d,y)) := \{f \in \mathcal{C}(c,d) \mid F(f)(x) = y\}.$$

The category of elements has a projection functor $\mathrm{el}(F) \to \mathcal{C}$ defined by $(c,x) \mapsto c$ on objects.

It is a fact that for $G \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ that there is a bijection

$$\mathrm{Nat}(F, G) \approx \lim_{(c,x)\in \mathrm{el}(F)} G(c)$$

defined by $\eta \mapsto ((c,x) \in \mathrm{el}(F) \mapsto \eta_c(x))$, and it is natural in F and G. See for example [Kel05, Eq. 3.30].

Lemma 1.1.15. *Suppose $\mathcal{C}$ is a small category. For all $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ we have*

$$F \approx \operatorname*{colim}_{(c,x)\in \mathrm{el}(F)^{\mathrm{op}}} \mathcal{C}(c,-),$$

which is to say that F is a colimit of representable copresheaves.

Proof. Let $G \in \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ be arbitrary. The following bijections are natural in G:

$$\begin{aligned}\mathrm{Nat}(\operatorname*{colim}_{(c,x)\in\mathrm{el}(F)^{\mathrm{op}}} \mathcal{C}(c,-), G) &\approx \lim_{(c,x)\in\mathrm{el}(F)} \mathrm{Nat}(\mathcal{C}(c,-), G)\\ &\approx \lim_{(c,x)\in\mathrm{el}(F)} G(c)\\ &\approx \mathrm{Nat}(F,G).\end{aligned}$$

Hence $\mathrm{Nat}(\mathrm{colim}_{(c,x)\in\mathrm{el}(F)^{\mathrm{op}}} \mathcal{C}(c,-), -) \approx \mathrm{Nat}(F,-)$, and thus by the Yoneda lemma we have that $\mathrm{colim}_{(c,x)\in\mathrm{el}(F)^{\mathrm{op}}} \mathcal{C}(c,-) \approx F$. ☐

***Warning* 1.1.16.** Colimits in the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ need not coincide with colimits in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$. For example, if $\mathcal{C}$ is finitely complete then the colimit of the empty diagram in $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ is the functor $c \mapsto \mathcal{C}(\top, -)$ where $\top \in \mathcal{C}$ is the terminal object, but as a colimit in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ this is instead $c \mapsto \varnothing$. We see in Lemma 1.1.19 that filtered colimits can be safely calculated as colimits in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$. ◊

Lemma 1.1.17. *Suppose $\mathcal{C}$ is a finitely complete small category and $F \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$. Then $\mathrm{el}(F)$ is a cofiltered small category.*

Proof. Let $D : I \to \mathrm{el}(F)$ be a finite diagram. For $i \in I$, write $(c_i, x_i) = D(i)$ for the components. We have that $\lim_{i\in I} c_i$ exists and that there is a bijection $f : \lim_{i\in I} F(c_i) \to F(\lim_{i\in I} c_i)$. It is easy to check that $(\lim_{i\in c_i} c_i, f(i \in I \mapsto x_i))$ is a cone for D. ☐

Lemma 1.1.18. *Suppose $\mathcal{C}$ is a finitely complete small category. For all $F \in \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ we have that F is a small filtered colimit of representable copresheaves.*

Proof. By Lemma 1.1.15, we have that F is a colimit of representable copresheaves over the category $\mathrm{el}(F)^{\mathrm{op}}$. It suffices to show that $\mathrm{el}(F)$ is a cofiltered category, which we did in Lemma 1.1.17. ☐

Lemma 1.1.19. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ be a diagram with I a small filtered category. Then $\mathrm{colim}_{i\in I}^{\mathrm{Fun}(\mathcal{C},\mathrm{Set})} D(i)$ is left exact. Hence, $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ admits all small filtered colimits.*

Proof. The limit $F = \mathrm{colim}_{i\in I}^{\mathrm{Fun}(\mathcal{C},\mathrm{Set})} D(i)$ exists since $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is cocomplete, where for $c \in \mathcal{C}$ we have $F(c) = \mathrm{colim}_{i\in I} D(i)(c)$. Let $E : J \to \mathcal{C}$ be a finite diagram. Then

$$\begin{aligned}F(\lim\nolimits_{j\in J}^{\mathcal{C}} E(j)) &= \mathrm{colim}_{i\in I}^{\mathrm{Set}} D(i)(\lim\nolimits_{j\in J}^{\mathcal{C}} E(j))\\ &\approx \mathrm{colim}_{i\in I}^{\mathrm{Set}} \lim\nolimits_{j\in J}^{\mathrm{Set}} D(i)(E(j))\\ &\approx \lim\nolimits_{j\in J}^{\mathrm{Set}} \mathrm{colim}_{i\in I}^{\mathrm{Set}} D(i)(E(j))\\ &= \lim\nolimits_{j\in J}^{\mathrm{Set}} F(E(j))\end{aligned}$$

since in Set finite limits commute with filtered colimits (see [ML98, IX.2.1]). Therefore F is left exact. ☐

Lemma 1.1.20. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ be a small diagram. Then $\lim_{i\in I}^{\mathrm{Fun}(\mathcal{C},\mathrm{Set})} D(i)$ is left exact. Hence, $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ admits all small limits.*

Proof. This is similar to Lemma 1.1.19 but we replace "colim" with "lim" and instead make use of the fact that small limits commute. □

Lemma 1.1.21. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I\times J \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})$ be such that I is finite and J is a small filtered category. Then*

$$\lim_{i\in I} \mathrm{colim}_{j\in J} D(i, j) \approx \mathrm{colim}_{j\in J} \lim_{i\in I} D(i, j).$$

Proof. Both of these (co)limits exist by Lemma 1.1.19 and Lemma 1.1.20. We have that for all $c \in \mathcal{C}$ that

$$\lim_{i\in I} \mathrm{colim}_{j\in J} D(i, j)(c) \approx \mathrm{colim}_{j\in I} \lim_{i\in J} D(i, j)(c),$$

and this is natural in c, hence these bijections define a natural isomorphism between the functors. □

Lemma 1.1.22. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I \to \mathcal{C}$ be a finite diagram. Then*

$$\mathrm{colim}_{i\in I^{\mathrm{op}}}^{\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathrm{Set})} \mathcal{C}(D(i), -) \approx \mathcal{C}(\lim_{i\in I} D(i), -).$$

Proof. Since $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ is cocomplete, we know $F = \mathrm{colim}_{i\in I^{\mathrm{op}}}^{\mathrm{Fun}(\mathcal{C},\mathrm{Set})} \mathcal{C}(D(i), -)$ exists, and what remains is to show that F is left exact. Let $E : J \to \mathcal{C}$ be a finite diagram. Then,

$$\begin{aligned} F(\lim_{j\in J} E(j)) &= \operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathcal{C}(D(i), \lim_{j\in J} E(j)) \\ &\approx \operatorname*{colim}_{i\in I^{\mathrm{op}}} \lim_{j\in J} \mathcal{C}(D(i), E(j)) \\ &\approx \lim_{j\in J} \operatorname*{colim}_{i\in I^{\mathrm{op}}} \mathcal{C}(D(i), E(j)) \\ &= \lim_{j\in J} F(E(j)), \end{aligned}$$

where the colimit and limit commute since I is cofiltered and J is finite. □

Lemma 1.1.23. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I \to \mathcal{C}$ be a small diagram such that $\mathrm{colim}_{i\in I} D(i)$ exists. Then*

$$\lim_{i\in I^{\mathrm{op}}}^{\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C},\mathrm{Set})} \mathcal{C}(D(i), -) \approx \mathcal{C}(\mathrm{colim}_{i\in I} D(i), -).$$

Proof. This is continuity of the hom functor. □

We now give a definition of the category of pro-objects in Definition 1.1.24, which will be followed up with the equivalent version from [Isa02] in Theorem 1.1.29.

Definition 1.1.24. Suppose $\mathcal{C}$ is a finitely complete small category. Then

$$\mathrm{Pro}(\mathcal{C}) := \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$$

is the *category of pro-objects in* $\mathcal{C}$. Using the Yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ we regard $\mathcal{C}$ as a subcategory of $\mathrm{Pro}(\mathcal{C})$ and identify $c \in \mathcal{C}$ with the pro-object $\mathcal{C}(c, -)$. $\Diamond$

Theorem 1.1.25. *Suppose $\mathcal{C}$ is a finitely complete small category.*

1. *The embedding $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ preserves finite limits and all small colimits.*
2. $\mathrm{Pro}(\mathcal{C})$ *admits all colimits (i.e., it is cocomplete).*
3. $\mathrm{Pro}(\mathcal{C})$ *is admits all small cofiltered limits.*
4. *Every object C in $\mathrm{Pro}(\mathcal{C})$ is a small cofiltered limit of objects in $\mathcal{C}$, in the sense that $C \approx \lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} F(i)$ for some small cofiltered diagram $F : I \to \mathcal{C}$.*
5. *For $C \in \mathrm{Pro}(\mathcal{C})$ and $d \in \mathcal{C}$, $\mathrm{Pro}(\mathcal{C})(C, d) \approx C(d)$, and this is natural in both C and d.*
6. *For $D : I \times J \to \mathrm{Pro}(\mathcal{C})$ a diagram such that I is finite and J is a small cofiltered category,*
$$\operatorname*{colim}_{i\in I} \lim_{j\in J} D(i, j) \approx \lim_{j\in J} \operatorname*{colim}_{i\in I} D(i, j).$$

Proof. (1) is the dual of Lemmas 1.1.22 and 1.1.23, (2) is the dual of Lemma 1.1.20, (3) is the dual of Lemma 1.1.19, (4) is the dual of Lemma 1.1.18, (5) is the Yoneda lemma, and (6) is the dual of Lemma 1.1.21. $\square$

Remark 1.1.26. The category $\mathrm{Pro}(\mathcal{C})$ is finitely complete, too, which is Corollary 1.1.41. $\blacksquare$

Lemma 1.1.27. *Suppose $\mathcal{C}$ is a finitely complete small category. Let $D : I \to \mathcal{C}$ and $E : J \to \mathcal{C}$ be small cofiltered diagrams. Then, identifying D with $\lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} D(i)$ and E with $\lim_{j\in J}^{\mathrm{Pro}(\mathcal{C})} E(j)$, we have that*

$$\mathrm{Pro}(\mathcal{C})(D, E) \approx \lim_{j\in J} \operatorname*{colim}_{i\in I} \mathcal{C}(D(i), E(j)).$$

Proof. This follows from exactness of the hom functor and the Yoneda lemma:

$$\begin{aligned}
\mathrm{Pro}(\mathcal{C})(D, E) &= \mathrm{Pro}(\mathcal{C})(\lim\nolimits_{i\in I}^{\mathrm{Pro}(\mathcal{C})} D(i), \lim\nolimits_{j\in J}^{\mathrm{Pro}(\mathcal{C})} E(j)) \\
&\approx \lim\nolimits_{j\in J}^{\mathrm{Set}} \mathrm{Pro}(\mathcal{C})(\lim\nolimits_{i\in I}^{\mathrm{Pro}(\mathcal{C})} D(i), E(j)) \\
&= \lim\nolimits_{j\in J}^{\mathrm{Set}} \mathrm{Nat}(\mathcal{C}(E(j), -), \operatorname{colim}\nolimits_{i\in I^{\mathrm{op}}}^{\mathrm{Fun}(\mathcal{C},\mathrm{Set})} \mathcal{C}(D(i), -)) \\
&\approx \lim\nolimits_{j\in J}^{\mathrm{Set}} \operatorname{colim}\nolimits_{i\in I}^{\mathrm{Set}} \mathcal{C}(D(i), E(j)).
\end{aligned}$$
$\square$

Remark 1.1.28. There's a relatively straightforward formulation of morphism composition from the point of view of the representation in Lemma 1.1.27. Suppose $D : I \to \mathcal{C}$, $E : J \to \mathcal{C}$, and $F : K \to \mathcal{C}$ are small cofiltered diagrams regarded as pro-objects. Let

$$f \in \lim_{j \in J} \operatorname{colim}_{i \in I} \mathcal{C}(D(i), E(j)) \qquad \text{and} \qquad g \in \lim_{k \in K} \operatorname{colim}_{j \in J} \mathcal{C}(E(j), F(k))$$

be morphisms. We construct

$$g \circ f \in \lim_{k \in K} \operatorname{colim}_{i \in I} \mathcal{C}(D(i), F(k))$$

using the following observation. The morphism f has the data that for all $j \in J$ there is an $i \in I$ and an element $f_j \in \mathcal{C}(D(i), E(j))$, and for g, for all $k \in K$ there is a $j \in J$ and an element $g_k \in \mathcal{C}(E(i), F(j))$.

Hence, the data for $g \circ f$ is that for all $k \in K$, we take the $j \in J$ and the $g_k \in \mathcal{C}(E(j), F(k))$ from g and then take the $i \in I$ and the $f_j \in \mathcal{C}(D(i), E(j))$ from f, and then for this $i \in I$ we set $(g \circ f)_k := g_k \circ f_j \in \mathcal{C}(D(i), F(k))$. This is reminiscent of the continuity of the composition of continuous functions. ■

We give the following alternative definition of the category of pro-objects, which appears in [Isa02].

Theorem 1.1.29 (Category of pro-objects, second version)**.** *Suppose $\mathcal{C}$ is a small category. The* category of pro-objects in $\mathcal{C}$*, which we temporarily denote* $\operatorname{Pro}'(\mathcal{C})$*, is the category whose objects are small cofiltered diagrams $I \to \mathcal{C}$ and whose morphisms sets for pro-objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ are given by*

$$\operatorname{Pro}'(\mathcal{C})(X, Y) := \lim_{j \in J} \operatorname{colim}_{i \in I} \mathcal{C}(X_i, Y_j).$$

Composition is defined as in Remark 1.1.28. There is an embedding $\mathcal{C} \to \operatorname{Pro}'(\mathcal{C})$ by sending $c \in \mathcal{C}$ to the diagram $C : * \to \mathcal{C}$ with $C(*) = c$.

If $\mathcal{C}$ is finitely complete, the functor $F : \operatorname{Pro}'(\mathcal{C}) \to \operatorname{Pro}(\mathcal{C})$ defined by $(C : I \to \mathcal{C}) \mapsto \lim^{\operatorname{Pro}(\mathcal{C})}_{i \in I} C_i$ is an equivalence.

Proof sketch. The inverse is that every pro-object is a cofiltered limit. □

Recall that for a functor $X : I \to J$, a description of the comma category $(X \downarrow j)$ for $j \in J$ is that the set of objects is $\prod_{i \in I} J(X(i), j)$ and the set of morphisms from $(i, f \in J(X(i), j))$ to $(i', f' \in J(X(i'), j))$ is the set of all $g \in I(i, i')$ such that the following diagram commutes:

$$\begin{tikzcd} X(i) \arrow[rr, "D(g)"] \arrow[dr, "f"'] & & X(i') \arrow[dl, "f'"] \\ & j & \end{tikzcd}$$

Recall also that a category I is *connected* if there is exactly one equivalence class for the equivalence relation generated by asserting that $i, j \in I$ are related if there exists a $k \in I$ and morphisms $k \to i$ and $k \to j$.

Definition 1.1.30. A functor $X : I \to J$ is *final* if for all $j \in J$ the category $(X \downarrow j)$ is nonempty and connected. $\Diamond$

The significance is the following theorem:

Theorem 1.1.31 ([ML98, Theorem IX.3.1])**.** *If $X : I \to J$ is a final functor and $F : J \to \mathcal{C}$ is a functor such that $\operatorname{colim}(F \circ X)$ exists, then $\operatorname{colim} F$ exists and the canonical map $\operatorname{colim}(F \circ X) \to \operatorname{colim} F$ is an isomorphism.*

Corollary 1.1.32. *Suppose $\mathcal{C}$ is a finitely complete small category, $F : J \to \mathcal{C}$ is a small cofiltered diagram, and $X : I \to J$ is a final functor between small cofiltered categories. Then*

$$\lim_{j\in J}^{\mathrm{Pro}(\mathcal{C})} F(j) \approx \lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} F(X(i)).$$

There is a useful tool for recognizing final functors between cofiltered categories:

Lemma 1.1.33 ([SP09, Lemma 3.26])**.** *Let $X : I \to J$ be a functor between small cofiltered categories. Then X is final if and only if for every $j \in J$ the category $(X \downarrow j)$ is cofiltered.*

Moreover, if X is full then X is final if and only if for every $j \in J$ there exists an $i \in I$ and a morphism $X(i) \to j$.

Remark 1.1.34. Suppose P is a poset category, $F \subseteq P$ is an order filter, and $F' \subseteq P$ is a cofiltered subcategory such that $F' \subseteq F$. If the inclusion functor $F' \hookrightarrow F$ is final, then the upward closure of F' is F. The upward closed axiom of order filters can be thought of as a normalization so that every inclusion that induces a final functor is an equality. ■

Theorem 1.1.35. *Suppose $\mathcal{C}$ is a finitely complete small category, $\mathcal{D}$ is a small category admitting cofiltered limits, and $f : \mathcal{C} \to \mathcal{D}$ is a functor. Then there is a functor $F : \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ that preserves cofiltered limits such that the composition $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ is naturally isomorphic to f. This extension is essentially unique.*

If $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D})$ is the full subcategory of functors that preserve small cofiltered limits, the functor $\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ induced by precomposition with $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$ is an equivalence.

Proof. Define F by

$$F(C) = \lim_{(c,x)\in \mathrm{el}(C)}^{\mathcal{D}} f(c).$$

We see that for $C \in \mathrm{Pro}(\mathcal{C})$ and $d \in \mathcal{D}$ that

$$\mathcal{D}(d, F(C)) \approx \lim_{(c,x)\in \mathrm{el}(C)}^{\mathrm{Set}} \mathcal{D}(d, f(c)) \approx \mathrm{Nat}(C, (c \mapsto \mathcal{D}(d, f(c)))),$$

and this is natural in C and d. Hence, if $\lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} C_i$ is a cofiltered limit,

$$\begin{aligned}
\mathcal{D}(d, F(\lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} C_i)) &\approx \mathrm{Nat}(\lim_{i\in I}^{\mathrm{Pro}(\mathcal{C})} C_i, (c \mapsto \mathcal{D}(d, f(c)))) \\
&\approx \lim_{i\in I}^{\mathrm{Set}} \mathrm{Nat}(C_i, (c \mapsto \mathcal{D}(d, f(c)))) \\
&\approx \lim_{i\in I}^{\mathrm{Set}} \mathcal{D}(d, F(C_i)) \\
&\approx \mathcal{D}(d, \lim_{i\in I}^{\mathcal{D}} F(C_i)).
\end{aligned}$$

Since this is natural in d, we have that $F(\lim^{\mathrm{Pro}(\mathcal{C})}_{i\in I} C_i) \approx \lim^{\mathcal{D}}_{i\in I} F(C_i)$, and so F preserves cofiltered limits.

Now for essential uniqueness of the extension. If $F' : \mathrm{Pro}(\mathcal{C}) \to \mathcal{D}$ is another extension, then since it preserves cofiltered limits we have that for all $C \in \mathrm{Pro}(\mathcal{C})$ that

$$\begin{aligned} F'(C) &\approx F'(\lim^{\mathrm{Pro}(\mathcal{C})}_{(c,x)\in \mathrm{el}(C)} c) \\ &\approx \lim^{\mathcal{D}}_{(c,x)\in \mathrm{el}(C)} F'(c), \end{aligned}$$

hence $F'(C)$ is essentially determined by its restriction to $\mathcal{C}$, which is to say it is essentially determined by f. □

Corollary 1.1.36. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are finitely complete small categories and $f : \mathcal{C} \to \mathcal{D}$ is a functor. Then there is an induced functor $F : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ that preserves cofiltered limits such that the compositions $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$ and $\mathcal{C} \to \mathcal{D} \hookrightarrow \mathrm{Pro}(\mathcal{D})$ are naturally isomorphic. This extension is essentially unique.*

If f is left exact, then so is the induced functor F, because F is right adjoint to the functor $f^* : \mathrm{Pro}(\mathcal{D}) \to \mathrm{Pro}(\mathcal{C})$ defined by $f^*(D) = D \circ f$ with D regarded as a functor.

Proof. We have the composition $\mathcal{C} \xrightarrow{f} \mathcal{D} \hookrightarrow \mathrm{Pro}(\mathcal{D})$, which is a functor from a finitely complete small category to a small category that admits cofiltered limits. Hence by Theorem 1.1.35 there is an essentially unique lift $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$.

Now suppose that f is left exact. The claim is that for all $C \in \mathrm{Pro}(\mathcal{C})$ and $D \in \mathrm{Pro}(\mathcal{D})$ that we have isomorphisms

$$\mathrm{Pro}(\mathcal{C})(f^*(D), C) \approx \mathrm{Pro}(\mathcal{D})(D, F(C))$$

natural in C and D. We calculate that

$$\begin{aligned} \mathrm{Pro}(\mathcal{D})(D, F(C)) &= \mathrm{Pro}(\mathcal{D})(D, \lim^{\mathrm{Pro}(\mathcal{D})}_{(c,x)\in \mathrm{el}(C)} f(c)) \\ &\approx \lim^{\mathrm{Set}}_{(c,x)\in \mathrm{el}(C)} \mathrm{Pro}(\mathcal{D})(D, f(c)) \\ &\approx \lim^{\mathrm{Set}}_{(c,x)\in \mathrm{el}(C)} D(f(c)) \\ &\approx \mathrm{Nat}(C, f^*(D)) \\ &= \mathrm{Pro}(\mathcal{C})(f^*(D), C). \end{aligned}$$ □

***Warning* 1.1.37.** Cofiltered limits in $\mathcal{C}$ need not coincide with cofiltered limits in $\mathrm{Pro}(\mathcal{C})$. For example, let $\mathscr{P}(\mathbb{R})$ be the poset category of all subsets of $\mathbb{R}$, which contains $\mathrm{Open}(\mathbb{R})$ as a subcategory. The inclusion $\mathrm{Open}(\mathbb{R}) \hookrightarrow \mathscr{P}(\mathbb{R})$ induces a functor $F : \mathrm{Pro}(\mathrm{Open}(\mathbb{R})) \to \mathscr{P}(\mathbb{R})$ that preserves cofiltered limits. Let $I \subseteq \mathrm{Open}(\mathbb{R})$ be the subcategory of all open subsets that contain 0, which is cofiltered. On one hand, $\lim^{\mathrm{Open}(\mathbb{R})}_{s\in I} s = \varnothing$ since this is the only open set contained in every open set from I, but on the other $\lim^{\mathrm{Pro}(\mathrm{Open}(\mathbb{R}))}_{s\in I} s \neq \varnothing$ since

$$F(\lim^{\mathrm{Pro}(\mathrm{Open}(\mathbb{R}))}_{s\in I} s) = \lim^{\mathscr{P}(\mathbb{R})}_{s\in I} s = \bigcap_{s\in I} s = \{0\}.$$

This F functor takes formal intersections of open sets to their intersection. ◊

Suppose $\mathcal{C}$ is a finitely complete small category and I is a small category. Consider the functor $\mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$ induced by $\mathcal{C} \hookrightarrow \mathrm{Pro}(\mathcal{C})$. Since $\mathrm{Pro}(\mathcal{C})$ admits small cofiltered limits, so does $\mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$, hence the functor induces an essentially unique functor $\mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$. In many useful situations this functor is an equivalence.

Definition 1.1.38. A category I is *loopless* if every endomorphism is an identity, and I is *cofinite* if I is loopless and for all $i \in I$ there are only finitely many j such that $I(i, j)$ is nonempty. $\diamondsuit$

Theorem 1.1.39 ([Isa02, 3.3, 3.6, 3.8])**.** *Suppose $\mathcal{C}$ is a finitely complete small category and I is a small category. Let $F : \mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$ denote the functor above.*

- *If I is finite, then F is an equivalence.*
- *If I is cofinite, then F is essentially surjective.*
- *If the morphism sets of I are finite, then F is essentially surjective.*

Lemma 1.1.40 (Level representations)**.** *Suppose $\mathcal{C}$ is a finitely complete small category and I a small category. Suppose the above functor $F : \mathrm{Pro}(\mathrm{Fun}(I, \mathcal{C})) \to \mathrm{Fun}(I, \mathrm{Pro}(\mathcal{C}))$ is essentially surjective.*

For every functor $C : I \to \mathrm{Pro}(\mathcal{C})$ there exists a functor $C' : I \times J \to \mathcal{C}$ with J a small cofiltered category such that C is equivalent to the functor $i \mapsto \lim^{\mathrm{Pro}(\mathcal{C})}_{j \in J} C'(i, j)$. This functor C' is known as a level representation *of C.*

Proof. Since F is essentially surjective, there is a pro-object in $\mathrm{Pro}(\mathrm{Fun}(I, C))$ whose image through F is isomorphic to C. We may represent this pro-object as a cofiltered limit over some cofiltered category J of elements of $\mathrm{Fun}(I, C)$, and then we may use the natural isomorphism $\mathrm{Fun}(J, \mathrm{Fun}(I, C)) \approx \mathrm{Fun}(I \times J, C)$. $\square$

Corollary 1.1.41. *Suppose $\mathcal{C}$ is a finitely complete small category. Then $\mathrm{Pro}(\mathcal{C})$ is finitely complete.*

Proof. Let $F : I \to \mathrm{Pro}(\mathcal{C})$ be a finite diagram. By Theorem 1.1.39 and Lemma 1.1.40 there is a level representation $F' : I \times J \to \mathcal{C}$ for some cofiltered small category J. Then one can check that $\lim^{\mathrm{Pro}(\mathcal{C})}_{i \in I} F(i)$ is $\lim^{\mathrm{Pro}(\mathcal{C})}_{j \in J} \lim^{\mathcal{C}}_{i \in I} F'(i, j)$. $\square$

1.1.2 Pro-open sets

For a topological space X, the poset category $\mathrm{Open}(X)$ is finitely complete since finite intersections of open sets are open. Hence, we have a category $\mathrm{Pro}(\mathrm{Open}(X))$ of *pro-open sets*.[1] Elements of $\mathrm{Pro}(\mathrm{Open}(X))$ may be regarded as formal intersections of open sets in that

[1]This is admittedly a misleading hyphenation, where "pro-(open set)" would be more accurate. Another option is "open pro-subset" since it does turn out that pro-open sets are a kind of pro-object in the category of subsets. A dear colleague suggested "proöpen set," but the only time one meets a dieresis is literature of the non-mathematical type.

each $U \in \mathrm{Pro}(\mathrm{Open}(X))$ is represented by some small cofiltered diagram $U : I \to \mathrm{Open}(X)$ such that $U = \lim_{i\in I}^{\mathrm{Pro}(\mathrm{Open}(X))} U_i$. Pro-open sets may be thought of as being "generalized open sets," we give a characterization in Lemma 1.1.46 that they are essentially order filters on $\mathrm{Open}(X)$. This may seem like an unnecessarily abstract approach for this object, but we feel that pro-objects motivate and give intuition for finding canonical constructions for germs.

Let us start by examining a basic invariant of pro-open sets. The inclusion $\mathrm{Open}(X) \hookrightarrow \mathscr{P}(X)$ into the poset of all subsets of X induces a functor

$$\ker : \mathrm{Pro}(\mathrm{Open}(X)) \to \mathscr{P}(X)$$

since $\mathscr{P}(X)$ admits all cofiltered limits.[2] Representing pro-open sets $U \in \mathrm{Pro}(\mathrm{Open}(X))$ by small cofiltered diagrams $I \to \mathrm{Open}(X)$, this functor has a simple description:

$$\ker U = \bigcap_{i\in I} U_i.$$

This functor is not in general an equivalence, but one can see that it is full and faithful when X is a T_1 space (Lemma 1.1.49).

We will now develop theory of pro-open sets. We would like to think of pro-open sets as if they were sets as much as possible, but to avoid some pitfalls we will use the following conventions:

- We use $\subseteq$ and $\cup$ for both pro-open sets and open sets.
- We use $\cap$ for finite intersections of both pro-open sets and open sets.
- We use limit notation for infinite intersections since $\mathrm{Open}(M)$ does not generally have them.
- When we work with arbitrary subsets A, we always write $\mathcal{N}(A)$ for the pro-open set generated by A since, for example, only $\mathcal{N}(A\cap B) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B)$ generally holds.

In this section we use the convention that open sets are lower case (u, v, ...) and pro-open sets are upper case (U, V, ...).

Lemma 1.1.42. *Suppose X is a topological space. The category $\mathrm{Pro}(\mathrm{Open}(X))$ is thin (i.e., it is a preorder), and the inclusion $\mathrm{Open}(X) \hookrightarrow \mathrm{Pro}(\mathrm{Open}(X))$ is full. Hence, we are justified in using the notation $U \subseteq V$ for $\mathrm{Pro}(\mathrm{Open}(X))(U,V) \neq \varnothing$.*

Proof. Suppose $U : I \to \mathrm{Open}(X)$ and $V : J \to \mathrm{Open}(X)$ are pro-objects. Then

$$\begin{aligned}\mathrm{Pro}(\mathrm{Open}(X))(U,V) &\approx \lim\nolimits_{j\in J} \mathrm{colim}_{i\in I} \mathrm{Open}(X)(U_i, V_j)\\ &\approx \lim\nolimits_{j\in J} \mathrm{colim}_{i\in I} \{* \mid U_i \subseteq V_j\}\\ &\approx \lim\nolimits_{j\in J} \{* \mid \exists i \in I, U_i \subseteq V_j\}\\ &\approx \{* \mid \forall j \in J, \exists i \in I, U_i \subseteq V_j\}.\end{aligned}$$

Hence, these morphism sets are subsingletons for all such U and V. □

[2] This usage of "kernel" matches the corresponding notion in filter theory.

Corollary 1.1.43. *Suppose X is a topological space and $U, V \in \mathrm{Pro}(\mathrm{Open}(X))$. Then $U \subseteq V$ if and only if for all $v \supseteq V$ there exists some $u \supseteq U$ such that $u \subseteq v$, where u and v are open sets.*

Corollary 1.1.44. *Suppose X is a topological space and $U \in \mathrm{Pro}(\mathrm{Open}(X))$. Then*

$$U \approx \lim_{u \supseteq U}^{\mathrm{Pro}(\mathrm{Open}(X))} u$$

where the limit ranges over all $u \in \mathrm{Open}(X)$ such that $U \subseteq u$.

Proof. One can check that, regarding U as an element of $\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Open}(X), \mathrm{Set})$, that $\mathrm{el}(U)$ is equivalent to the full subcategory of $\mathrm{Open}(X)$ generated by all $u \in \mathrm{Open}(X)$ such that $U \subseteq u$. Hence $\lim_{(u,x)\in \mathrm{el}(U)}^{\mathrm{Pro}(\mathrm{Open}(X))} u \approx \lim_{u \supseteq U}^{\mathrm{Pro}(\mathrm{Open}(X))} u$. □

Corollary 1.1.45. *Suppose X is a topological space and $f : \mathrm{Open}(X) \to \mathcal{C}$ is a functor to a small category admitting cofiltered limits. The essentially unique functor $F : \mathrm{Pro}(\mathrm{Open}(X)) \to \mathcal{C}$ that extends f is given by $F(U) = \lim_{u \supseteq U}^{\mathcal{D}} f(u)$.*

Lemma 1.1.46. *Suppose X is a topological space. There is a one-to-one correspondence between order filters on $\mathrm{Open}(X)$ and isomorphism classes of $\mathrm{Pro}(\mathrm{Open}(X))$. In particular, letting $\mathcal{F}$ denote the set of order filters on $\mathrm{Open}(X)$, the function*

$$\begin{aligned} \mathcal{F} &\to \mathrm{Pro}(\mathrm{Open}(X)) \\ F &\mapsto (u \mapsto \{* \mid u \in F\}) \end{aligned}$$

is injective and essentially surjective, and the function

$$\begin{aligned} \mathrm{Pro}(\mathrm{Open}(X)) &\to \mathcal{F} \\ U &\mapsto \{u \in \mathrm{Open}(X) \mid U \subseteq u\} \end{aligned}$$

is surjective and essentially injective.

Proof. Suppose we have $U \in \mathrm{Pro}(\mathrm{Open}(X))$ represented by $U : I \to \mathrm{Open}(X)$ with I a small cofiltered category. For all $u \in \mathrm{Open}(X)$,

$$U(u) \approx \mathrm{Pro}(\mathrm{Open}(X))(U, u) \approx \{* \mid \exists i \in I, U_i \subseteq u\}.$$

Hence, U is isomorphic to a pro-object whose values are either $\varnothing$ or $\{*\}$. We can use these objects as the skeleton of the category $\mathrm{Pro}(\mathrm{Open}(X))$. It suffices to show that this skeleton is in bijective correspondence with $\mathcal{F}$.

For U in the skeleton, U is determined by the set $\{u \in \mathrm{Open}(X) \mid U \subseteq u\}$ (and is, in particular, isomorphic to the object set for $\mathrm{el}(U)$), and one can check that this is an order filter on $\mathrm{Open}(X)$ from the conclusion of the above calculation.

Conversely, given an order filter $F \subseteq \mathrm{Open}(X)$, let $U \in \mathrm{Pro}(\mathrm{Open}(X))$ be the pro-object defined by $U(u) = \{* \mid u \in F\}$ for all $u \in \mathrm{Open}(X)$. Upward closure implies functoriality of U, nonemptiness of F implies $F(X) = \{*\}$ is the terminal object, and downward directedness implies F preserves binary products, thus F is left exact since $\mathrm{Open}(X)$ is a poset category. □

Remark 1.1.47. Here is how we can use reindexing lemmas to yield a similar result, which also reveals that the significance of filters is that they serve as the indexing categories for pro-open sets. We start by showing that pro-objects can all be represented by inclusion functors $J \hookrightarrow \mathrm{Open}(X)$ where J is cofiltered. Given a pro-object $U : I \to \mathrm{Open}(X)$, let $J = \{U(i) \mid i \in I\}$, which is a cofiltered subcategory of $\mathrm{Open}(X)$, and so the inclusion functor $V : J \hookrightarrow \mathrm{Open}(X)$ defines a pro-object. Let $X : I \to J$ be the functor from restricting the codomain of U, hence $U = V \circ X$. By Lemma 1.1.33, to check that X is final it suffices to check that $(X \downarrow j)$ is cofiltered for each $j \in J$. Objects of this category are inclusions $X(i) \subseteq j$ for $i \in I$ and morphisms are inclusions $X(i) \subseteq X(j)$ induced from morphisms $f : i \to j$, so it is easy to see that $(X \downarrow j)$ is cofiltered. Thus, by Corollary 1.1.32 U and V define the same pro-object.

Now, let $V : J \hookrightarrow \mathrm{Open}(X)$ be a pro-object where $J \subseteq \mathrm{Open}(X)$ is a cofiltered subcategory. Replacing J with the upwards closure of J does not change the resulting pro-object. Therefore, we may assume J is an order filter on $\mathrm{Open}(X)$. ■

Definition 1.1.48. Suppose X is a topological space. Define the functor $\mathcal{N} : \mathscr{P}(X) \to \mathrm{Pro}(\mathrm{Open}(X))$ by

$$\mathcal{N}(A) = \lim\nolimits_{u \in (A \downarrow \mathrm{Open}(X))}^{\mathrm{Pro}(\mathrm{Open}(X))} u,$$

where, using the notation for the coslice category, the limit ranges over open subsets u that contain A. For $A \subseteq X$, we call $\mathcal{N}(A)$ the *neighborhood pro-open set of* A. When U is a pro-open set, we write $x \in U$ for the relation $\mathcal{N}\{x\} \subseteq U$. ◊

If $u \in \mathrm{Open}(X)$, then note that $\mathcal{N}(u) = u$ as pro-open sets. Furthermore, for $A \subseteq X$ then one can check that $\mathcal{N}(A) \subseteq u$ holds if and only if $A \subseteq u$ does.

Lemma 1.1.49. *A topological space X is T_1 if and only if $\mathcal{N} : \mathscr{P}(X) \to \mathrm{Pro}(\mathrm{Open}(X))$ is full and faithful.*

Proof. A characterization of T_1 spaces is that every set is the intersection of all the open sets containing it, which is equivalent to $\ker \circ \mathcal{N}$ being the identity. □

Lemma 1.1.50. *If X is a topological space and $\{A_i\}_i$ is a family of subsets of X, then*

$$\mathcal{N}(\bigcup_i A_i) = \bigcup_i \mathcal{N}(A_i).$$

Proof. For every $u \in \mathrm{Open}(X)$,

$$\begin{aligned} \bigcup_i A_i \subseteq u &\iff \forall i, A_i \subseteq u \\ &\iff \forall i, \mathcal{N}(A_i) \subseteq u \\ &\iff \mathrm{colim}_i^{\mathrm{Pro}(\mathrm{Open}(X))} \mathcal{N}(A_i) \subseteq u. \end{aligned}$$

The conclusion follows from the Yoneda lemma. □

Lemma 1.1.51. *If X is a topological space and $A, B \subseteq X$, then $\mathcal{N}(A \cap B) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B)$.*

Proof. This follows from the facts that $\mathcal{N}(A \cap B) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(A \cap B) \subseteq \mathcal{N}(B)$. □

***Warning* 1.1.52.** It is not generally the case that $\mathcal{N}(A \cap B) = \mathcal{N}(A) \cap \mathcal{N}(B)$. For example, if $X = [0,1] \subset \mathbb{R}$, $A = \{0\}$, and $B = (0,1)$, then on one hand $\mathcal{N}(A \cap B) = \varnothing$, but on the other $\mathcal{N}(A) \cap \mathcal{N}(B) \neq \varnothing$ since every open neighborhood of 0 intersects $(0,1)$.

However, if $F : I \to \mathscr{P}(X)$ is a small cofiltered diagram, then one can at least say that, for all $u \in \mathrm{Open}(X)$, $\lim_{i \in I} \mathcal{N}(F(i)) \subseteq u$ if and only if there exists some $i \in I$ such that $F(i) \subseteq u$. ♢

Lemma 1.1.53. *If X is a topological space, then X is a normal space if and only if for all closed sets $A, B \subseteq X$ then $\mathcal{N}(A \cap B) = \mathcal{N}(A) \cap \mathcal{N}(B)$.*

Proof. First the converse. Suppose $A, B \subseteq X$ are disjoint closed sets. By hypothesis, $\mathcal{N}(A) \cap \mathcal{N}(B) = \varnothing$. Hence, there exist open neighborhoods $u \supseteq A$ and $v \supseteq B$ such that $u \cap v = \varnothing$, as required for a normal space.

Second, suppose X is normal. It suffices to show that $\mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(A \cap B)$. Suppose $u \supseteq A \cap B$ is an arbitrary open neighborhood. Then $A - u$ and $B - u$ are disjoint closed sets. Since X is normal, there exist disjoint open neighborhoods $v_A \supseteq A - u$ and $v_B \supseteq A - u$. Hence, $A \subseteq v_A \cup u$, $B \subseteq v_B \cup u$, and $(v_A \cup u) \cap (v_b \cup u) = u$, which means $\mathcal{N}(A) \cap \mathcal{N}(B) \subseteq u$, as required. □

1.1.3 Category of smooth manifold germs

We now apply the generalities from Sections 1.1.1 and 1.1.2 to smooth manifolds, and in Section 1.1.4 we will resume our discussion of jets. The category we construct here is a generalization of the category of smooth manifolds with halations from [SP09], and the goal is to be able to work with the many kinds of germs we come across. Our category should generalize to sites, and we leave it to future work to properly develop this within the context of topos theory.

The overall idea is to define functions on smooth manifold germs rather than define germs of functions on smooth manifolds, lifting the concept to the domain. In the process, we also generalize germs to mean arbitrary open filters on a manifold rather than the usual case of the neighborhood filter at a point.

Definition 1.1.54. A *smooth manifold germ* is a pair (M, U) where M is a smooth manifold (without boundary) and $U \in \mathrm{Pro}(\mathrm{Open}(M))$. ♢

Recall that for M and N smooth manifolds, $C^\infty(M, N)$ denotes the set of smooth maps $M \to N$. We can formally lift this to smooth manifold germs in the following way. We may regard C^∞ as being a functor $\mathrm{Open}(M)^{\mathrm{op}} \to \mathrm{Open}(N) \to \mathrm{Set}$, where, in particular, if we have open submanifolds $U \subseteq U' \subseteq M$ and $V \subseteq V' \subseteq N$, then we have a commutative diagram of functions induced by these inclusions:

$$\begin{array}{ccc} C^\infty(U', V) & \longrightarrow & C^\infty(U, V) \\ \downarrow & & \downarrow \\ C^\infty(U', V') & \longrightarrow & C^\infty(U, V') \end{array}$$

Then for smooth manifold germs (M, U) and (N, V) we define the set of *smooth maps of manifold germs* to be

$$C^\infty((M, U), (N, V)) = \lim_{v \supseteq V} \operatorname*{colim}_{u \supseteq U} C^\infty(u, v).$$

This is the essentially unique functor $\mathrm{Pro}(\mathrm{Open}(M))^{\mathrm{op}} \times \mathrm{Pro}(\mathrm{Open}(N)) \to \mathrm{Set}$, which we also denote by C^∞. Of course, if $U : I \to \mathrm{Open}(M)$ and $V : J \to \mathrm{Open}(N)$ are small cofiltered diagrams representing the smooth manifold germs, then we have the alternative representation

$$C^\infty((M, U), (N, V)) \approx \lim_{j \in J} \operatorname*{colim}_{i \in I} C^\infty(U(i), V(j)).$$

We now represent maps of smooth manifold germs in a more concrete way.

Lemma 1.1.55. *Suppose (M, U) and (N, V) are smooth manifold germs. Let X be the set of all pairs $(u, f \in C^\infty(u, N))$ with $u \supseteq U$ such that for all open $v \supseteq V$ we have $f^{-1}(v) \supseteq U$. Define the equivalence relation $\sim$ on X by $(u, f) \sim (u', f')$ if there exists some open $u'' \supseteq U$ contained in $u \cap u'$ such that $f|_{u''} = f'|_{u''}$. Then*

$$C^\infty((M, U), (N, V)) \approx X/\sim$$

where $f \in C^\infty((M, U), (N, V))$ is represented as an element of X by taking its projection to $\mathrm{colim}_{u \supseteq V} C^\infty(u, N)$ *and taking a representative, which is precisely an element of X.*

Proof. This is a matter of unfolding definitions. □

There is a composition law for smooth maps of manifold germs. Suppose (M, U), (M', U'), and (M'', U'') are smooth manifold germs. We define

$$C^\infty((M', U'), (M'', U'')) \times C^\infty((M, U), (M', U')) \to C^\infty((M, U), (M'', U''))$$

using the same principle as in Remark 1.1.28. We now give a description from the point of view of Lemma 1.1.55. Suppose $\varphi \in C^\infty((M, U), (M', U'))$ and $\psi \in C^\infty((M', U'), (M'', U''))$, and represent φ by $f \in C^\infty(u, M')$ with $u \supseteq U$ open and ψ by $g \in C^\infty(u', M'')$ with $u' \supseteq U'$ open. We have $U \subseteq f^{-1}(u')$, so $U \subseteq u \cap f^{-1}(u')$. The composition $\psi \circ \varphi$ is represented by $g \circ (f|_{u \cap f^{-1}(u')})$, which one can check has the property that preimages of open supersets of U'' are supersets of U.

Definition 1.1.56. The *category of smooth manifold germs* is the category whose objects are smooth manifold germs and whose morphisms are smooth maps of smooth manifold germs. ♢

Remark 1.1.57. If Man were finitely complete, another way we could have defined this category is by identifying the relevant full subcategory of $\mathrm{Pro}(\mathsf{Man})$. In light of Theorem 1.1.29, we still can by defining $\mathrm{Pro}(\mathsf{Man})$ as formal cofiltered limits, and this category still has all the same shapes of limits and colimits as one would expect. From this perspective we can construct limits and colimits in this $\mathrm{Pro}(\mathsf{Man})$ and then check whether they are smooth manifold germs. ■

Suppose $U \in \mathrm{Pro}(\mathrm{Open}(M))$ and $u \supseteq U$ is open. Considering u as an smooth manifold in its own right, we can define a pro-open set $U' \in \mathrm{Pro}(\mathrm{Open}(u))$ by the property that for all $v \in \mathrm{Open}(u)$, $U' \subseteq w$ if and only if $U \subseteq w$. Or, in other words, the inclusion $u \subseteq M$ induces a functor $\mathrm{Open}(u) \hookrightarrow \mathrm{Open}(M)$, and, by regarding U as an element of $\mathrm{Fun}^{\mathrm{lex}}(\mathrm{Open}(M), \mathrm{Set})$, then U' is from precomposing with this inclusion. We write $U|_u$ for this restriction, with the understanding that it is only valid when $U \subseteq u$.

An essential feature of smooth manifold germs is that, up to isomorphism, one may restrict the pro-open set to any open set containing it:

Lemma 1.1.58. *Suppose (M, U) is a smooth manifold germ and $u \supseteq U$ is open. Then $(M, U) \approx (u, U|_u)$.*

Proof. The map $u \hookrightarrow M$ induces a map $(u, U|_u) \to (M, U)$ of smooth manifold germs. It also induces a map $(M, U) \to (u, U|_u)$, which is the two-sided inverse of the above map. □

The main type of smooth manifold germs we consider are from the neighborhood pro-open sets of subsets of the manifolds.

Definition 1.1.59. For M and N smooth manifolds and $A \subseteq M$ and $B \subseteq N$ subsets, we define the notation $C^\infty_A(M, N)_B = C^\infty((M, \mathcal{N}(A)), (N, \mathcal{N}(B)))$. If we omit either A or B, we mean to take $A = M$ or $A = N$ respectively.

For $f \in C^\infty(M, N)$ such that for all open $V \supseteq B$ then $f^{-1}(V) \supseteq A$, we write $[f]_{A,B} \in C^\infty_A(M, N)_B$ for its *germ*. More generally, if $U \supseteq A$ is open and $f \in C^\infty(U, N)$ satisfies the same condition then we write $[f]_{A,B} \in C^\infty_A(M, N)_B$ as well for its germ. We write $[f]_A \in C^\infty_A(M, N)$ for the special case of $B = N$. ♢

Remark 1.1.60. The special case $C^\infty_p(M, N)$ for $p \in M$ is the usual set of smooth map germs at p. More generally, for $A \subseteq M$ the set $C^\infty_A(M, N)$ is known as the smooth map germs along A (see [Nes20, Chapter 13] for example). Specialized to this situation, the set is given by

$$C^\infty_A(M, N) = \operatorname*{colim}_{U \in A \downarrow \mathrm{Open}(M)} C^\infty(U, N)$$

where $A \downarrow \mathrm{Open}(M)$ is all the open sets that contain A, slightly abusing comma category notation since A is not necessarily an open set. ■

Lemma 1.1.61. *Suppose M and N are smooth manifolds and $A \subseteq M$ and $B \subseteq N$.*

1. *If B is closed, then for all $[f]_{A,B} \in C^\infty_A(M,N)_B$, $f(A) \subseteq B$.*
2. *For all $U \supseteq A$ open and $f \in C^\infty(M,N)$, if $f(A) \subseteq B$ then we have the corresponding germ $[f]_{A,B} \in C^\infty_A(M,N)_B$.*
3. *If B is closed, then*

$$C^\infty_A(M,N)_B \approx \operatorname*{colim}_{U \in A \downarrow \mathrm{Open}(M)} \{f \in C^\infty(U,N) \mid f(A) \subseteq B\}.$$

Proof. (1) Suppose $x \in \operatorname{dom} f$ is such that $f(x) \notin B$. Since N is a T_1 space, there exists an open set $V \subseteq N$ such that $B \subseteq V$ and $f(x) \notin V$. Hence, $x \notin f^{-1}(V)$, and since $A \subseteq f^{-1}(V)$ we have that $x \notin A$. Therefore for every $x \in A$, $f(x) \in B$.

(2) If $f(A) \subseteq B$ then for all open $V \supseteq B$ we have $f^{-1}(V) \supseteq A$, hence f satisfies the requirement to project to a germ.

(3) This follows from (1) and (2). □

Example 1.1.62. For $p \in M$ and $q \in N$, the set $C^\infty_p(M,N)_q$ is equivalently the set of germs of smooth maps of pointed spaces. ♢

Lemma 1.1.63. *Suppose M is a smooth manifold, $n \in \mathbb{N}$, and $A \subseteq M$ and $B \subseteq \mathbb{R}^n$ are closed subsets. Then the map*

$$\{f \in C^\infty(M,\mathbb{R}^n) \mid f(A) \subseteq B\} \to C^\infty_A(M,\mathbb{R}^n)_B$$

defined by $f \mapsto [f]_{A,B}$ is a surjection.

Proof. Let $[f]_{A,B} \in C^\infty_A(M,\mathbb{R}^n)$ be arbitrary, where $f \in C^\infty(U,\mathbb{R}^n)$ for some open $U \supseteq A$ and $f(A) \subseteq B$. We have that A and $M \setminus U$ are disjoint closed sets, so since M is normal there exist disjoint open sets $V, W \subseteq M$ such that $A \subseteq V$, that $M \setminus U \subseteq W$, and that $\overline{V}$ and $\overline{W}$ are disjoint. Using bump functions (see, for example [Lee13, 2.25]), there exists a $\varphi \in C^\infty(U,\mathbb{R})$ such that $\varphi(\overline{V}) = 1$ and $\varphi(\overline{W}) = 0$. Since $(\varphi \cdot f)|_V = f|_V$, we have that $[\varphi \cdot f]_{A,B} = [f]_{A,B}$, and since $(\varphi \cdot f)|_{W \cap U} = 0$, there exists a function $f' \in C^\infty(M,N)$ such that $f'|_U = \varphi \cdot f$ and $f'|_W = 0$. Because $[f']_{A,B} = [\varphi \cdot f]_{A,B}$, we conclude that $[f']_{A,B} = [f]_{A,B}$. □

Corollary 1.1.64. *Let M, n, A, and B are as in Lemma 1.1.63. Then*

$$C^\infty_A(M,\mathbb{R}^n)_B \approx \{f \in C^\infty(M,\mathbb{R}^n) \mid f(A) \subseteq B\}/\sim$$

where $\sim$ is the equivalence relation defined by having $f \sim g$ if there exists some open neighborhood U of A such that $f|_U = g|_U$.

1.1.4 Jets and the local ring of germs of functions

In this section, we review the local ring of germs of smooth functions on a manifold and use it to provide a standard algebraic interpretation of fibers of the jet bundle, and we give cases where the ring of germs is isomorphic to a localization of the ring of smooth functions. More generally, we review germs of smooth functions along subsets of a manifold, which may be less familiar (see [Nes20, Chapter 13] for example). We do this from the perspective of pro-objects, using what we have set up in Section 1.1.3.

Let M be a smooth manifold. The set $C^\infty(M) := C^\infty(M,\mathbb{R})$ has a ring structure since $\mathbb{R}$ is a smooth ring. For $A \subseteq M$, then $C^\infty_A(M)$, the *ring of germs of smooth functions along* A. Recall that

$$C^\infty_A(M) := \operatorname*{colim}_{U \in A \downarrow \mathrm{Open}(M)} C^\infty(U),$$

where the colimit ranges over open neighborhoods of A, abusing comma category notation.

For $p \in M$, the special case $C^\infty_p(M)$ is the *ring of germs of smooth functions at* p (or the stalk of C^∞ at p). When it is not ambiguous, we may write C^∞_p for $C^\infty_p(M)$.

Remark 1.1.65. Thinking of C^∞ as a functor $\mathrm{Open}(M)^{\mathrm{op}} \to \mathbb{R}\text{-Alg}$, then since $\mathbb{R}$-Alg is cocomplete there is an induced functor $C^\infty : \mathrm{Pro}(\mathrm{Open}(M))^{\mathrm{op}} \to \mathbb{R}\text{-Alg}$. From this point of view, the formula for germs along a subset $A \subseteq M$ is

$$C^\infty_A(M) = C^\infty(\mathcal{N}(A)) = C^\infty(\lim^{\mathrm{Pro}(\mathrm{Open}(M))}_{U \in A\downarrow \mathrm{Open}(M)} U) = \mathrm{colim}^{\mathbb{R}\text{-Alg}}_{U \in A \downarrow \mathrm{Open}(M)} C^\infty(U),$$

using the fact that cofiltered limits are carried to filtered colimits. Hence, we're able to work with smooth functions defined "on" A by having them come with a particular infinitesimal extension to a neighborhood of A. ■

Lemma 1.1.66. *Let M be a smooth manifold and $A \subseteq M$ a closed set. The canonical projection $C^\infty(M) \to C^\infty_A(M)$ is surjective. In particular, $C^\infty_A(M)$ is the quotient of $C^\infty(M)$ by the equivalence relation where f and g in $C^\infty(M)$ are equivalent if there exists an open set $U \subseteq M$ such that $f|_U = g|_U$.*

Proof. This is Lemma 1.1.63 and Corollary 1.1.64 specialized to $B = N = \mathbb{R}$. □

Remark 1.1.67. This lemma is not generally true when A is not closed. For example, if $U = \mathbb{R} \setminus \{0\}$ then $C^\infty_U(\mathbb{R}) = C^\infty(U)$ has functions that are not restrictions of functions from $C^\infty(\mathbb{R})$ such as $x \mapsto x^{-1}$. ■

It is worth discussing the relationship between rings of germs and localizations of the ring of smooth functions. For $A \subseteq M$, let

$$S_A = \{f \in C^\infty(M) \mid \text{for all } x \in A,\ f(x) \neq 0\},$$

which is a multiplicative set. Recall that $S_A^{-1}C^\infty(M)$ denotes the localization of $C^\infty(M)$ by S_A, and it is the universal ring such that whenever $h : C^\infty(M) \to R$ is a ring homomorphism with the property that that $h(s)$ is a unit for all $s \in S_A$, then h factors through $S_A^{-1}C^\infty(M)$.

Concretely, elements of $S_A^{-1}C^\infty(M)$ may be written as formal fractions f/s with $f \in C^\infty(M)$ and $s \in S_A$, with an equivalence relation such that $f/s = f'/s'$ if and only if there exists some $t \in S_A$ such that $t(fs' - f's) = 0$.

Lemma 1.1.68. *Let M be a smooth manifold and $A \subseteq M$ a closed set. Then there is an isomorphism*

$$h : S_A^{-1}C^\infty(M) \to C_A^\infty(M)$$

defined by $h(f/s) = [f/s]_A$, regarding f/s as a function defined on the open set $s^{-1}(\mathbb{R} \setminus \{0\})$.

Proof. Note that for all $s \in S_A$ that s is nonzero on the open set $s^{-1}(\mathbb{R} \setminus \{0\})$, hence s is a unit in $C_A^\infty(M)$. Therefore, the canonical map factors as

$$C^\infty(M) \to S_A^{-1}C^\infty(M) \xrightarrow{h} C_A^\infty(M),$$

where h is defined as in the lemma statement. By Lemma 1.1.66, the canonical map $C^\infty(M) \to C_A^\infty(M)$ is a surjection, and thus so is h. It suffices to show that h is an injection. Let $f/s \in S_A^{-1}C^\infty(M)$ be such that $h(f/s) = 0$. Then there is some open neighborhood U of A such that $U \subseteq s^{-1}(\mathbb{R} \setminus \{0\})$ and $f/s|_U = 0$. We can multiply this through by s to get that $f|_U = 0$. Let $\varphi \in C^\infty(M)$ be such that $\varphi(A) = 1$ and the support of φ is contained within U (see [Lee13, 2.25] for instance). Then $\varphi \in S_A$ and $\varphi(1f - s0) = 0$, and so $f/s = 0$ in $S^{-1}C^\infty(M)$. Since f/s was arbitrary, h is injective. □

For $p \in M$, the ring $C_p^\infty(M)$ is a local ring whose maximal ideal $\mathfrak{m}_p(M)$ (also written $\mathfrak{m}_p$) consists of those germs that vanish at p. Note that $\mathfrak{m}_p(M) = C_p^\infty(M)_0$, regarding $C_p^\infty(M)_0$ as a subset of $C_p^\infty(M)$. The preimage of $\mathfrak{m}_p(M)$ in $C^\infty(M)$, which we also denote by $\mathfrak{m}_p(M)$ when it does not cause confusion, is also a maximal ideal and consists of all smooth functions that vanish at p.

For $f : M \to \mathbb{R}^n$ a smooth function, whether there exists an open neighborhood $U \subseteq M$ of p on which f is invertible is a property of the germ $[f]_p \in C_p^\infty(M, \mathbb{R}^n)$ due to the inverse function theorem, since the germ determines $df_p : T_pM \to T_{f(p)}\mathbb{R}^n$. When there exists such an open neighborhood, then $f|_U$ is a chart centered at p, and with $\pi_1, \ldots, \pi_n : \mathbb{R}^n \to \mathbb{R}$ being the standard projections, the germs $[\pi_1 \circ f]_p, \ldots, [\pi_n \circ f]_p$ define a system of *local coordinate germs*. Germs $x_1, \ldots, x_n \in C_p^\infty(M)$ define a system of local coordinate germs if and only if the germ $x_1 \times \cdots \times x_n \in C_p^\infty(M, \mathbb{R}^n)$ is nonsingular at p.

The following lemma and corollary are surprisingly useful. We give a generalization of the usual Hadamard lemma for families of functions.

Lemma 1.1.69 (Parameterized Hadamard lemma)**.** *Suppose M is a smooth manifold of dimension m, Q is a smooth manifold, and $p \in M$. Let $x_1, \ldots, x_m \in \mathfrak{m}_p(M)$ be a system of local coordinate germs.*

Suppose $[f]_{Q\times p} \in \mathfrak{m}_{Q\times p}(Q \times M)$. Then there are germs $f_1, \ldots, f_m \in C_{Q\times p}^\infty(Q \times M)$ such that

$$f = f_1x_1 + \cdots + f_mx_m$$

with the property that $s \mapsto f_i(s,p)$ is constant for all i.

Furthermore, if there is a $k \in \mathbb{N}$ such that $[f]_{Q\times p} \in \mathfrak{m}_{Q\times p}(Q\times M)^k$, then there are germs $f_I \in C^\infty_{Q\times p}(Q\times M)$ for $I \in \mathbb{N}^m$ ranging over multiindices with $|I| = k$ such that

$$f = \sum_{|I|=k} f_I x_I$$

with the property that $s \mapsto f_I(s,p)$ is constant for all I.

Proof. Define the smooth map $h : Q \times \mathbb{R}^m \times [0,1] \to \mathbb{R}$ by $h(s,x,t) = f(s,tx)$. Then since $f(s,0) = 0$ for all $s \in Q$,

$$\begin{aligned} f(s,x) = h(s,x,1) - h(s,x,0) = \int_0^1 \frac{\partial h}{\partial t}(s,x,t)\,dt &= \int_0^1 \sum_{i=1}^m x_i \frac{\partial f}{\partial x_i}(s,tx)\,dt \\ &= \sum_{i=1}^m \left(\int_0^1 \frac{\partial f}{\partial x_i}(s,tx)\,dt \right) x_i. \end{aligned}$$

Hence, let $f_i(s,x) = \int_0^1 \frac{\partial f}{\partial x_i}(s,tx)\,dt$ for each i, which are smooth germs in s and x. Note that $\frac{\partial f_i}{\partial s}(s,0) = \int_0^1 \frac{\partial^2 f}{\partial x_i \partial s}(s,0)\,dt = 0$ since $\frac{\partial f}{\partial s}(s,0) = 0$, hence $s \mapsto f_i(s,0)$ is constant.

For the case of $[f]_{Q\times p} \in \mathfrak{m}_{Q\times p}(Q\times M)^k$ we may iteratively apply the above k times. □

Corollary 1.1.70 (Hadamard lemma)**.** *Let M be a smooth manifold of dimension m, $p \in M$, and $f \in \mathfrak{m}_p$. Let $x_1, \ldots, x_m \in \mathfrak{m}_p$ be a system of local coordinate germs at p. Then there are germs $f_1, \ldots, f_m \in C^\infty_p$ such that*

$$f = f_1 x_1 + \cdots + f_m x_m.$$

Furthermore, if $f \in \mathfrak{m}_p^k$ for some $k \in \mathbb{N}$, then there are germs $f_I \in C^\infty_p$ for $I \in \mathbb{N}^m$ ranging over multiindices with $|I| = k$ such that

$$f = \sum_{|I|=k} f_I x_I.$$

We now resume our discussion of jets, giving some well-known algebraic characterizations. The function $j^k : C^\infty(M,N) \to J^k(M,N)$ depends only on the germ of a map, so it induces a well-defined surjective function $j^k_p : C^\infty_p(M,N) \to J^k_p(M,N)$ for each $p \in M$. Similarly, j^k_p gives functions $C^\infty_p(M,N)_q \to J^k_p(M,N)_q$ for each $p,q \in M$.

Lemma 1.1.71. *Let M be a smooth manifold and $p \in M$. Give $J^k_p(M,\mathbb{R})$ the ring structure induced by the ring structure of $\mathbb{R}$. The kernel of $C^\infty_p \to J^k_p(M,\mathbb{R})$ is $\mathfrak{m}_p^{k+1}$, hence*

$$C^\infty_p / \mathfrak{m}_p^{k+1} \cong J^k_p(M,\mathbb{R}).$$

Proof. This is an invariant way of truncating a Taylor series to kth order, as can be checked in coordinates. □

Given a smooth map $f : M \to N$ with $q = f(p)$, there is an induced map $f^* : C^\infty_q \to C^\infty_p$ defined by $[g]_q \mapsto [g \circ f]_p$. Since $f^*\mathfrak{m}_q \subseteq \mathfrak{m}_p$, we additionally have $f^*\mathfrak{m}_q^{k+1} \subseteq \mathfrak{m}_p^{k+1}$, and hence there is an induced map $f^* : C^\infty_q/\mathfrak{m}_q^{k+1} \to C^\infty_p/\mathfrak{m}_p^{k+1}$ of rings.

Lemma 1.1.72. *Let M and N be smooth manifolds with $p \in M$ and $q \in N$. Every element $j^k f_p \in J^k_p(M,N)_q$ induces a ring homomorphism $(j^k f_p)^* : J^k_q(N,\mathbb{R}) \to J^k_p(M,\mathbb{R})$, and every ring homomorphism $J^k_q(N,\mathbb{R}) \to J^k_p(M,\mathbb{R})$ is uniquely induced in this way. In particular, the rule $j^k f_p \mapsto f^*$ defines a bijection*

$$J^k_p(M,N)_q \to \mathrm{Hom}_{\mathbf{Ring}}(C^\infty_q/\mathfrak{m}_q^{k+1}, C^\infty_p/\mathfrak{m}_p^{k+1}).$$

1.1.5 The Malgrange Preparation Theorem

For the purpose of analyzing singularities, the Malgrange Preparation Theorem is a useful tool since it can be used to show that, for suitably large k, singularity types satisfying a stability condition are determined by their k-jets — in particular, they have a polynomial form up, modulo diffeomorphisms of the domain and codomain. Even if it were not helpful for obtaining exact descriptions of singularities, having a finite-dimensional space helps limit the search for singularity types. We reproduce the statements of theorems here, which are drawn from [GG73]. We also prove parameterized versions of the theorem that will be used in Section 1.4.

For the following, we let M and N be smooth manifolds, let $p \in M$, let $f \in C^\infty_p(M,N)$, and let $q = f(p)$. Recall that a $C^\infty_p(M)$-module A is also a $C^\infty_q(N)$-module by the action $\varphi a = f^*(\varphi)a$, where $\varphi \in C^\infty_q(N)$ and $a \in A$.

Theorem 1.1.73 (Malgrange Preparation Theorem [GG73, IV.3.6] and [Mat68])**.** *Let $S \subseteq M$ be finite and let A be a finitely generated $C^\infty_S(M)$-module. Then A is a finitely generated $C^\infty_q(N)$-module if and only if $A/\mathfrak{m}_q(N)A$ is a finite-dimensional vector space over $\mathbb{R}$.*

Theorem 1.1.74 ([GG73, IV.3.10])**.** *Let A be a finitely generated $C^\infty_p(M)$-module, and let $e_1, \dots, e_k \in A$. The elements $e_1, \dots, e_k$ generate A as a $C^\infty_q(N)$-module if and only if the images of $e_1, \dots, e_k$ generate $A/\mathfrak{m}_p^{k+1}(M)A$ as a $C^\infty_q(N)$-module.*

Corollary 1.1.75 ([GG73, IV.3.11])**.** *If the images of $e_1, \dots, e_k$ generate the vector space $A/(\mathfrak{m}_p^{k+1}(M)A + \mathfrak{m}_q(N)A)$, then $e_1, \dots, e_k$ generate A as a $C^\infty_q(N)$-module.*

Proof. This follows from the preceding and Nakayama's Lemma via Corollary A.1.2. □

The following parameterized and global versions of the preparation theorem are described in [AGZV12, I.4.2] in the case of A a ring of smooth functions, which we generalize here.

Corollary 1.1.76. *Let S be a smooth manifold, $t_0 \in S$, and $f \in C^\infty_{(p,t_0)}(M \times S, N)$. Let A be a finitely generated $C^\infty_{(p,t_0)}(M \times S)$-module, which we give the structure of a $C^\infty_{(q,t_0)}(N \times S)$-module via the map $(x,t) \mapsto (f(x,t),t)$. Then A is a finitely generated $C^\infty_{(q,t_0)}(N \times S)$-module if and only if $A/(\mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A)$ is a finite-dimensional vector space over $\mathbb{R}$.*

Furthermore, $e_1, \ldots, e_k$ generate A as a $C^\infty_{(q,t_0)}(N \times S)$-module if and only if their images generate the vector space $A/(\mathfrak{m}_p^{k+1}(M)A + \mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A)$.

Proof. This is essentially Theorem 1.1.73, where $\mathfrak{m}_{(q,t_0)}(N \times S)A = \mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A$ by Lemma 1.1.69. For the last part, we combine Theorem 1.1.74 and Corollary 1.1.75 and use the fact that if $\varphi a \in \mathfrak{m}^{k+1}_{(p,t_0)}(M \times S)A$, then using Lemma 1.1.69 to write $\varphi(x,t) = \sum_{|I|+|J|=k+1} \varphi_{I,J}(x,t)x^I t^J$, we can see that $\varphi_{I,J}(x,t)x^I t^J a \in \mathfrak{m}_{t_0}(S)A$ when $|J| > 0$, hence $\mathfrak{m}^{k+1}_{(p,t_0)}(M \times S)A + \mathfrak{m}_{t_0}(S)A = \mathfrak{m}_p^{k+1}(M)A + \mathfrak{m}_{t_0}(S)A$. □

For M, N, and S smooth manifolds and $p \in M$, we think of $C^\infty_{p\times S}(M \times S, N)$ as being smooth map germs parameterized by S. For each $t_0 \in S$, there is a natural map $C^\infty_{p\times S}(M \times S, N) \to C^\infty_{(p,t_0)}(M \times S, N)$ defined by $[f]_{p\times S} \mapsto [f]_{(p,t_0)}$.

Lemma 1.1.77. *Let S be a compact smooth manifold and $f \in C^\infty_{p\times S}(M \times S, N)$. Let A be a finitely generated $C^\infty_{p\times S}(M \times S)$-module, which we give the structure of a $C^\infty_{q\times S}(N \times S)$-module via the map $(x,t) \mapsto (f(x,t),t)$. Then A is a finitely generated $C^\infty_{q\times S}(N \times S)$-module if and only if for all $t_0 \in S$, $A/(\mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A)$ is a finite-dimensional vector space over $\mathbb{R}$.*

Furthermore, $e_1, \ldots, e_k$ generate A as a $C^\infty_{q\times S}(N \times S)$-module if and only if for all $t_0 \in S$ their images generate the vector space $A/(\mathfrak{m}_p^{k+1}(M)A + \mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A)$.

Proof. The forward directions of each part are clear. We first show the converse of the last part. Suppose $e_1, \ldots, e_k \in A$ are such that for all $t_0 \in S$ their images generate $A/(\mathfrak{m}_p^{k+1}(M)A + \mathfrak{m}_{t_0}(S)A + \mathfrak{m}_q(N)A)$. Letting $a \in A$, we will show that $a = \sum_{i=1}^k \varphi_i e_i$ for some germs $\varphi_i \in C^\infty_{q\times S}(N \times S)$.

For $t_0 \in S$, by Lemma 1.1.68 the ring $C^\infty_{(p,t_0)}(M \times S)$ is the localization of $C^\infty_{p\times S}(M \times S)$ at the complement of $\mathfrak{m}_{(p,t_0)}(M \times S)$. Let A_{t_0} denote the localization of A at this set, making it into a $C^\infty_{(p,t_0)}(M \times S)$-module. Since localization commutes with quotients, we have that $A_{t_0}/(\mathfrak{m}_p^{k+1}(M)A_{t_0} + \mathfrak{m}_{t_0}(S)A_{t_0} + \mathfrak{m}_q(N)A_{t_0})$ is a vector space spanned by the images of $e_1, \ldots, e_k$. Hence, by Corollary 1.1.76, A_{t_0} is generated by the images of $e_1, \ldots, e_k$ as a $C^\infty_{(q,t_0)}(N \times S)$-module. There is a closed set $C_{t_0} \subseteq S$ containing an open neighborhood of t_0 and functions $\varphi_{t_0,i} \in C^\infty_{q\times C_{t_0}}(N \times S)$ such that the element $\sum_{i=1}^k \varphi_{t_0,i}[e_i]_{C_{t_0}}$ of $A_{C_{t_0}}$ is equal to the image $[a]_{C_{t_0}}$.

Since S is compact, there is a finite subcollection $C_{t_1}, \ldots, C_{t_r}$ for $t_1, \ldots, t_r \in S$ whose interiors cover S. Let $\{\chi_{t_i}\}_{1\le i\le r}$ be a partition of unity subordinate to the subcover. Then,

$$\sum_{i=1}^k \left(\sum_{j=1}^r \chi_{t_j} \varphi_{t_j,i} \right) e_i = a,$$

where we have $\chi_{t_j}\varphi_{t_j,i} \in C^\infty_{q\times S}(N \times S)$ by asserting it vanishes outside of $N \times C_{t_0}$. Note that the sum $\sum_{j=1}^r \chi_{t_j}\varphi_{t_j,i}$ makes sense because we can restrict the N components of the domains of each $\varphi_{t_j,i}$ to be the same since there are finitely many of them. Thus A is generated by $e_1, \ldots, e_k$.

For the converse of the first part, use Corollary 1.1.76 to get finite sets of local generators of A at each $t_0 \in S$, use compactness in a similar way to get generators associated to a finite collection of closed sets whose interiors cover S, then use a similar bump function argument to yield elements of A whose localizations are the elements. The collection of these is a finite generating set for A. □

1.2 The Thom Transversality Theorem

Recall that a smooth map $f : M \to N$ is *transverse to a submanifold* $W \subseteq N$ *on* $S \subseteq M$, denoted $f \pitchfork W$ on S, if for every $p \in S \cap f^{-1}(W)$, then $T_{f(p)}N = T_{f(p)}W + df_p(T_pM)$. Or, equivalently, if for such p the composition

$$q_{f(p)} \circ df_p : T_pM \to T_{f(p)}N/T_{f(p)}W$$

is surjective, where $q : TN \to TN/TW$ is the quotient of vector bundles over W. When S is omitted, we assume $S = M$. A standard fact about transversality is that if $f \pitchfork W$ on M, then $f^{-1}(W)$ is an embedded submanifold of M whose codimension in M is the codimension of W in N. Given the Whitney C^∞ topology on smooth maps $M \to N$ (Definition 1.2.1), the set of maps transverse to W is dense (and in fact residual). A powerful generalization of this transversality theorem is the Thom Transversality Theorem (Theorem 1.2.9), which additionally lets us control the jets of a map. This is essential in our classification of singularities — the only singularity types we need to consider are those whose jet extensions are transverse to every orbit in the jet manifolds.

The treatment here closely follows [GG73]. We first need to recall two topologies on $C^\infty(M, N)$.

Definition 1.2.1. Let M and N be smooth manifolds and $C^\infty(M, N)$ the set of smooth mappings $M \to N$. The *Whitney* C^∞ *topology* on $C^\infty(M, N)$ is the topology whose basis is given by the collection of sets

$$B(U) = \{f \in C^\infty(M, N) \mid j^k f(M) \subseteq U\}$$

for each $k \in \mathbb{N}$ and open $U \subseteq J^k(M, N)$.

The *weak topology* on $C^\infty(M, N)$ is the topology whose basis is given by the collection of sets

$$B'(K, U) = \{f \in C^\infty(M, N) \mid j^k f(K) \subseteq U\}$$

for each $k \in \mathbb{N}$, compact $K \subseteq M$, and open $U \subseteq J^k(M, N)$.

When no topology is specified for $C^\infty(M, N)$, we assume the Whitney C^∞ topology. ♢

Remark 1.2.2. Suppose $J^k(M,N)$ is given a metric compatible with M and N. A sequence of functions $f_1, f_2, \ldots \in C^\infty(M,N)$ converges to $f \in C^\infty(M,N)$ in the Whitney C^∞ topology if for each k there is a compact subset $K_k \subseteq M$ such that (1) the sequence stabilizes outside K_k and (2) $j^k f_i$ converges uniformly to $j^k f$ on K_k. ■

Lemma 1.2.3. *Let L, M, and N be smooth manifolds.*

- $j^k : C^\infty(M,N) \to C^\infty(M, J^k(M,N))$ *defined by* $f \mapsto j^k f$ *is continuous.*
- $C^\infty(L,M) \times C^\infty(L,N) \to C^\infty(L, M \times N)$ *defined by* $(f,g) \mapsto (x \mapsto (f(x), g(x)))$ *is a homeomorphism.*
- *The composition map* $C^\infty(M,N) \times C^\infty(L,M) \to C^\infty(L,N)$ *is continuous if L is compact or if* $C^\infty(L,M)$ *is replaced by the open subset of proper mappings.*

Definition 1.2.4. Let X be a topological space. A subset $S \subseteq X$ is *residual* if it is the countable intersection of dense open subsets of X. X is a *Baire space* if every residual set is dense. ♢

The Whitney C^∞ topology lacks a countable base at every point and is not metrizable. It is, however, at least a Baire space, which is sufficient for our purposes.

Lemma 1.2.5 ([Hir76, 2.4.4])**.** *Let M and N be smooth manifolds and $S \subseteq C^\infty(M,N)$. If S is closed with respect to the weak topology, then S as a subspace of $C^\infty(M,N)$ with the Whitney C^∞ topology is a Baire space.*

Corollary 1.2.6. *Let M and N be smooth manifolds. Then $C^\infty(M,N)$ with the Whitney C^∞ topology is a Baire space.*

We will also need a way to control the boundary conditions for the Thom transversality theorem, which the following definition and lemma provides.

Definition 1.2.7. Let M and N be smooth manifolds, $C \subseteq M$ a closed subspace, and $f : M \to N$. Define $C^\infty(M,N; f|C)$ to be the set of $g \in C^\infty(M,N)$ such that $g|_C = f|_C$. ♢

Lemma 1.2.8. *Given the hypotheses of Definition 1.2.7, $C^\infty(M,N; f|C)$ as a subspace of $C^\infty(M,N)$ is a Baire space.*

Proof. By Lemma 1.2.5, we only need to show that $C^\infty(M,N; f|C)$ is closed with respect to the weak topology. Let $g \in C^\infty(M,N)$ be in the complement of $C^\infty(M,N; f|C)$. With an $x \in C$ such that $g(x) \neq f(x)$, let $V \subset N$ be a neighborhood of $g(x)$ disjoint from $f(x)$. The set $M \times V$ is open in $J^0(M,N) = M \times N$, and $B'(\{x\}, M \times V)$ is a basis neighborhood of g disjoint from $C^\infty(M,N; f|C)$. □

We are now ready to state the transversality theorem, which includes a generalization by Schommer-Pries to incorporate boundary conditions.

Theorem 1.2.9 (Thom Transversality Theorem)**.** *Let M and N be smooth manifolds and W a submanifold of $J^k(M,N)$. Then*

$$T_W = \{f \in C^\infty(M,N) \mid j^k f \pitchfork W\}$$

is a residual subset of $C^\infty(M,N)$. If W is closed, then T_W is open.

Furthermore, if $C \subseteq M$ is a closed subset and $g : M \to N$ a smooth map such that $j^k g \pitchfork W$ on C, then $T_W \cap C^\infty(M,N;g|C)$ is a residual subset of $C^\infty(M,N;g|C)$.

Proof. See [GG73, Theorem II.4.9] or [Hir76, 3.2.8] for the first part. The second part is an adaptation of [SP09, Theorem 1.7] and [GG73, Theorem II.4.11], and we sketch the details here. The beginning of the first part of the theorem is to choose a countable covering of W by subsets $W_1, W_2, \dots$ open in W such that each W_i satisfies (1) $\overline{W_i}$ is a subset of W, (2) $\overline{W_i}$ is compact, (3) there exist coordinate neighborhoods U_i of M and V_i of N such that $(\alpha \times \beta)(\overline{W_i}) \subset U_i \times V_i$, and (4) $\overline{U_i}$ is compact. Each W_i is used to define an open subset $\{f \in C^\infty(M,N) \mid j^k f \pitchfork W \text{ on } (j^k f)^{-1}(\overline{W_i})\}$, and the intersection of all of these is T_W, so the argument reduces to showing these are each dense. The argument proceeds by using a jet bundle version of the parametric transversality theorem [GG73, II.4.7] to show that for every $f \in C^\infty(M,N)$ the set of perturbations to f that make $j^k f \pitchfork W$ on $(j^k f)^{-1}(\overline{W_i})$ is dense, and notably this perturbation is supported on U_i.

Hence, given $f \in C^\infty(M,N;g|C)$, since $j^k g$ is already transverse to W on C, we only need $W_1, W_2, \dots$ to cover $W \cap \alpha^{-1}(M \setminus C)$. We can arrange for each U_i to be disjoint from C, and with such a choice it follows that the perturbations considered in the proof stay within $C^\infty(M,N;g|C)$. □

Remark 1.2.10. That the sets produced by the Thom Transversality Theorem are residual means that, for any countable collection of jet transversality constraints, the set of smooth functions satisfying them is dense. More precisely, given natural numbers k_i and submanifolds $W_i \subseteq J^{k_i}(M,N)$ for all $i \in \mathbb{N}$, then $\bigcap_i T_{W_i} \subseteq C^\infty(M,N)$ is dense since it is a residual subset of a Baire space. ■

We will also need John Mather's multijet transversality theorem to put singularities with intersecting images into general position.

Definition 1.2.11. Let M and N be smooth manifolds, and let M^r denote the r-fold product $M \times M \times \cdots \times M$. The *fat diagonal* $\mathbf{\Delta}^r_M \subset M^r$ is the set $\{p \in M^r \mid p_i = p_j \text{ for some } i \neq j\}$, hence the submanifold $M^r \setminus \mathbf{\Delta}^r_M$ is the space of r-tuples of distinct points of M.[3]

Given a multiindex $\mathbf{k} \in \mathbb{N}^r$, the *k-multijet bundle* $J^{\mathbf{k}}(M,N)$ is the pullback of the product bundle $\prod_{i=1}^r J^{k_i}(M,N) \to M^r$ to $M^r \setminus \mathbf{\Delta}^r_M$. Elements of $J^{\mathbf{k}}(M,N)$ are called $\mathbf{k}$-*multijets*. There is a *source map* $\alpha : J^{\mathbf{k}}(M,N) \to M^r \setminus \mathbf{\Delta}^r_M$ and *target map* $\beta : J^{\mathbf{k}}(M,N) \to N^r$ defined componentwise.

For a smooth map $f : M \to N$, the $\mathbf{k}$-*multijet extension* $j^{\mathbf{k}} f : M^r \setminus \mathbf{\Delta}^r_M \to J^{\mathbf{k}}(M,N)$ is given by $j^{\mathbf{k}} f_p = (j^{k_1} f_{p_1}, \dots, j^{k_r} f_{p_r})$. ◇

[3]The fat diagonal is not to be confused with the *diagonal* $\Delta^r_M := \{(p,\dots,p) \in M^r \mid p \in M\} \subseteq M^r$.

Theorem 1.2.12 (Multijet Transversality Theorem)**.** *Let M and N be smooth manifolds, $\mathbf{k} \in \mathbb{N}^r$ a multiindex, and W a submanifold of $J^{\mathbf{k}}(M,N)$. Then*

$$T_W = \{f \in C^\infty(M,N) \mid j^{\mathbf{k}} f \pitchfork W\}$$

is a residual subset of $C^\infty(M,N)$.

Furthermore, if $C \subseteq M$ is a closed subset and $g : M \to N$ a smooth map such that $j^{\mathbf{k}} g \pitchfork W$ on C, then $T_W \cap C^\infty(M,N; g|C)$ is a residual subset of $C^\infty(M,N; g|C)$.

Proof. The proof of the first part when $k_1 = \cdots = k_r$ is given in [GG73, II.4.13], and the following is a description for arbitrary multiindices. Let $\mathbf{k}' \in \mathbb{N}^r$ be a multiindex where $k_1' = \cdots = k_r'$ and $k_i \leq k_i'$ for all i. Consider the map $J^{\mathbf{k}'}(M,N) \to J^{\mathbf{k}}(M,N)$ defined componentwise by the projections $J^{k_i'}(M,N) \to J^{k_i}(M,N)$. Since each factor is a submersion, the product is a submersion, and we can take the preimage of W to get a submanifold W' of $J^{\mathbf{k}'}(M,N)$. For $f : M \to N$, $j^{\mathbf{k}} f \pitchfork W$ if and only if $j^{\mathbf{k}'} f \pitchfork W'$, so the first part of the theorem follows.

For the second part, a similar modification to the one in Theorem 1.2.9 can be made to ensure perturbations have support disjoint from C. □

1.3 Foliated manifolds and distributions

Recall that a *distribution* is a subbundle of a tangent bundle, and that an *integrable* distribution is one that can be locally given as the tangent bundle of a submanifold.

In our version of the Thom–Boardman singularities, we begin with families of integrable distributions on the domain and codomain manifolds. The primary source of these distributions is from the standard foliation on a product of manifolds, where if $M = M_1 \times M_2 \times \dots M_m$, then at each point $p \in M$ we get a filtration of distributions

$$0 \subseteq T_p M_1 \subseteq T_p(M_1 \times M_2) \subseteq \cdots \subseteq T_p(M_1 \times M_2 \times \cdots \times M_m) = T_p M,$$

where the imprecise notation $T_p(M_1 \times \cdots \times M_k)$ means the tangent space of the level set containing p of the projection $M_1 \times \cdots \times M_n \twoheadrightarrow M_{k+1} \times \cdots \times M_m$. To classify singularities means to enumerate the orbits of $C_p^\infty(M,N)$ under the actions of diffeomorphisms of M and N that preserve relevant structure — for example, diffeomorphisms that act invariantly on a collection of distributions.

In this section, the aim is to precisely describe the diffeomorphisms for the foliation described above. This is admittedly done in strictly more generality than necessary. We review the definition of smooth manifolds via pseudogroups (see [GG73] for reference), we define foliated manifolds in these terms, and we show that foliated manifolds according to this definition do give the usual notion of a foliated manifold. For a foliated manifold whose leaves form a flag at each point, the transition maps have differentials taking values in a fixed parabolic subgroup of the general linear group. We also give a description of the tangent space of the diffeomorphism group as a restricted kind of vector field, which is used in the definition of infinitesimal stability of germs (Section 1.4).

Definition 1.3.1. For spaces X and Y, a *local homeomorphism from X to Y* is a homeomorphism $\varphi : U \to V$ where $U \subseteq X$ and $V \subseteq Y$ are open subsets, which we may write as $\varphi : X \circ\!\!\to Y$ with $\operatorname{dom}\varphi = U$ and $\operatorname{cod}\varphi = V$. A *local homeomorphism of X* is a local homeomorphism from X to X. $\Diamond$

Definition 1.3.2. A *pseudogroup* on $\mathbb{R}^n$ is a collection Γ of local homeomorphisms of $\mathbb{R}^n$ with the following properties:

1. $\mathrm{id}_{\mathbb{R}^n} \in \Gamma$,
2. if $\varphi, \psi \in \Gamma$ with $\operatorname{dom}\varphi = \operatorname{im}\psi$ then $\varphi \circ \psi \in \Gamma$,
3. if $\varphi \in \Gamma$, then $\varphi^{-1} \in \Gamma$,
4. if $\varphi \in \Gamma$ and $U \subseteq \operatorname{dom}\varphi$ is open, then $\varphi|_U \in \Gamma$, and
5. if φ is a local homeomorphism of $\mathbb{R}^n$ along with an open cover $\{U_\alpha\}_{\alpha\in I}$ of $\operatorname{dom}\varphi$ such that $\varphi|_{U_\alpha} \in \Gamma$ for every $\alpha \in I$, then $\varphi \in \Gamma$. $\Diamond$

Remark 1.3.3. More abstractly, we can understand pseudogroups in the following way. Consider the sheaf F on $\mathbb{R}^n$ where for $U \subseteq \mathbb{R}^n$ an open set, $F(U)$ consists of continuous injective maps $U \to \mathbb{R}^n$, which by invariance of domain are homeomorphisms onto their open images. Each $F(U)$ is a disjoint union of the sets $F(U, V)$ of those mappings $\varphi : U \to \mathbb{R}^n$ with $\operatorname{im}\varphi = V$ for $V \subseteq \mathbb{R}^n$ open. There is a map $(-)^{-1} : F(U, V) \to F(V, U)$ from taking inverse functions, and hence F defines a groupoid on the open subsets of $\mathbb{R}^n$. Thus a pseudogroup on $\mathbb{R}^n$ is, equivalently, a subsheaf Γ of F such that Γ defines a subgroupoid of the groupoid for F. $\blacksquare$

Definition 1.3.4. Let M be a topological n-manifold (a second-countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$). Let A be a set of local homeomorphisms from M to $\mathbb{R}^n$, called *charts*, and let Γ be a pseudogroup on $\mathbb{R}^n$. A is a *Γ-atlas on M* if

1. $M = \bigcup_{\varphi\in A} \operatorname{dom}\varphi$, and
2. for all $\varphi, \psi \in A$, the *transition function* $\psi \circ \varphi^{-1}|_{\varphi(\operatorname{dom}\varphi\cap\operatorname{dom}\psi)}$ is in Γ.

Two Γ-atlases are *compatible* if their union is a Γ-atlas. A *Γ-structure* on M is an equivalence class of compatible Γ-atlases. (Note: a Γ-structure is equivalently a maximal Γ-atlas.) $\Diamond$

Definition 1.3.5. For $k \in \mathbb{N} \cup \{\infty\}$, let $\operatorname{diff}^k(\mathbb{R}^n)$ be the pseudogroup of local homeomorphisms on $\mathbb{R}^n$ that are in class C^k. In particular, $\operatorname{diff}^\infty(\mathbb{R}^n)$ consists of *local diffeomorphisms*. $\Diamond$

Definition 1.3.6. A *differentiable manifold of class C^k* is manifold with a $\operatorname{diff}^k(\mathbb{R}^n)$-structure. A *smooth manifold* is a differentiable manifold of class C^∞. $\Diamond$

Definition 1.3.7. For G a subgroup of $\mathrm{GL}(\mathbb{R}^n)$ and $k \in \mathbb{N} \cup \{\infty\}$, define the pseudogroup

$$\Gamma_G^k = \{\varphi \in \operatorname{diff}^k(\mathbb{R}^n) \mid \text{for all } x \in \operatorname{dom}\varphi,\ d\varphi_x \in G\}. \qquad \lozenge$$

Definition 1.3.8. Suppose that M and N are n-manifolds, that N has a Γ_G^k-structure, and that $f : M \to N$ is a homeomorphism. This induces a *pullback* Γ_G^k-structure on M whose charts are given by $\varphi \circ f : M \hookrightarrow \mathbb{R}^n$ for all charts $\varphi : N \hookrightarrow \mathbb{R}^n$.

If both M and N have a Γ_G^k-structure, we say that a homeomorphism $f : M \to N$ is a *diffeomorphism* if the pullback structures with respect to both f and f^{-1} are compatible with respect to the structures on M and N, respectively.

We write $\mathrm{Diff}(M, N)$ for the set of diffeomorphisms $M \to N$. The *diffeomorphism group* is $\mathrm{Diff}(M) := \mathrm{Diff}(M, M)$. $\lozenge$

Definition 1.3.9. Let V be a finite-dimensional vector space and $\{U_i\}_{i \in I}$ an I-indexed family of subspaces. Then the stabilizer subgroup of this collection of subspaces is denoted

$$\mathrm{GL}(V; \{U_i\}_i) = \bigcap_{i \in I} \{g \in \mathrm{GL}(V) \mid gU_i \subseteq U_i\}.$$

A *filtration of length* m is a nondecreasing sequence of m subspaces $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_m$ ending with $U_m = V$, and a *(partial) flag* is a filtration of increasing subspaces. For the special case of a filtration, we write

$$\mathrm{GL}(U_1 \subseteq \cdots \subseteq U_m) := \mathrm{GL}(V; \{U_i\}_i). \qquad \lozenge$$

Remark 1.3.10. The subgroup $\mathrm{GL}(U_1 \subseteq \cdots \subseteq U_m)$ is known to be a parabolic subgroup of $\mathrm{GL}(V)$; in the case that $\dim U_i = i$ for all i then it is a Borel subgroup. Borel subgroups of $\mathrm{GL}(V)$ are all conjugate to the group of upper triangular matrices. ■

Example 1.3.11. If U is a k-dimensional subspace of an n-dimensional vector space V, then the Grassmannian $\mathrm{Gr}_k(V)$ of k-dimensional subspaces of V can be identified with the coset space $\mathrm{GL}(V)/\mathrm{GL}(U \subseteq V)$. In more generality, $\mathrm{GL}(V)/\mathrm{GL}(U_1 \subset \cdots \subset U_n)$ when $\dim U_i = i$ for all i can be identified with the flag variety on V: cosets of this subgroup are in one-to-one correspondence with the set of all flags $0 \subset U_1' \subset \cdots \subset U_n' = V$. $\lozenge$

Example 1.3.12. Consider a filtration $U_1 \subseteq \cdots \subseteq U_m = \mathbb{R}^m$. Choosing a basis for $\mathbb{R}^m$ such that each subspace is the span of some prefix of the basis, then elements of $\mathrm{GL}(U_1 \subseteq \cdots \subseteq U_m)$ with respect to this basis are in upper-block-diagonal form:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix}$$

Letting $k_i = \dim(U_i/U_{i-1})$ for each $1 \leq i \leq m$ (and setting $U_0 = 0$), then each A_{ii} is a $k_i \times k_i$ invertible matrix and each A_{ij} is a $k_i \times k_j$ matrix. $\lozenge$

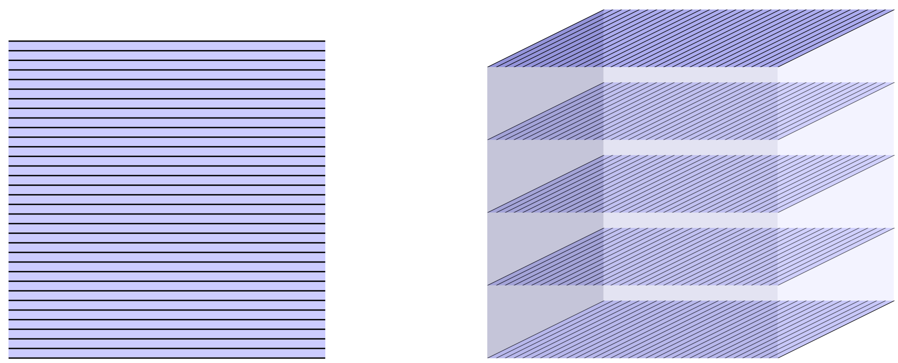


Figure 1.1: Illustrations of the product foliations of $[0,1]^2$ and $[0,1]^3$. The foliated cube (right) is thought of as a stack of 2-dimensional leaves that themselves are foliated squares, and a foliated square (left) is thought of as a stack of 1-dimensional leaves.

Definition 1.3.13. Let $\{U_i\}_{i\in I}$ be an I-indexed family of subspaces of $\mathbb{R}^n$. Define the pseudogroup

$$F(\{U_i\}_i) = \Gamma^\infty_{\mathrm{GL}(\mathbb{R}^n;\{U_i\}_i)}.$$

As a special case, for $k_1, \dots, k_n \in \mathbb{N}$ with $k_1 \leq \cdots \leq k_n$, define

$$F(k_1 \leq \cdots \leq k_n) = F(\{\mathbb{R}^{k_i}\}_{i=1}^n),$$

where include $\mathbb{R}^{k_i} \subseteq \mathbb{R}^n$ in the usual way by $(x_1, \dots, x_{k_i}) \mapsto (x_1, \dots, x_{k_i}, 0, \dots, 0)$.

A *smooth $\{U_i\}_i$-foliated manifold* is a manifold with an $F(\{U_i\}_i)$-structure, a A *smooth $(k_1 \leq \cdots \leq k_m)$-foliated manifold* is a manifold with an $F(k_1 \leq \cdots \leq k_m)$-structure, and a *smooth codimension-k foliated n-manifold* is a smooth $(n-k \leq n)$-foliated manifold. ♢

This is not the usual definition of a foliated manifold, but it is equivalent, as we will see shortly. While we do this in generality, in the sequel we only use the following standard foliation on a product structure:

Definition 1.3.14. Let $X_1, \dots, X_m$ be smooth manifolds of dimensions $k_1, \dots, k_m$, respectively. The *product foliation* on $X_1 \times \cdots \times X_m$ is the $F(k_1 \leq k_1+k_2 \leq \cdots \leq k_1+k_2+\cdots+k_m)$-structure generated by the product charts $\varphi_1 \times \cdots \times \varphi_m$ with φ_i a chart on X_i for all i.

We write $\mathrm{Diff}(X_1 \subseteq X_1 \times X_2 \subseteq \cdots \subseteq X_1 \times \cdots \times X_m)$ to indicate the diffeomorphism group for the product foliation. ♢

Remark 1.3.15. The product foliation is sensitive to the order of the product: $X_1 \times X_2$ and $X_2 \times X_1$ generally have different product foliations despite being diffeomorphic as smooth manifolds. The 1-dimensional leaves (Definition 1.3.19) of $\mathbb{R} \times \mathbb{R}$ are either horizontal or vertical depending on the product order (see Figure 1.1). ■

What the definition of a $\{U_i\}_i$-foliated manifold M gives us is an I-indexed family of distributions for M:

Lemma 1.3.16. *Let $U \subseteq \mathbb{R}^m$ be a subspace, and let M be a $(U \subseteq \mathbb{R}^m)$-foliated manifold. There is a distribution $V \subseteq TM$ with the property that whenever φ is a chart and $p \in \operatorname{dom}\varphi$, then $V_p = (d\varphi_p)^{-1}(U)$.*

Proof. We can use $p \mapsto (d\varphi_p)^{-1}(U)$ to locally define a distribution over $\operatorname{dom}\varphi$, so what we need to show is that the distribution is independent of the chart. Let ψ be another chart for M, and let $p \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$. If $v \in (d\varphi_p)^{-1}(U)$, then

$$d\psi_p(v) = d(\psi \circ \varphi^{-1})_p(v).$$

Since transition functions are in the $F(U \subseteq \mathbb{R}^m)$ pseudogroup, by definition this is in U, hence $(d\varphi_p)^{-1}(U) \subseteq (d\psi_p)^{-1}(U)$. By symmetry, this is an equality. □

Lemma 1.3.17. *Let U be a subspace of $\mathbb{R}^n$, $f \in F(U \subseteq \mathbb{R}^n)$, and $x \in \operatorname{dom} f$. Suppose $u \in U$ is such that $x + ut \in \operatorname{dom} f$ for all $t \in [0,1]$. Then*

$$f(x+u) - f(x) \in U.$$

Consequentially, f restricts to a local homeomorphism between the affine subspaces $x + U$ and $f(x) + U$.

Proof. Letting $h(t) = f(x + tu)$, then

$$f(x+u) - f(x) = h(1) - h(0) = \int_0^1 \frac{\partial h}{\partial t}\,dt = \int_0^1 df_{x+tu}(u)\,dt.$$

For all $t \in [0,1]$, since $df_{x+tu} \in \mathrm{GL}(U \subseteq \mathbb{R}^n)$, then $df_{x+tu}(u) \in U$. Hence, $f(x+u) - f(x) \in U$.

Therefore, by restricting to a star-shaped open neighborhood V of u in $\operatorname{dom} f$, then f carries V to $f(x) + V$. □

Lemma 1.3.18. *Let $U \subseteq \mathbb{R}^m$ be a subspace, and let M be a $(U \subseteq \mathbb{R}^m)$-foliated manifold. The distribution V from Lemma 1.3.16 is integrable.*

Proof. Let $p \in M$ and φ be a chart for M with $p \in \operatorname{dom}\varphi$. Let P be a connected component of $\varphi^{-1}(\varphi(p) + U)$ containing p. The affine subspace $\varphi(p) + U$ is a submanifold and φ is a local homeomorphism, so P is a submanifold of M. Let $q \in P$ and $v \in T_qP$. Then since P has the local defining function $M \hookleftarrow \!\!\!\!\circ\; \mathbb{R}^m \twoheadrightarrow \mathbb{R}^m/U$ from φ and the quotient, we see $d\varphi_q \in U$, and hence $T_qP \subseteq (d\varphi_q)^{-1}(U)$ By consideration of dimension, this is an equality.

Now we will use Lemma 1.3.17 to show that, locally, P does not depend on the choice of chart. Let ψ be another chart containing p, and define P' to be the connected component of $\psi^{-1}(\psi(p) + U)$ containing p. Consider the transition function $\psi \circ \varphi^{-1}$, which is an element of the pseudogroup $F(U \subseteq \mathbb{R}^m)$. By the lemma, the transition function locally sends $\varphi(p) + U$ to $\psi(p) + U$. Therefore, there is some open neighborhood of p inside of which P and P' coincide. □

Definition 1.3.19. Let $U \subseteq \mathbb{R}^m$ be a subspace, and let M be a $(U \subseteq \mathbb{R}^m)$-foliated manifold. Given a chart φ for M and $x \in \operatorname{cod}\varphi$, a *plaque* P is a connected component P of $\varphi^{-1}(x+U)$, which is a submanifold of dimension $\dim U$. Let φ_P denote the restriction of $\varphi|_P$ to $P \to x+U$, and let M' be M with the final topology such that the maps φ_P are continuous for all plaques P. A *leaf* is a connected component $L \subseteq M'$, which is a smooth manifold of dimension $\dim U$ with an atlas generated by the collection of the φ_P functions for all plaques P with $P \subseteq L$. ♢

According to this definition, if L is a leaf of M, then the restriction to L of the map $M' \hookrightarrow M$ is a smooth immersion. Furthermore, for $p \in L$, we have $T_pL = V_p$, where V is the distribution associated to the $(U \subseteq \mathbb{R}^m)$-foliation from Lemma 1.3.16. This means that for a $(\{U_i\}_i)$-foliated manifold, at every point p, there is a leaf L_i containing p associated to each U_i for $i \in I$. In fact, given a chart φ, then letting P_i be the plaque for U_i at p for each $i \in I$, the plaques $\{P_i\}_i$ are locally modeled by $\{U_i\}_i$.

If the subspaces are closed under intersection, then so are the leaves. In that case, for each $i \in I$, leaf L_i inherits a $F(\{U_j\}_{j\leq i})$-structure, where U_j is regarded as a subspace of U_i. Every leaf of L_i is an intersection of L_i with a leaf of M.

Example 1.3.20. If $X_1, \ldots, X_m$ are smooth manifolds of respective dimensions $k_1, \ldots, k_m$, then the leaves of the product foliation for $X_1 \times \cdots \times X_m$ that contain a point $p \in X_1 \times \cdots \times X_m$ are

$$\begin{aligned}\{p\} &\subseteq X_1 \times \{(p_2, \ldots, p_m)\} \\ &\subseteq X_1 \times X_2 \times \{(p_3, \ldots, p_m)\} \\ &\vdots \\ &\subseteq X_1 \times \cdots \times X_{m-1} \times \{(p_m)\} \\ &\subseteq X_1 \times \cdots \times X_m.\end{aligned}$$

♢

Lemma 1.3.21. *Let $\{U_i\}_i$ be an I-indexed family of subspaces of $\mathbb{R}^m$, and let M be a $\{U_i\}_i$-foliated manifold. Given $p \in M$, there is a trivialization of the tangent bundle in an open neighborhood W of p that trivializes each distribution V_i (where V_i is the distribution associated to U_i from Lemma 1.3.16). That is, if $\pi : TM \to M$ is the projection, the distribution in $\pi^{-1}(W) \approx W \times T_p$ appears as $W \times V_p$.*

Proof. Any chart for the $F(\{U_i\}_i)$ structure suffices. □

Suppose M is a $\{U_i\}_i$-foliated manifold and N is a $\{V_j\}_j$-foliated manifold. For the local classification of singularities for smooth maps $M \to N$, in coordinates the problem is to classify germs $C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)_0$ of functions that send 0 to 0, modulo the actions of $F(\{U_i\}_i)_{0,0}$ and $F(\{V_i\}_i)_{0,0}$ (the sets of germs of elements in the respective pseudogroups that send 0 to 0). The following lemma is stated in terms of charts, but consider also its application in understanding germs of elements of the foliation pseudogroups, particularly when the subspaces form a filtration.

Lemma 1.3.22. *Let k and m be natural numbers such that $0 \leq k \leq m$. Suppose φ and ψ are charts for a smooth $(k \leq m)$-foliated manifold M, and let $f = \psi \circ \varphi^{-1}|_{\varphi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi)}$ be the transition function. Let $(x_1, \dots, x_m)$ be local coordinates for φ and $(y_1, \dots, y_m)$ be local coordinates for ψ. Then for $k+1 \leq i \leq m$, locally f^*y_i is a function of $x_{k+1}, \dots, x_m$.*

Proof. By the lemma, the transition function induces a well-defined local homeomorphism of $\mathbb{R}^m/\mathbb{R}^k$, and with respect to the local coordinates $(x_{k+1}, \dots, x_m)$ and $(y_{k+1}, \dots, y_m)$ to parameterize the quotient, $f^*(y_i)$ is a function of $x_{k+1}, \dots, k_m$. □

Corollary 1.3.23. *Suppose φ, ψ are charts for a smooth $(k_1 \leq \cdots \leq k_m)$-foliated manifold M. Let the local coordinates with respect to φ and ψ respectively be*

$$(x_1, \dots, x_{k_m}) \text{ and } (y_1, \dots, y_{k_m}),$$

and let $f = \psi \circ \varphi^{-1}|_{\varphi(\operatorname{dom}\varphi \cap \operatorname{dom}\psi)}$ be the transition function. If $1 \leq i \leq k_m$, $1 \leq j \leq m$, and $k_{j-1} < i \leq k_j$ (setting $k_0 = 0$), then locally f^*y_i is a function of only $x_{k_{j-1}+1}, \dots, x_{k_m}$.

Example 1.3.24. For a $(2 \leq 3)$-foliated manifold, transition functions are locally of the form

$$\begin{aligned} f^*y_1 &= f_1(x_1, x_2, x_3) \\ f^*y_2 &= f_2(x_1, x_2, x_3) \\ f^*y_3 &= f_3(x_3) \end{aligned}$$

for locally defined smooth functions f_1, f_2, and f_3. ◊

Remark 1.3.25. We will have occasion to consider families of diffeomorphisms, and one thing we would like is a notion of the tangent space for the space of diffeomorphisms.

Suppose $G \subseteq \mathrm{GL}(\mathbb{R}^m)$ is a smooth Lie group and M is an m-manifold with a Γ_G^∞-structure. A smooth family of diffeomorphisms indexed by a smooth manifold C is a function $g : C \to \mathrm{Diff}(M)$ that is smooth in the sense that the map $M \times C \to M$ defined by $(x,t) \mapsto g_t(x)$ is smooth.

Let $g : (-\varepsilon, \varepsilon) \to \mathrm{Diff}(M)$ be a smooth family of diffeomorphisms Let $\varphi : M \hookrightarrow \mathbb{R}^m$ be a chart centered at $p \in M$ with $\varphi(p) = 0$ and coordinates $(x_1, \dots, x_m)$, and let $\psi : M \hookrightarrow \mathbb{R}^m$ be a chart centered at $q = g_0(p)$ with coordinates $(y_1, \dots, y_m)$. With this, let $g' : U \times (-\delta, \delta) \to \mathbb{R}^m$ be g with respect to the charts, with $\delta > 0$ and the open neighborhood $0 \in U \subseteq \operatorname{dom}\varphi$ chosen so g' is well-defined.

By the definitions of Γ_G^∞ and of diffeomorphisms, $d(g'_t)_x \in G$ for each $t \in (-\delta, \delta)$ and $x \in \operatorname{cod}\varphi$. For fixed x, we may regard this as being a path in G, so by taking the time derivative we get a path $\frac{\partial d(g'_t)_x}{\partial t}(x,t) \in \mathrm{Lie}(G)$ in the Lie algebra, which we think of as a subspace sitting inside $\mathrm{Lie}(\mathrm{GL}(\mathbb{R}^n)) = \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$. We may reorder the differentials to get $d\left(\left.\frac{\partial g'}{\partial t}\right|_t\right)_x \in \mathrm{Lie}(G)$ for all t, where we write $\left.\frac{\partial g'}{\partial t}\right|_t : \mathbb{R}^m \to \mathbb{R}^m$ for the derivative evaluated at t.

We may think of $\left.\frac{\partial g'}{\partial t}\right|_t$ as being a vector field on U, and this vector field may be regarded as being the time derivative of the family g in a neighborhood of p. ■

Definition 1.3.26. Let $G \subseteq \mathrm{GL}(\mathbb{R}^m)$ be a smooth Lie group, let M be an m-manifold with a Γ_G^∞-structure, and let $g \in \mathrm{Diff}(M)$. The *tangent space* $T_g \mathrm{Diff}(M)$ *at* g is defined to be the set of all smooth vector fields $v \in C^\infty(TM)$ such that for all $p \in M$ and all charts $\varphi : M \circ\!\!\rightarrow \mathbb{R}^m$ and $\psi : M \circ\!\!\rightarrow \mathbb{R}^m$ with $p \in \mathrm{dom}\,\varphi$ and $f(p) \in \mathrm{cod}\,\psi$, then $h = d\psi \circ v \circ \varphi^{-1} : \mathbb{R}^m \circ\!\!\rightarrow \mathbb{R}^m$ satisfies $dh_x \in \mathrm{Lie}(G)$ for all x in a neighborhood of $\varphi(p)$. $\lozenge$

Example 1.3.27. If $\mathbb{R}^m$ has a Γ_G^∞-structure with the identity map as a chart, the tangent space $T_{\mathrm{id}} \mathrm{Diff}(\mathbb{R}^m)$ is the set of all smooth vector fields $v \in C^\infty(T\mathbb{R}^m)$ such that $dv_x \in \mathrm{Lie}(G)$ for all $x \in \mathbb{R}^m$. $\lozenge$

Lemma 1.3.28. *Let M be a $(k_1 \leq \cdots \leq k_m)$-foliated k_m-manifold and let $v \in C^\infty(T\mathbb{R}^m)$. Give TM local coordinates $(x_1, \ldots, x_{k_m}, y_1, \ldots, y_{k_m})$, where $(x_1, \ldots, x_{k_m})$ parameterize an open subset of M and, for each i, y_i corresponds to $\partial/\partial x_i$. In these coordinates, $v \in T_{\mathrm{id}} \mathrm{Diff}(M)$ if and only if v satisfies the conclusion of Corollary 1.3.23, that $v^* y_i$ is a function of only $x_{k_{j-1}+1}, \ldots, x_{k_m}$ where $1 \leq i \leq k_m$, $1 \leq j \leq m$, and $k_{j-1} < i \leq k_j$ (setting $k_0 = 0$).*

Proof. The condition that $v \in T_{\mathrm{id}} \mathrm{Diff}(M)$ is that $dv_x \in \mathrm{Lie}(\mathrm{GL}(k_1 \leq \cdots \leq k_m))$ for all x. This Lie algebra is the set of all upper-block-diagonal matrices of the form

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix}$$

where A_{st} is a $k_s \times k_t$ matrix. Thus, for $1 \leq i, j \leq k_m$ with (i, j) in the zero region of this matrix, $\frac{\partial v^* y_i}{\partial x_j}(x) = 0$ for all x, implying $v^* y_i$ has no functional dependence on x_j. Conversely, if v is of this form then we can see that dv_x lies in the Lie algebra for all x. $\square$

Remark 1.3.29. There is a coordinate-free statement of this lemma for germs of diffeomorphisms at $p \in M$. With $k_0 = 0$, for each $0 \leq i < m$ there is a $(k_m - k_i)$-dimensional smooth manifold germ M_i (the "plaque space" at p) and a germ in $C_p^\infty(M, M_i)$ projecting onto it — every open neighborhood of p contains a connected subneighborhood consisting of k_i-plaques, and the quotient by plaques in this neighborhood gives a $(k_m - k_i)$-dimensional smooth manifold; by taking the formal limit of these quotients over all such open neighborhoods of p one obtains the manifold germ M_i. Each M_i has a tangent bundle TM_i defined as a formal inverse limit, and for each for each $0 \leq i \leq j < m$, we have a canonical smooth map germ $M_i \to M_j$ and a bundle map germ $TM_i \to TM_j$. Letting I be the poset category with objects $\{0, 1, \ldots, m-1\}$ and morphisms $i \to j$ when $i \leq j$, then we can view both $\{M_i\}_i$ and $\{TM_i\}_i$ as functors from I to the category of smooth manifold germs.

The tangent space $T_{\mathrm{id}} \mathrm{Diff}(M)$ near p is, equivalently, the set of natural transformations from $\{M_i\}_i$ to $\{TM_i\}_i$ whose components for each i are sections of the projections $TM_i \to M_i$. That is to say, an element of the tangent space is a collection of smooth section germs

$s_i : M_i \to TM_i$ such that the following diagram commutes:

$$\begin{tikzcd} TM_0 \arrow[r, two heads] & TM_1 \arrow[r, two heads] & \cdots \arrow[r, two heads] & TM_{m-1} \\ M_0 \arrow[r, two heads] \arrow[u, "s_0"] & M_1 \arrow[r, two heads] \arrow[u, "s_1"] & \cdots \arrow[r, two heads] & M_{m-1} \arrow[u, "s_{m-1}"] \end{tikzcd}$$

The map s_0 determines the rest. Then, since M_0 is the manifold germ of M centered at p and TM_0 is the corresponding bundle germ, we can identify this set of natural transformations as a subspace of $C_p^\infty(TM)$. ■

Definition 1.3.30. Let M be an m-manifold with Γ_G^∞-structure. A *local diffeomorphism* is a local homeomorphism $\varphi : M \circ\!\!\to M$ such that the restriction $\operatorname{dom}\varphi \to \operatorname{cod}\varphi$ is a diffeomorphism. For $p, q \in M$, we may consider the collection of all local diffeomorphisms sending p to q, and the stalk of this directed system is $\operatorname{Diff}_p(M)_q$, the *diffeomorphism germs sending p to q*. When $p = q$, we write $\operatorname{Diff}_p(M)$, the *group of diffeomorphism germs at p*. ♢

Lemma 1.3.31. *Let G be a smooth Lie group, let M be an m-manifold with a Γ_G^∞-structure, and let $p \in M$. Let $v : [0,1] \to T_{\mathrm{id}} \operatorname{Diff}(M)$ be a smooth family of vector fields on M each vanishing at p, and let $g_0 \in \operatorname{Diff}_p(M)$ be a diffeomorphism germ. Then there is a smooth family $g : [0,1] \to \operatorname{Diff}_p(M)$ of diffeomorphism germs solving the differential equation*

$$\tfrac{d}{dt} g(x,t) = v_t(g(x,t)) \qquad\qquad g(x,0) = g_0(x),$$

where $v_t(g(t,x))$ is from interpreting v_t as being an element of $C^\infty(TM)$ and we regard $\frac{d}{dt} g_t(x)$ as an element of $T_{g_t(x)}M$.

Proof. The existence of a solution is from taking the flow of the vector field starting from g_0, restricting the domains of the diffeomorphisms to a neighborhood of p as necessary for there to exist a global flow. The flow has the property that p remains fixed through time since the vector fields vanish there. Since the spatial derivative of the vector field locally lies in $\operatorname{Lie}(G)$, by reversing the discussion from Remark 1.3.25 the resulting spatial derivative of the solution lies in G, so each g_t is a local diffeomorphism centered at p. □

1.4 Stability

In general, if G is a topological group acting on a topological space X, we say a point $x \in X$ is *stable* if its orbit Gx is an open set. The sense in which such a point is stable is that small perturbations of the point can be compensated for by the group action.

The ideal when it comes to a suitable notion of a "generic" smooth map $M \to N$ is a *stable map*, which is a stable element of $C^\infty(M,N)$ with respect to the action by $\operatorname{Diff}(M) \times \operatorname{Diff}(N)$, where M and N might have additional structure (in the sense of Definition 1.3.7, in which case the diffeomorphism groups are of structure-preserving diffeomorphisms). For $g \in \operatorname{Diff}(M)$, $h \in \operatorname{Diff}(N)$, and $f \in C^\infty(M,N)$, we use the action $(g,h) \cdot f = hfg^{-1}$.

Table 1.1: Density of stable maps in $C^\infty(M, N)$. The region where stable maps are not dense is, roughly, when $7 \leq n \leq \frac{7}{6}m - \frac{8}{6}$ (see [Mat71] or [GG73, p. 163]).

	dim N									
dim M	1	2	3	4	5	6	7	8	9	10
1	•	•	•	•	•	•	•	•	•	•
2	•	•	•	•	•	•	•	•	•	•
3	•	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	•	•	•
6	•	•	•	•	•	•	•	•	•	•
7	•	•	•	•	•	•	•	•	•	•
8	•	•	•	•	•		•	•	•	•
9	•	•	•	•	•	•				•
10	•	•	•	•	•	•				

Unfortunately, even for smooth manifolds M and N, stable maps are not in general dense in $C^\infty(M, N)$. Mather in [Mat71] gave the exact conditions on the dimensions of M and N for which stable maps are dense (the dimension pairs for low dimensions are given in Table 1.1). This provides a baseline for stable maps with respect to smooth manifolds with additional structure — diffeomorphisms for foliated manifolds have orbits as least as fine, making achieving stability no more easier. The table gives hope that in low dimensions achieving the ideal is tractable.

As we discussed in Remark 1.3.25 and Definition 1.3.26, we may regard the tangent space $T_{\mathrm{id}} \operatorname{Diff}(M)$ of $\operatorname{Diff}(M)$ at the identity to be some subspace of $C^\infty(TM)$, the space of smooth sections of TM (where if M is just a smooth manifold, then $T_{\mathrm{id}} \operatorname{Diff}(M)$ is $C^\infty(TM)$ itself). See, for example, [GG73] for a description of the tangent space of the smooth diffeomorphism group as a Fréchet manifold.

For $f \in C^\infty(M, N)$, we may regard the tangent space $T_f C^\infty(M, N)$ to be $C^\infty(f^*TN)$. Taking the differential of the action of the diffeomorphism groups on $C^\infty(M, N)$ yields a map

$$\begin{aligned} C^\infty(TM) \oplus C^\infty(TN) &\to C^\infty(f^*TN) \\ (v, w) &\mapsto -f_*v + f^*w \end{aligned}$$

where f_*v and f^*w are the vector fields defined by, for $x \in M$,

$$(f_*v)_x = df_x(v_x) \qquad \text{and} \qquad (f^*w)_x = w_{f(x)}.$$

If f is stable, then this map is surjective. The converse yields the following definition:

Definition 1.4.1. Let M and N be smooth manifolds, possibly with additional structure. Let $\theta(M) \subseteq C^\infty(TM)$ and $\theta(N) \subseteq C^\infty(TN)$ respectively be the tangent spaces of the structure-preserving diffeomorphism groups for M and N. We say that $f \in C^\infty(M,N)$ is *infinitesimally stable* if the map

$$\begin{aligned} \theta(M) \oplus \theta(N) &\to C^\infty(f^*TN) \\ (v,w) &\mapsto -f_*v + f^*w \end{aligned}$$

is surjective.

For $p \in M$, we say that the germ $f \in C^\infty_p(M,N)$ is *(locally) infinitesimally stable* if the corresponding map $\theta_p(M) \oplus \theta_p(N) \to C^\infty_p(f^*TN)$ of germs of sections is surjective. $\diamondsuit$

The implicit function theorem does not apply to Fréchet manifolds, so infinitesimal stability does not immediately imply stability. For smooth manifolds with no additional structure, Mather proved that if f is proper and infinitesimally stable then it is indeed stable. It should be noted that local infinitesimal stability of all germs of a smooth map is not sufficient for infinitesimal stability since it ignores interactions between germs with the same image.

Local notions of stability are still useful since they are a necessary condition for stability, so we will define what it means for germs to be stable. After showing that infinitesimal stability implies local stability for smooth maps to $\mathbb{R}^n$ with a product foliation, as a corollary we will have that local infinitesimal stability implies local stability for germs from a smooth manifold to a $(k_1 \leq \cdots \leq k_r)$-foliated manifold.

Definition 1.4.2. Let M be a manifold with a Γ^∞_G-structure and N a manifold with a Γ^∞_H-structure. Let $f \in C^\infty_p(M,N)_q$ and $f' \in C^\infty_{p'}(M,N)_{q'}$ be map germs. We say f and f' are *equivalent* if there exist diffeomorphism germs $\varphi \in \mathrm{Diff}_p(M)_{p'}$ and $\psi \in \mathrm{Diff}_q(M)_{q'}$ such that $\psi \circ f = f' \circ \varphi$ (see Definition 1.3.30). $\diamondsuit$

After a perturbation of a stable map, its map germs can be said to move according to the diffeomorphisms that carry the map back to its original configuration, so the correct notion of stability of map germs needs to account for the fact that small perturbations can move their centers. This definition is from Poénaru [Poé74, II.III.1]:

Definition 1.4.3. With M and N as in Definition 1.4.2, let $f \in C^\infty(M,N)$ and $p \in M$. The germ $[f]_p \in C^\infty_p(M,N)$ is *(locally) stable* if for all small enough neighborhoods $U \subseteq M$ of p, there exists a neighborhood $V \subseteq C^\infty(U,N)$ of $f|_U$ such that for all $f' \in V$, there exists a point $p' \in U$ such that $[f]_p$ is equivalent to $[f']_{p'}$. $\diamondsuit$

For infinitesimal stability implying stability, we rely heavily on the work of Dufour and Buchner [Duf75,Duf77,Buc77]. They each extended the work of Mather to apply to stability of *diagrams* of smooth maps. The global version can be generalized to the case where each leaf space is a manifold, but we leave it in terms of product foliations due to its applicability to n-categorical decompositions. The more general local version is given as a corollary.

Theorem 1.4.4. *Let M be a smooth m-manifold, and let $\mathbb{R}^n$ be given the standard product $(\mathbb{R}^{k_1} \subseteq \mathbb{R}^{k_2} \subseteq \cdots \subseteq \mathbb{R}^{k_r})$-foliation, where $k_r = n$. If $f \in C^\infty(M, \mathbb{R}^n)$ is proper and infinitesimally stable, then it is stable.*

For $p \in M$ and $f \in C^\infty_p(M, \mathbb{R}^n)$, if f is locally infinitesimally stable, then it is locally stable.

Proof. Letting $k_0 = 0$, for each $0 \leq i \leq r-2$ let $\pi_i : \mathbb{R}^n/\mathbb{R}^{k_i} \to \mathbb{R}^n/\mathbb{R}^{k_i+1}$ be the canonical quotient. Consider the diagram (called a *cascade* by Dufour):

$$M \xrightarrow{f} \mathbb{R}^n \xrightarrow{\pi_0} \mathbb{R}^n/\mathbb{R}^{k_1} \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{r-2}} \mathbb{R}^n/\mathbb{R}^{k_{r-1}}.$$

Letting $k_0 = 0$, there is an action of $\mathrm{Diff}(M) \times \prod_{0\leq i\leq r-1} \mathrm{Diff}(\mathbb{R}^n/\mathbb{R}^{k_i})$ on $C^\infty(M, \mathbb{R}^n) \times \prod_{0\leq i\leq r-2} C^\infty(\mathbb{R}^n/\mathbb{R}^{k_i}, \mathbb{R}^n/\mathbb{R}^{k_i+1})$ defined by

$$(g, g_0, \dots, g_{r-1}) \cdot (f, h_0, \dots, h_{r-2}) = (g_0 f g^{-1}, g_1 h_0 g_0^{-1}, \dots, g_{r-1} h_{r-2} g_{r-2}^{-1}).$$

Ignoring the foliation structure, the differential of this action at the diagram above is

$$\begin{aligned} T : C^\infty(TM) \oplus \bigoplus_{0\leq i\leq r-1} C^\infty(T(\mathbb{R}^n/\mathbb{R}^{k_i})) &\to C^\infty(f^*T\mathbb{R}^n) \oplus \bigoplus_{0\leq i\leq r-2} C^\infty(\pi_i^* T(\mathbb{R}^n/\mathbb{R}^{k_{i+1}})) \\ (v, \{w_i\}_i) &\mapsto (-f_* v + f^* w_0, \{-(\pi_i)_* w_i + (\pi_i)^* w_{i+1}\}_i). \end{aligned}$$

If this map is surjective, the cascade is *infinitesimally stable*. Infinitesimally stable cascades of proper maps are stable by [Duf75].

We will show that f is infinitesimally stable in this sense and then relate stability of the diagram back to stability of f. To show the above map is surjective, suppose $\{\varphi, \{\psi_i\}_i\}$ is an arbitary element of the codomain of T. We claim that there is an element $\{v, \{w_i\}_i\}$ of the domain of T such that $T(\{v, \{w_i\}_i) + \{\varphi, \{\psi_i\}_i\} = \{\varphi', \{0\}_i\}$ for some $\varphi \in C^\infty(f^*T\mathbb{R}^n)$. Start with $\{v, \{w_i\}_i\} = \{0, \{0\}_i\}$. Let $0 \leq i \leq r-2$ be least such that $\{\varphi', \{\psi'_i\}_i\} = T(\{v, \{w_i\}_i) + \{\varphi, \{\psi_i\}_i\}$ has $\psi'_i \neq 0$. The equation $(\pi_i)_* w'_i = \psi'_i$ can be solved for w'_i since $(\pi_i)_*$ is surjective. Replacing w_i with $w_i + w'_i$, then we may assume $\psi'_j = 0$ for all $i \leq j \leq r-2$, and then we may recurse until $\varphi'_j = 0$ for all j.

We will find some $\{a, \{b_i\}_i\}$ in the domain of T such that $T(\{a, \{b_i\}_i\}) = \{\varphi', \{0\}_i\}$. For each $0 \leq i \leq r-2$, we have the condition $-(\pi_i)_* b_i + (\pi_i)^* b_{i+1} = 0$. Using standard $(y_1, \dots, y_n)$ coordinates for $\mathbb{R}^n$, we have for each i representations

$$b_i = \sum_{k_i < j \leq n} \beta_{ij}(y_{k_i+1}, \dots, y_n) \tfrac{\partial}{\partial y_j}.$$

The condition is thus

$$-\sum_{k_{i+1} < j \leq n} \beta_{ij}(y_{k_i+1}, \dots, y_n) \tfrac{\partial}{\partial y_j} + \sum_{k_{i+1} < j \leq n} \beta_{i+1,j}(y_{k_{i+1}+1}, \dots, y_n) \tfrac{\partial}{\partial y_j} = 0.$$

Hence, for each i and j, $\beta_{ij}(y_{k_i+1},\dots,y_n) = \beta_{i+1,j}(y_{k_{i+1}+1},\dots,y_n)$, which inductively implies that $\beta_{0j}(y_1,\dots,y_n) = \beta_{ij}(y_{k_i+1},\dots,y_n)$ for all i and j such that $0 \le i \le r-2$ and $k_i < j \le n$. We can see that (1) $\{\beta_{0j}\}_{1\le j\le n}$ determines the rest and (2) β_{0j} is a function of only $y_{k_i+1},\dots,y_n$, where i is such that $k_i < j \le k_{i+1}$. Thus, $\beta_0 \in C^\infty(T\mathbb{R}^n)$ must satisfy Lemma 1.3.28 and may be regarded as being an element of $T_{\mathrm{id}}\operatorname{Diff}(\mathbb{R}^{k_1} \subseteq \cdots \subseteq \mathbb{R}^{k_r})$. Since by hypothesis f is infinitesimally stable, there exists some $a \in C^\infty(TM)$ and $b_0 \in C^\infty(T\mathbb{R}^n)$ such that $f_*a + f^*b_0 = \varphi'$ and that b_0 satisfies the above constraints. Hence, $T(\{a, \{b_i\}_i\}) = \{\varphi', \{0\}_i\}$. Subtracting the correction from before, we have $T(\{a, \{b_i\}_i\} - \{v, \{w_i\}_i\}) = \{\varphi, \{\varphi_i\}_i\}$. Therefore, T is surjective and the cascade is infinitesimally stable.

Since the cascade is thus stable, its orbit under the action of the product of diffeomorphism groups is open. Thus, restricting this orbit to $C^\infty \times \{0\}_{0\le i\le r-2}$ yields an open orbit under the $\operatorname{Diff}(M) \times \operatorname{Diff}(\mathbb{R}^{k_1} \subseteq \cdots \subseteq \mathbb{R}^{k_r})$ action, and therefore f is stable.

Now for the statement about local infinitesimal stability. The notion of local infinitesimal stability of a cascade is the same except we work with germs everywhere. For $[f]_p \in C_p^\infty(M,\mathbb{R}^n)$ locally infinitesimally stable, we may assume the domain of f is suitably restricted such that it is infinitesimally stable. This local result is incidental to the underlying proofs of the above global result. A quick overview: there is a large enough $k \in \mathbb{N}$ such that any germ with a jet in the same orbit as $j^k f_p$ in $J^k(M,\mathbb{R}^n)$ is equivalent, and there is a small enough neighborhood of f such that any g in it has a jet extension $j^k g$ transversely intersecting this orbit near $j^k f_p$. □

Corollary 1.4.5. *Let M be a smooth m-manifold, let $p \in M$, and let N be a smooth $(k_1 \le \cdots \le k_r)$-foliated n-manifold (with $k_r = n$). If the germ $f \in C_p^\infty(M,N)$ is locally infinitesimally stable, then it is locally stable.*

Proof. Working in coordinates, this follows immediately from Theorem 1.4.4. □

Definition 1.4.6. A map germ $f \in C_p^\infty(M,N)$ is *finitely determined* if there is some $k \in \mathbb{N}$ such that every $f' \in C_p^\infty(M,N)$ with $j^k f_p = j^k f'_p$ is an equivalent germ. We say such a germ is *k-determined.* ◊

A k-determined map germ is, in local coordinates, equivalent to a polynomial, which is a desirable property because it helps bound the effort that goes into classifying singularities. If there is a universal bound on k for all locally stable germs, then the classification of finitely determined germs reduces to the study of orbits of a Lie group action on a finite-dimensional manifold (the jet space), and [Mat69b, Lemma 3.1] can be used to classify orbits.

Theorem 1.4.7 (Tougeron's Finite Determinacy Theorem)**.** *When M and N are both smooth manifolds with no additional structure, every locally infinitesimally stable germ in $C_p^\infty(M,N)$ is* $(\dim N + 1)$*-determined.*

Proof. See, for example, [AGZV12, I.6.3]. □

Theorem 1.4.8 (Nakai)**.** *If M is a smooth manifold and N is a $(k_1 \leq \cdots \leq k_r)$-foliated manifold. There is some $d \in \mathbb{N}$ such that every locally infinitesimally stable germ is d-determined.*

Proof. In Theorem 1.4.4 we analyzed local infinitesimal stability in terms of local infinitesimal stability (and thus local stability) of cascades. According to [Nak89, Theorem 0.3.1], since these are finite convergent trees, local stability is finitely determined. This d is not given explicitly, however. □

Remark 1.4.9. Based on examples, it appears that a locally stable germs $f \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)$, where $\mathbb{R}^n$ has the product $(1 \leq \cdots \leq n)$-foliation can be put into a polynomial form such that f_i has degree at most $n + 2 - i$ for all $1 \leq i \leq n$. It is plausible that there is a finite determinacy theorem with these degree bounds.

Even without this finer result, we still can check that germs are "locally infinitesimally k-stable," which is whether the map $\theta_p(M) \oplus \theta_q(N) \to C_p^\infty(f^*TN)/\mathfrak{m}_p(M)^{k+1}$ is surjective (i.e., whether the equations for local infinitesimal stability can be solved to order k). ■

We conclude by giving technical lemmas for more relaxed algebraic conditions for determining local infinitesimal stability of Morse 2-functions and Morse 3-functions.

Lemma 1.4.10. *Let $f \in C_p^\infty(M, \mathbb{R}^2)_0$ for M a smooth manifold and $\mathbb{R}^2$ with the standard product foliation. Let (y_1, y_2) be standard coordinates for $\mathbb{R}^2$, and use y_2 for coordinates for $\mathbb{R}^2/\mathbb{R}$. If*

$$\begin{aligned} C_p^\infty(f^*T\mathbb{R}^2) = f_* C_p^\infty(TM) + f^* C_0^\infty(\mathbb{R}^2)\tfrac{\partial}{\partial y_1} + f^* C_0^\infty(\mathbb{R}^2/\mathbb{R})\tfrac{\partial}{\partial y_2} \\ + \mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) C_p^\infty(f^*T\mathbb{R}^2) + \mathfrak{m}_0(\mathbb{R}^2)^2 C_p^\infty(f^*T\mathbb{R}^2), \end{aligned}$$

then f is locally infinitesimally stable (and thus locally stable).[4]

Equivalently, f is locally infinitesimally stable if

$$C_p^\infty(f^*T\mathbb{R}^2) = f_* C_p^\infty(TM) + (f_1^2, f_2) C_p^\infty(f^*T\mathbb{R}^2) + \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R} f_1 \tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2},$$

where $(f_1^2, f_2) \subseteq C_p^\infty(M)$ is the ideal generated by f_1^2 and f_2.

Proof. Wassermann in [Was75, Corollary 1.8] gives a useful version of Nakayama's Lemma (Lemma A.1.1) that can apply to this situation, though we give a proof using the more general Lemma A.1.3. Similar to Theorem 1.4.4, we consider the cascade

$$M \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}^2/\mathbb{R}.$$

[4] N.B. In $\mathfrak{m}_0(\mathbb{R}^2)^2 C_p^\infty(f^*T\mathbb{R}^2)$ for example, since $C_p^\infty(f^*T\mathbb{R}^2)$ is a $C_p^\infty(M)$-module then we are effectively multiplying by the ideal $C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^2)^2$.

This cascade induces a system of rings and modules:

$$C_0^\infty(\mathbb{R}^2/\mathbb{R}) \xrightarrow{\pi^*} C_0^\infty(\mathbb{R}^2) \xrightarrow{f^*} C_p^\infty(M)$$

$$\begin{tikzcd}
C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) \arrow[dr, "\pi^*"] & C_0^\infty(T\mathbb{R}^2) \arrow[d, "-\pi_*"] \arrow[dr, "f^*"] & C_p^\infty(TM) \arrow[d, "-f_*"] \\
& C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) & C_p^\infty(f^*T\mathbb{R}^2)
\end{tikzcd}$$

We define the $\mathbb{R}$-modules A and B by

$$\begin{aligned}
A &= C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) \oplus C_0^\infty(T\mathbb{R}^2) \oplus C_p^\infty(TM) \\
B &= C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) \oplus C_p^\infty(f^*T\mathbb{R}^2)
\end{aligned}$$

and, similarly to Theorem 1.4.4, define $T : A \to B$ according to the diagram, which in symbols is

$$T(u, v, w) = (\pi^*u - \pi_*v, f^*v - f_*w).$$

Define the following Jacobson ideals:

$$\begin{aligned}
I_1 &= \mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) && \subseteq \mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) \\
I_0 &= C_0^\infty(\mathbb{R}^2)\mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) + \mathfrak{m}_0(\mathbb{R}^2)^2 && \subseteq \mathfrak{m}_0(\mathbb{R}^2) \\
I &= C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) + C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^2)^2 && \subseteq \mathfrak{m}_p(M)
\end{aligned}$$

We can see that

$$\begin{aligned}
C_0^\infty(\mathbb{R}^2)\pi^*(I_1) &= C_0^\infty(\mathbb{R}^2)\mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) && \subseteq I_0 \\
C_p^\infty(M)f^*(I_0) &= C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^2/\mathbb{R}) + C_p^\infty(M)\mathfrak{m}_0^2(\mathbb{R}^2) && = I,
\end{aligned}$$

hence these Jacobson ideals satisfy the hypotheses of the lemma.

We now check the adequacy conditions for the special case where (2) does not hold for the "$i = 2$" condition. In the following, note that $C_p^\infty(M)f^*C_0^\infty(T\mathbb{R}^2) = C_p^\infty(f^*T\mathbb{R}^2)$ and $C_0^\infty(\mathbb{R}^2)\pi^*C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) = C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R}))$.

- Condition at $C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R}))$. Suppose we are given a submodule $N \subseteq C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R}))$ such that

  $$C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) = N + \pi^*C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) + I_0C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})).$$

  It suffices to prove that $C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) = N + \pi^*C_0^\infty(T(\mathbb{R}^2/\mathbb{R}))$. Letting $L = C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R}))$, the assumption is equivalently that the map

  $$\pi^* : C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) \to L/(N + \mathfrak{m}_0(\mathbb{R}^2/\mathbb{R})L + \mathfrak{m}_0(\mathbb{R}^2)^2L)$$

  of $C^\infty(\mathbb{R}^2/\mathbb{R})$-modules is a surjection. Since $C^\infty(\mathbb{R}^2/\mathbb{R})$ is generated by the one element $\frac{\partial}{\partial y_2}$ then by Corollary 1.1.75 we have that $\pi^* : C_0^\infty(T(\mathbb{R}^2/\mathbb{R})) \to L/N$ is a surjection as well, and hence $L = N + \pi^*C_0^\infty(T(\mathbb{R}^2/\mathbb{R}))$.

- Condition at $C_p^\infty(f^*T\mathbb{R}^2)$. Suppose we are given that

$$C_p^\infty(f^*T\mathbb{R}^2) = f_*C_p^\infty(TM) + f^*C_0^\infty(T\mathbb{R}^2) + IC_p^\infty(f^*T\mathbb{R}^2).$$

  It suffices to prove the following:

$$\begin{aligned} C_p^\infty(f^*T\mathbb{R}^2) &= f_*C_p^\infty(TM) + f^*C_0^\infty(T\mathbb{R}^2) \\ IC_p^\infty(f^*T\mathbb{R}^2) &= If_*C_p^\infty(TM) + f^*I_0C_0^\infty(T\mathbb{R}^2) \end{aligned}$$

  Since $I \subseteq C_p(M)\mathfrak{m}_0(\mathbb{R}^2)$, then Theorem 1.1.73 applies to the assumption, yielding the first equation. Multiplying the first equation by I_0 yields the second equation.

Hence, the adequacy conditions are satisfied.

Now we check that $B = T(A) + I_0C_0^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) + IC_p^\infty(f^*T\mathbb{R}^2)$. Similar to the argument in the proof of Theorem 1.4.4, since π_* is surjective we only need to check that

$$C_p^\infty(f^*T\mathbb{R}^2) \subseteq T(A) + I_0C^\infty(\pi^*T(\mathbb{R}^2/\mathbb{R})) + IC_0^\infty(f^*T\mathbb{R}^2).$$

One can see that this is satisfied if

$$C_p^\infty(f^*T\mathbb{R}^2) = f_*C_p^\infty(TM) + f^*C_0^\infty(\mathbb{R}^2)\tfrac{\partial}{\partial y_1} + f^*C_0^\infty(\mathbb{R}^2/\mathbb{R})\tfrac{\partial}{\partial y_2} + IC_p^\infty(f^*T\mathbb{R}^2),$$

which is our hypothesis, hence by Lemma A.1.3 we have that $B = T(A)$. Like in Theorem 1.4.4, since $B = T(A)$ then f is infinitesimally stable (and thus locally stable).

For the equivalent formulation, we compute that

$$\begin{aligned} \mathfrak{m}_0(\mathbb{R}^2/\mathbb{R})C_p^\infty(f^*T\mathbb{R}^2) + \mathfrak{m}_0(\mathbb{R}^2)^2C_p^\infty(f^*T\mathbb{R}^2) &= (f_2)C_p^\infty(f^*T\mathbb{R}^2) + (f_1,f_2)^2C_p^\infty(f^*T\mathbb{R}^2) \\ &= (f_1^2, f_2)C_p^\infty(f^*T\mathbb{R}^2). \end{aligned}$$

The module $f^*C_0^\infty(\mathbb{R}^2)\frac{\partial}{\partial y_1}$ consists of elements $\varphi(f_1,f_2)\frac{\partial}{\partial y_1}$, and by the Hadamard Lemma (Lemma 1.1.69) there are constants $c_0, c_1 \in \mathbb{R}$ and functions $\varphi_{11}, \varphi_{12}, \varphi_2 \in C_0^\infty(\mathbb{R}^2)$ such that

$$\varphi(f_1,f_2) = c_0 + c_1f_1 + \varphi_{11}(f_1,f_2)f_1^2 + \varphi_{12}(f_1,f_2)f_1f_2 + \varphi_2(f_1,f_2)f_2.$$

Hence, $f^*C_0^\infty(\mathbb{R}^2)\frac{\partial}{\partial y_1} \subseteq \mathbb{R}\frac{\partial}{\partial y_1} + \mathbb{R}f_1\frac{\partial}{\partial y_1} + (f_1^2,f_2)C_p^\infty(f^*T\mathbb{R}^2)$. Similarly, $f^*C_0^\infty(\mathbb{R}^2/\mathbb{R})\frac{\partial}{\partial y_2} \subseteq \mathbb{R}\frac{\partial}{\partial y_2} + (f_1^2,f_2)C_p^\infty(f^*T\mathbb{R}^2)$. Therefore, the hypothesis may equivalently be written as

$$C_p^\infty(f^*T\mathbb{R}^2) = f_*C_p^\infty(TM) + (f_1^2,f_2)C_p^\infty(f^*T\mathbb{R}^2) + \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R}f_1\tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2}. \qquad \square$$

Corollary 1.4.11. *Let $f \in C_p^\infty(M,\mathbb{R}^2)_0$ for M a smooth manifold and $\mathbb{R}^2$ with the standard product foliation. Then f is locally infinitesimally stable if*

$$\begin{aligned} C_p^\infty(f^*T\mathbb{R}^2) = f_*C_p^\infty(TM) &+ (f_1^2,f_2)C_p^\infty(f^*T\mathbb{R}^2) + \mathfrak{m}_p(M)^4C_p^\infty(f^*T\mathbb{R}^2) \\ &+ \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R}f_1\tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2}. \end{aligned}$$

Hence, local infinitesimal stability is determined by the 4-jet of f.

Proof. Let $A = f_* C_p^\infty(TM) + (f_1^2, f_2)C_p^\infty(f^*T\mathbb{R}^2)$. Since we assume that $C_p^\infty(f^*T\mathbb{R}^2)/(A + \mathfrak{m}_p(M)^4 C_p^\infty(f^*T\mathbb{R}^2))$ is a vector space of dimension at most three, by Corollary A.1.2 we deduce that

$$\mathfrak{m}_p(M)^3 C_p^\infty(f^*T\mathbb{R}^2) \subseteq A.$$

Thus, $\mathfrak{m}_p(M)^4 C_p^\infty(f^*T\mathbb{R}^2) \subseteq A$, so we have that

$$C_p^\infty(f^*T\mathbb{R}^2) = f_* C_p^\infty(TM) + (f_1^2, f_2)C_p^\infty(f^*T\mathbb{R}^2) + \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R}f_1\tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2}.$$

By Lemma 1.4.10, f is locally infinitesimally stable.

We now consider the claim that local infinitesimal stability is determined by the 4-jet. Suppose $f' \in C_p^\infty(M, \mathbb{R}_0^2)$ is another germ with $j^4 f_p = j^p f'_p$. Let $(x_1, \ldots, x_m)$ be coordinates for M and (y_1, y_2) the standard coordinates for $\mathbb{R}^2$. Given that $j^4 f_p = j^4 f'_p$, then in coordinates there are functions $\varphi_1, \varphi_2 \in \mathfrak{m}_p(M)^5$ such that $f' = f + \varphi$. For $u_1, u_2, w_1, w_2 \in C_p^\infty(M)$, then

$$\begin{aligned}(u_1\tfrac{\partial f'}{\partial x_1} + u_2\tfrac{\partial f'}{\partial x_2} + w_1(f'_1)^2\tfrac{\partial}{\partial y_1} + w_2 f'_2\tfrac{\partial}{\partial y_1}) - (u_1\tfrac{\partial f}{\partial x_1} + u_2\tfrac{\partial f}{\partial x_2} + w_1 f_1^2\tfrac{\partial}{\partial y_1} + w_2 f_2\tfrac{\partial}{\partial y_1})\\ = u_1\tfrac{\partial \varphi}{\partial x_1} + u_2\tfrac{\partial \varphi}{\partial x_2} + w_1(2f_1\varphi_1 + \varphi_1^2)\tfrac{\partial}{\partial y_1} + w_2\varphi_2\tfrac{\partial}{\partial y_2}.\end{aligned}$$

This difference is an element of $\mathfrak{m}_p(M)^4 C_p^\infty(f^*T\mathbb{R}^2) \subseteq \mathfrak{m}_p(M)A$. Defining $A' = f'_* C_p^\infty(TM) + ((f'_1)^2, f'_2)C_p^\infty((f')^*T\mathbb{R}^2)$, then calculation implies $A' \subseteq A$ since each element of A' is a sum of elements of A. We also see from this that $A \subseteq A' + \mathfrak{m}_p(M)A$. Since $A' + m_p(M)A \subseteq A + m_p(M)A$ and $A + m_p(M)A = A$, then $A' + \mathfrak{m}_p(M)A = A$. By Nakayama's Lemma, $A' = A$, hence $C_p^\infty(M, \mathbb{R}^2)/A' = C_p^\infty(M, \mathbb{R}^2)/A$, so by Lemma 1.4.10 we see f' is locally infinitesimally stable too. □

Lemma 1.4.12. *Let $f \in C_p^\infty(M, \mathbb{R}^n)_0$ for M a smooth manifold and $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ with the standard product foliation for some fixed $0 \le k \le n$. Let $(y_1, \ldots, y_n)$ be the standard coordinates for $\mathbb{R}^n$, and use $(y_{k+1}, \ldots, y_n)$ for coordinates for $\mathbb{R}^n/\mathbb{R}^k$. If*

$$\begin{aligned}C_p^\infty(f^*T\mathbb{R}^n) = f_* C_p^\infty(TM) + \sum_{1\le i\le k} f^* C_0^\infty(\mathbb{R}^n)\tfrac{\partial}{\partial y_i} + \sum_{k+1\le i\le n} f^* C_0^\infty(\mathbb{R}^n/\mathbb{R}^k)\tfrac{\partial}{\partial y_i}\\ + \mathfrak{m}_0(\mathbb{R}^n/\mathbb{R}^k)C_p^\infty(f^*T\mathbb{R}^n) + \mathfrak{m}_0(\mathbb{R}^n)^{n-k+1}C_p^\infty(f^*T\mathbb{R}^n),\end{aligned}$$

then f is locally infinitesimally stable (and thus locally stable).

Proof. The proof is along the same lines as Lemma 1.4.10. We consider the cascade $M \xrightarrow{f} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/\mathbb{R}^k$, which induces the following system of rings and modules:

$$C_0^\infty(\mathbb{R}^n/\mathbb{R}^k) \xrightarrow{\pi^*} C_0^\infty(\mathbb{R}^n) \xrightarrow{f^*} C_0^\infty(M)$$

$$\begin{tikzcd} C_0^\infty(T(\mathbb{R}^n/\mathbb{R}^k)) \arrow[dr, "\pi^*"] & C_0^\infty(T\mathbb{R}^n) \arrow[d, "-\pi_*"] \arrow[dr, "f^*"] & C_p^\infty(TM) \arrow[d, "-f_*"] \\ & C_0^\infty(\pi^*T(\mathbb{R}^n/\mathbb{R}^k)) & C_p^\infty(f^*T\mathbb{R}^n) \end{tikzcd}$$

We define the $\mathbb{R}$-modules A and B and the homomorphism $T : A \to B$ using the obvious generalization. Define the following Jacobson ideals:

$$\begin{aligned}
I_1 &= \mathfrak{m}_0(\mathbb{R}^n/\mathbb{R}^k) && \subseteq \mathfrak{m}_0(\mathbb{R}^n/\mathbb{R}^k) \\
I_0 &= C_0^\infty(\mathbb{R}^n)\mathfrak{m}_0(\mathbb{R}^n/\mathbb{R}^k) + \mathfrak{m}_0(\mathbb{R}^n)^{n-k+1} && \subseteq \mathfrak{m}_0(\mathbb{R}^2) \\
I &= C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^n/\mathbb{R}^k) + C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^n)^{n-k+1} && \subseteq \mathfrak{m}_p(M)
\end{aligned}$$

With these choices, the argument from Lemma 1.4.10 carries through, where we use $n-k+1$ instead of 2 since the $n-k$ elements $\frac{\partial}{\partial y_{k+1}}, \ldots, \frac{\partial}{\partial y_n}$ generate $C_0^\infty(T(\mathbb{R}^n/\mathbb{R}^k))$ as a $C_0^\infty(\mathbb{R}^n/\mathbb{R}^k)$ module. □

Corollary 1.4.13. *Let $f \in C_p^\infty(M, \mathbb{R}^{n+1})_0$ for M a smooth manifold and $\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}$ with the standard product foliation. Then f is locally infinitesimally stable if*

$$\begin{aligned}
C_p^\infty(f^*T\mathbb{R}^{n+1}) = f_*C_p^\infty(TM) + IC_p^\infty(f^*T\mathbb{R}^{n+1}) + \mathfrak{m}_p(M)^{2n+2}C_p^\infty(f^*T\mathbb{R}^{n+1}) \\
+ \sum_{1\le i\le n+1} \mathbb{R}\tfrac{\partial}{\partial y_i} + \sum_{1\le i,j\le n} \mathbb{R}y_i\tfrac{\partial}{\partial y_j},
\end{aligned}$$

where $I \subseteq C_p^\infty(M)$ is the ideal $I = (f_1, \ldots, f_n)^2 + (f_{n+1})$.

Proof. This follows from the same sorts of calculations in Lemma 1.4.12, and then using the application of Nakayama's lemma from Corollary 1.4.11, where now the quotient has dimension at most $n + (n+1)$. □

Lemma 1.4.14. *Let $f \in C_p^\infty(M, \mathbb{R}^3)_0$ for M a smooth manifold and $\mathbb{R}^3$ given the standard product foliation. Let (y_1, y_2, y_3) be coordinates for $\mathbb{R}^3$, and use (y_2, y_3) for coordinates for $\mathbb{R}^3/\mathbb{R}$ and (y_3) for coordinates for $\mathbb{R}^3/\mathbb{R}^2$. If*

$$\begin{aligned}
C_p^\infty(f^*T\mathbb{R}^3) = f_*C_p^\infty(TM) + f^*C_0^\infty(\mathbb{R}^3)\tfrac{\partial}{\partial y_1} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R})\tfrac{\partial}{\partial y_2} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R}^2)\tfrac{\partial}{\partial y_3} \\
+ \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2)C_p^\infty(f^*T\mathbb{R}^3) + \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R})^2C_p^\infty(f^*T\mathbb{R}^3),
\end{aligned}$$

then f is locally infinitesimally stable (and thus locally stable).

Proof. The proof is similar to Lemma 1.4.10, but now there is a system of four rings, and we use the following Jacobson ideals:

$$\begin{aligned}
I_2 &= \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2) && \subseteq \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2) \\
I_1 &= C_0^\infty(\mathbb{R}^3/\mathbb{R})\mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2) + \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R})^2 && \subseteq \mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}) \\
I_0 &= C_0^\infty(\mathbb{R}^3)\mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2) + C_0^\infty(\mathbb{R}^3)\mathfrak{m}_0(\mathbb{R}^3/\mathbb{R})^2 && \subseteq \mathfrak{m}_0(\mathbb{R}^3) \\
I &= C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^3/\mathbb{R}^2) + C_p^\infty(M)\mathfrak{m}_0(\mathbb{R}^3/\mathbb{R})^2 && \subseteq \mathfrak{m}_p(M). \quad \square
\end{aligned}$$

1.5 The Thom–Boardman singularities

The higher-Morse-theoretic decomposition we use relies on a generalization of the classification of singularities of smooth maps developed by Thom and Boardman in [Tho56, Boa67]. This purpose of this section is to review this theory and to lay the groundwork for the generalization.

Suppose M and N are smooth manifolds of respective dimensions m and n. For $f \in C^\infty(M,N)$ in a dense subset, Thom and Boardman defined, for all $r \in \mathbb{N}$ and $i_1,\dots,i_r \in \mathbb{N}$, locally closed submanifolds $S[i_1][i_2]\dots[i_r](f) \subseteq M$ that stratify M.[5] (Strictly speaking, the submanifolds nest, and only those of these that end in a single $i_r = 0$ form a stratification of M.)

It starts with the simplest classification of points of M, which is how far the differential df drops rank:

$$S[i_1](f) = \{p \in \operatorname{dom} f \mid \operatorname{drk} df_p = i_1\}.$$

(The *dropped rank* is $\operatorname{drk} df_p = \min(\dim T_pM, \dim T_{f(p)}N) - \operatorname{rk} df_p$.) For f in a dense subset, $S[i_1](f)$ is a submanifold of M, and so one can define

$$S[i_1][i_2](f) = S[i_2](f|_{S[i_1](f)}).$$

Again, for f in a dense subset, $S[i_1][i_2](f)$ is a submanifold of $S_i(f)$. In this way one recursively defines

$$S[i_1][i_2]\dots[i_r](f) = S[i_r](f|_{S[i_1][i_2]\dots[i_{r-1}](f)}).$$

The result is a stratification of M such that within each stratum S the differential $d(f|_S)$ has constant rank. If $p \in S[i_1][i_2]\dots[i_r](f)$, we say that f has a $S[i_1][i_2]\dots[i_r]$ *Thom–Boardman singularity* at p. It is a local classification in that the type of a Thom-Boardman singularity depends on no more than the germ of f at p, and the classification is invariant under diffeomorphisms of M and N.

It should be said that what is meant by a classification of singularities is, generally speaking, a characterization of the orbits of smooth map germs $[f]_p \in C^\infty_p(M,N)_q$ up to the actions of the groups $\operatorname{Diff}_p(M)$ and $\operatorname{Diff}_q(N)$ of smooth map germs of diffeomorphisms that preserve relevant structure, for instance orientations or foliations. The Thom–Boardman singularity type is not a complete classification of singularities (for example [Por72]), but it can be a useful first approach. In small enough (co)dimensions, the Thom–Boardman singularity type can be enough to derive a finite list of possible normal forms for a singularity. As an example from Morse theory, nondegenerate critical points of a smooth map $f : \mathbb{R}^m \to \mathbb{R}$ are the $S[1][0]$ singularities, and one can determine that they are locally modeled by quadratic forms. The signature of a quadratic form is a singularity invariant, and the signature is sufficient to further stratify $S[1][0]$ into the m remaining singularity types.

We will be considering a variation on Thom–Boardman singularities, taking into account foliations on the domain and codomain, and to see how to do this it is worth going into

[5] The traditional notation is $S_{i_1 i_2 \dots i_r}$, however in anticipation of singularity types with more complicated symbols we use 1×1 matrices for consistency.

detail about how to show the above stratification of M can be carried out. A beautiful algebraic approach by Boardman [Boa67] and Mather [Mat73] involves defining subvarieties of the jet bundles $J^k(M,N)$ for each Thom–Boardman singularity type and then applying Theorem 1.2.9 (the Thom Transversality Theorem). These subvarieties stratify the jet bundles themselves, and thus if $f : M \to \mathbb{N}$ is in the dense set for which the jet extension $j^k f : M \to J^k(M,N)$ is transverse to the subvarieties, the preimage of the stratification stratifies M.

We will use the Porteous intrinsic derivative to define the relevant submanifolds of the jet bundles [Por71], which we have found to be adaptable in refining singularity types based on foliations and other data.

Here is the high-level strategy for the construction of the jet bundle submanifolds for the Thom–Boardman singularities. Suppose we have defined a submanifold $S \subseteq J^k(M,N)$ to classify singularities, where we say a smooth map $f : M \to N$ has an S singularity at $p \in M$ if $j^k f_p \in S$. By the Thom Transversality Theorem, for a dense set of smooth maps, when f has an S singularity at p its jet extension intersects S transversely, and so the S points of such an f form a submanifold $S(f) \subseteq M$. When $j^k f_p \in S$, the condition $j^k f \pitchfork S$ at p is whether the map

$$q_{j^k f_p} \circ d(j^k f)_p : T_p M \to N_{j^k f_p} S$$

is surjective, where $NS = TJ^k(M,N)/TS$ is the normal bundle of S, pulling back $TJ^k(M,N)$ to be a bundle over S. In the case that $j^k f$ intersects S transversely, these assemble into a map of vector bundles over $S(f)$, and we can identify the tangent bundle of $S(f)$ with the kernel bundle $K \subseteq TM$ over $S(f)$ of this map. There also is another way in which the maps assemble into a map of bundles. The differential of $j^k f$ at p is determined by $j^{k+1} f_p$, so, lifting S to $S' \subseteq J^{k+1}(M,N)$, the differential defines a map of bundles over S:

$$\begin{aligned} D : S' &\to \mathrm{Hom}(TM, NS) \\ j^{k+1} f_p &\mapsto q_{j^k f_p} \circ d(j^k f)_p \end{aligned}$$

where TM stands for the pullback of TM along $\alpha : J^k(M,N) \to M$ and then pulling it back to S. Inside S' is the subset of $(k+1)$-jets whose k-jet extension intersects S transversely, and in the ideal case this set forms a submanifold $S'' \subseteq S'$. Now that we have this bundle map, we can make use of the structure of $\mathrm{Hom}(TM, NS)$ to stratify S''. For each $j^{k+1} f_p \in S''$, we can consider $S(f)$ to be a submanifold locally at p, and in particular $d(f|_{S(f)})_p : T_p S(f) \to T_{f(p)} N$ is well-defined, which by our identification from earlier is the restriction of df_p to the kernel K_p. The kernels assemble into a vector bundle K over S, so we may consider the restriction map $\mathrm{Hom}(TM, NS) \to \mathrm{Hom}(K, NS)$. Let $S[i] \subseteq S''$ denote the set of $(k+1)$-jets whose image in $\mathrm{Hom}(K, NS)$ has dropped rank i. In the ideal case, for all $i \in \mathbb{N}$ the set $S[i]$ is a submanifold, and we say these $S[i]$ manifolds refine the S-type singularity.[6] We then recursively refine submanifolds in this way.

[6] We ignore $S' \setminus S''$ since these jets only appear for smooth functions whose jet extension is not transverse to S, and such functions are in the complement of a dense set by the Thom Transversality Theorem.

To be able to pull this off, there are two ingredients: (1) we need to be able to efficiently characterize the normal bundle NS, and (2) we need to be able to show that the rank decomposition of $\mathrm{Hom}(K, NS)$ pulls back to a decomposition of S''. The key is that for each S we consider:

- the composition $S'' \to \mathrm{Hom}(K, NS)$ is a submersion onto its image, and we can characterize this image;
- the intersection of this image with the submanifold $\mathrm{Hom}^i(K, NS) \subseteq \mathrm{Hom}(K, NS)$ of homomorphisms that drop rank by i is still a submanifold; and
- as a submanifold of the image, we can characterize the normal bundle of this intersection to get a normal bundle for $S[i]$ in S''.

The process naturally gives a direct sum decomposition of the normal bundle of each singularity type, with the next direct summand coming from the Porteous intrinsic derivative.

The main way we will modify the above process is to consider dropped ranks in more than just $\mathrm{Hom}(K, NS)$. For example, foliations of the codomain give a filtration of T_pM by taking preimages of tangent spaces through df_p, beyond just the filtration $0 \subseteq K_p \subseteq T_pM$, and we can consider the submanifold of $\mathrm{Hom}(TM, NS)$ from specifying dropped ranks for each subspace. If we have submanifolds of the domain, then for singularities along that submanifold we can also consider the intersection of the filtration with its tangent space. To do this process effectively, we need to develop methods to calculate normal bundles in these situations.

We have not said anything about classifying multi-singularities arising from sets of points mapping to the same point, but in essence the only difference is that we use multijets and the Multijet Transversality Theorem (theorem 1.2.12).

1.5.1 Submanifolds of linear maps with rank conditions

For our classification of singularities, we need to be able to calculate normal bundles for certain intersections of submanifolds of $\mathrm{Hom}(V, W)$, where V and W are finite-dimensional vector spaces.

For $a \in \mathrm{Hom}(V, W)$, let $\operatorname{drk} a = \min(\dim V, \dim W) - \operatorname{rk} a$ be the amount by which a drops rank, and for $r \in \mathbb{N}$ define the topological subspaces

$$\mathrm{Hom}^r(V, W) = \{a \in \mathrm{Hom}(V, W) \mid \operatorname{drk} a = r\}.$$

These manifolds are analyzed in [GG73], where the normal bundle at $a \in \mathrm{Hom}^r(V, W)$ is shown to be naturally isomorphic to $\mathrm{Hom}(\ker a, \operatorname{coker} a)$, but we are interested in a more general setup. Given a finite index set I, suppose for all $i \in I$ that $V_i \subseteq V$ and $W_i \subseteq W$ are subspaces and $r_i \in \mathbb{N}$ (note: subspaces are allowed to repeat). For $a \in \mathrm{Hom}(V, W)$ and

$i \in I$ we let $a_i : V_i \to W/W_i$ denote the composition $V_i \hookrightarrow V \xrightarrow{a} W \twoheadrightarrow W/W_i$. The subset in particular that we are interested in is

$$\operatorname{Hom}(V, W; \{r_i\}_i) := \bigcap_{i \in I} \{a \in \operatorname{Hom}(V, W) \mid a_i \in \operatorname{Hom}^{r_i}(V_i, W/W_i)\}. \tag{1.5.1}$$

While this notation is ambiguous, $\{V_i\}_i$ and $\{W_i\}_i$ will be clear from context.

When the collections $\{V_i\}_i$ and $\{W_i\}_i$ are closed under intersections, we will show that $\operatorname{Hom}(V, W; \{r_i\}_i)$ is a submanifold and give calculations for normal bundles in Theorem 1.5.15 and Corollary 1.5.20. Note that the condition $\operatorname{drk} a_i = r_i$ is equivalent to $\dim(V_i \cap a^{-1}(W_i)) = k_i$ for $k_i = r_i + \max(0, \dim V_i + \dim W_i - \dim W)$, which is the dimension of the kernel of a_i. We will find it more convenient to work with the kernel dimensions $\{k_i\}_i$ in this section. Whether we are using the dropped-rank convention or the kernel convention is determined by whether we are using $\{r_i\}_i$ or $\{k_i\}_i$.

Recall that the *Grassmannian* $\operatorname{Gr}_n(V)$, for V a finite-dimensional vector space and $0 \leq n \leq \dim V$, is the moduli space of n-dimensional subspaces of V. The smooth manifold structure for $\operatorname{Gr}_n(V)$ is given by the following: for $U \in \operatorname{Gr}_n(V)$ and a choice of section $s : V/U \to V$ of $q : V \twoheadrightarrow V/U$, there is a chart for $\operatorname{Gr}_n(V)$ centered at U whose inverse is given by

$$\begin{aligned} \Gamma : \operatorname{Hom}(U, V/U) &\to \operatorname{Gr}_n(V) \\ f &\mapsto \{u + sfu \mid u \in U\}. \end{aligned}$$

This should be thought of as giving the graph of f in the coordinate system associated to the splitting of V given by s and q. The tangent bundle for $\operatorname{Gr}_n(V)$ is canonically isomorphic to $\operatorname{Hom}(U, V/U)$, where by abuse of notation "$U \to \operatorname{Gr}_n(V)$" is the tautological vector bundle whose fiber at a point of $\operatorname{Gr}_n(V)$ is the point itself as a vector space.

Supposing for all $i \in I$ we have subspaces $V_i \subseteq V$ and numbers $k_i \in \mathbb{N}$, we define the following Schubert-variety-like subset of $\operatorname{Gr}_n(V)$:

$$\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) = \bigcap_{i \in I} \{U \in \operatorname{Gr}_n(V) \mid \dim(U \cap V_i) = k_i\}.$$

The first thing we will show is that when $\{V_i\}_i$ is closed under intersections that this is a smooth submanifold of $\operatorname{Gr}_n(V)$, and as a corollary we obtain a description of its tangent bundle.

Definition 1.5.1. A *semilattice* I is a poset such that for all $i, j \in I$ there exists a greatest lower bound of $\{i, j\}$, denoted by $i \wedge j \in I$ called the *meet*. A semilattice is *bounded* if it has a maximum element $\top \in I$. ◊

Remark 1.5.2. A bounded semilattice is a poset that, as a category, is finitely complete. In other words, it is closed under meets of arbitrary finite subsets. If $S \subseteq I$ is finite, we write $\wedge S$ for the greatest lower bound of S, which we also write as $\bigwedge_{i \in S} i$. ■

Remark 1.5.3. Given a family $\{V_i\}_{i\in I}$ of distinct subspaces of a vector space V, if the family is closed under pairwise intersections then we may give I the structure of a semilattice defined by (1) for all $i, j \in I$, then $i \leq j$ if and only if $V_i \subseteq V_j$ and (2) for all $i, j \in I$, then $i \wedge j$ is the element such that $V_{i\wedge j} = V_i \cap V_j$. If the family has a maximal subspace, then I is a bounded semilattice.

Another way to give this data is as a binary-product-preserving functor $I \to \mathrm{Subspace}(V)$ from a semilattice I to the poset category of subspaces of V. If I is a bounded semilattice, then the corresponding concept is instead a left exact functor $I \to \mathrm{Subspace}(V)$. ■

Definition 1.5.4. Let $\{V_i\}_i$ be an I-indexed family of subspaces of V, let $n \in \mathbb{N}$, and for each $i \in I$ let $k_i \in \mathbb{N}$. Then we define

$$\mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i) = \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)/\sim$$

where $\sim$ is the relation where $U \sim U'$ if and only if, for all $i \in I$, $U \cap V_i = U' \cap V_i$. ♢

The purpose of this space is to record only the portion of U that is within the subspaces. Certainly, if there is an $i \in I$ such that $V_j \subseteq V_i$ for all $j \in I$, then the map

$$\mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i) \to \mathcal{V}_{k_i}(V_i; \{V_i\}_i; \{k_i\}_i)$$

defined by $[U] \mapsto U \cap V_i$ is a homeomorphism.

For each $i \in I$, the restriction of $U \in \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$ to V_i defines the element $U \cap V_i$ of $\mathrm{Gr}_{k_i}(V_i)$, and more specifically of $\mathcal{V}_{k_i}(V_i; \{V_j\}_{j\leq i}, \{k_j\}_{j\leq i})$. This is functorial in the sense that, whenever $i' \leq i$, there is an induced map

$$\mathcal{V}_{k_i}(V_i; \{V_j\}_{j\leq i}, \{k_j\}_{j\leq i}) \to \mathcal{V}_{k_{i'}}(V_{i'}; \{V_j\}_{j\leq i'}, \{k_j\}_{j\leq i'})$$

defined by intersecting the element of the Grassmannian with $V_{i'}$. The problem we now consider is in organizing the data of how these spaces fit together. We think of the amount of "new" information about U that $U \cap V_i$ provides by considering how much of it can be recovered from the images $U \cap V_{i'}$ for all $i' < i$. Then by walking up the I semilattice, we construct coordinate charts from only the new information that arises at each step.

Definition 1.5.5. Suppose I is a poset. A *downward set* is a subset $S \subseteq I$ such that if $i \in I$, $j \in S$ and $i \leq j$ then $i \in S$. For $S \subseteq I$, the *downward set generated by* S is the minimal downward set containing $S \subseteq I$ and is denoted by

$${\downarrow} S := \{j \in I \mid j \leq i \text{ for some } i \in S\}.$$

We write $I{\downarrow}S := {\downarrow}S$ if we wish to make I explicit. The set ${\downarrow}^\circ S; = ({\downarrow}S) \setminus S$ (or $I{\downarrow}^\circ S$) is the *open downward set generated by* S. ♢

To organize the data, rather than considering individual indices, we instead consider restrictions to entire downward sets. For downward sets $S \subseteq S' \subseteq I$, there is an induced map

$$\mathcal{V}_n^\circ(V;\{V_i\}_{i\in S'};\{k_i\}_{i\in S'}) \to \mathcal{V}_n^\circ(V;\{V_i\}_{i\in S};\{k_i\}_{i\in S}).$$

We will show that this map is a fiber bundle and describe the fibers. In particular, we will do so for the case that S' has one more element than S, and the rest follows by induction.

In the following, for $i \in I$ we will have k'_i denote the dimension of $U \cap V_i / \sum_{j<i} U \cap V_j$ for $U \in \mathcal{V}_n(V;\{V_j\}_j,\{k_j\}_j)$. This does not depend on the choice of U and these numbers are the unique solution to the equations $k_i = \sum_{j\le i} k'_j$ for all $i \in I$. They can be calculated by inclusion-exclusion on the semilattice as

$$k'_i = k_i - \sum_{\varnothing \ne T \subseteq \downarrow^\circ i} (-1)^{|T|+1} k_{\wedge T}.$$

Lemma 1.5.6. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections, let $n \in \mathbb{N}$, and for all $i \in I$ let $k_i \in \mathbb{N}$. Let $m \in I$ be some maximal element. Define the map*

$$\begin{aligned}\gamma : \mathcal{V}_n^\circ(V;\{V_i\}_{i\ne m};\{k_i\}_{i\ne m}) &\to \mathrm{Gr}_{k_m-k'_m}(V_m)\\ [U] &\mapsto \sum_{i<m} U \cap V_i\end{aligned}$$

and form the fiber bundle $E \to \mathrm{Gr}_{k_m-k'_m}(V_m)$ whose fiber over W is given by $\mathrm{Gr}_{k'_m}(V_m/W)$. The map

$$\mathcal{V}_n^\circ(V;\{V_i\}_i;\{k_i\}_i) \to \mathcal{V}_n^\circ(V;\{V_i\}_{i\ne m};\{k_i\}_{i\ne m})$$

induced by the inclusion $I \setminus \{m\} \subseteq I$ of downward sets is a fiber bundle isomorphic to an open subspace of $\gamma^* E$. In particular, the fiber above $[U] \in \mathcal{V}_n^\circ(V;\{V_i\}_{i\ne m};\{k_i\}_{i\ne m})$ is the open subspace

$$\mathcal{V}_{k'_m}(V_m/\gamma([U]);\{(\gamma([U])+V_i)/\gamma([U])\}_{i<m};\{0\}_{i<m}) \subseteq \mathrm{Gr}_{k'_m}(V_m/\gamma([U]))$$

and the inclusion of this fiber into $\mathcal{V}_n^\circ(V;\{V_i\}_i;\{k_i\}_i)$ is given by

$$\begin{aligned}\eta_{[U]} : \mathcal{V}_{k'_m}(V_m/\gamma([U]);\{(\gamma([U])+V_i)/\gamma([U])\}_{i<m};\{0\}_{i<m}) &\to \mathcal{V}_n^\circ(V;\{V_i\}_i;\{k_i\}_i)\\ W &\mapsto W + \sum_{i\ne m} U \cap V_i\end{aligned}$$

These bundles fit into a commutative diagram mapping fibers injectively to fibers:

$$\begin{tikzcd}
\mathcal{V}_n^\circ(V;\{V_i\}_i;\{k_i\}_i) \arrow[r] \arrow[d] & E \arrow[d] \\
\mathcal{V}_n^\circ(V;\{V_i\}_{i\ne m};\{k_i\}_{i\ne m}) \arrow[r, "\gamma"] & \mathrm{Gr}_{k_m-k'_m}(V_m)
\end{tikzcd}$$

Proof. We see that γ is well-defined since $k'_m = \dim V_m / \sum_{i<m} U \cap V_i$ and $\dim V_m = k_m$. The fiber bundle E is from taking the tautological vector bundle W' for $\mathrm{Gr}_{k_m - k'_m}(V_m)$, forming the vector bundle V_m/W', and then taking $\mathrm{Gr}_{k'_m}$ of the fibers.

Next we show $\eta_{[U]}$ is well-defined. Given W in the domain of $\eta_{[U]}$, then by choosing some splitting $V_m \approx \gamma([U]) \oplus V_m/\gamma([U])$ we may lift W to being a k'_m-dimensional subspace of V_m whose intersection with $\gamma([U])$ is trivial. Hence, $W + \gamma([U])$ is a k_m-dimensional subspace, and since the difference in any two sections of $V_m \twoheadrightarrow V_m/\gamma([U])$ has its image lying in U_m, the choice of section does not change the resulting subspace. Thus adding $\sum_{i \neq m} U \cap V_i$ to W yields a well-defined subspace. The subspace is not n-dimensional, however, but $\eta_{[U]} \cap \sum_i V_i = \eta_{[U]}$ is of the correct dimension if it satisfies all the subspace dimension constraints, which we now check.

We see that $\eta_{[U]}(W) \cap \gamma([U]) = \gamma([U])$ for all W. Thus, for $i < m$, since $\gamma([U]) \cap U \cap V_i = U \cap V_i$, we have $\eta_{[U]}(W) \cap U \cap V_j = U \cap V_j$. From this we get the inequality

$$\dim(\eta_{[U]}(W) \cap V_j) \geq \dim(\eta_{[U]}(W) \cap U \cap V_j) = \dim(U \cap V_j) = k_j.$$

The function $W \mapsto \dim(\eta_{[U]}(W) \cap V_j)$ is upper semi-continuous, and thus the set of those $W \in \mathrm{Gr}_{k'_m}(V_m/\gamma([U]))$ for which this inequality is an equality for all k_j is open. Supposing these inequalities are equalities, then we see

$$W \cap (\gamma([U]) + V_i)/\gamma([U]) = ((W + \gamma([U])) \cap V_i)/\gamma([U]) = (\gamma([U]) + U \cap V_i)/\gamma([U]),$$

which is 0-dimensional, and thus W lies inside the domain of $\eta_{[\eta]}$. For injectivity, note that we can recover W by intersecting with V_m and quotienting by $\sum_{i \neq m} V_i$. □

Corollary 1.5.7. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections, let $n \in \mathbb{N}$, and for all $i \in I$ let $k_i \in \mathbb{N}$. Then $\mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i)$ is a smooth submanifold. In particular, $\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$ is a smooth manifold.*

Proof. If I is the empty set, then this is $\mathrm{Gr}_n(\{0\})$, which is a smooth manifold (though potentially empty). Now we proceed by induction on the cardinality of I. Let $m \in I$ be maximal. By the lemma, we may regard $\mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i)$ as being an open subspace of the pullback of a smooth fiber bundle over a Grassmannian. By induction, γ is a smooth map, and so the pullback inherits a smooth structure. □

Corollary 1.5.8. *Assume the same hypotheses as the lemma, with $m \in I$ maximal. Let $[U] \in \mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i)$. The fiber bundle from the previous lemma yields the following short exact sequence:*

$$\begin{aligned} 0 \to \mathrm{Hom}&\left(\frac{U \cap V_m}{\sum_{i<m} U \cap V_i}, \frac{U + V_m}{U}\right) \\ &\to T_{[U]}\mathcal{V}_n^\circ(V; \{V_i\}_i; \{k_i\}_i) \\ &\to T_{[U]}\mathcal{V}_n^\circ(V; \{V_i\}_{i \neq m}; \{k_i\}_{i \neq m}) \to 0 \end{aligned}$$

Proof. A representation of $[U]$ in the fiber is $U' = (U \cap V_m)/\gamma([U])$, and

$$\frac{V_m/\gamma([U])}{(U \cap V_m)/\gamma([U])} \cong \frac{V_m}{U \cap V_m} \cong \frac{U + V_m}{U},$$

which we can use to calculate the tangent space for the fiber:

$$\begin{aligned} &T_{U'}\mathcal{V}_{k'_m}(V_m/\gamma([U]); \{(\gamma([U]) + V_i)/\gamma([U])\}_{i<m}; \{0\}_{i<m}) \\ &\quad = T_{U'} \operatorname{Gr}_{k'_m}(V_m/\gamma([U])) \\ &\quad \cong \operatorname{Hom}(U', (V_m/\gamma([U]))/U') \\ &\quad \cong \operatorname{Hom}(U', (U + V_m)/U) \end{aligned}$$

With this we can write the short exact sequence for the tangent spaces in the desired way. □

Corollary 1.5.9. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections and includes V, let $n \in \mathbb{N}$, and for all $i \in I$ let $k_i \in \mathbb{N}$. For $U \in \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$, recursive application of the previous lemma yields the following isomorphism:*

$$T_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \cong \bigoplus_{i \in I} \operatorname{Hom}\left(\frac{U \cap V_i}{\sum_{j<i} U \cap V_j}, \frac{U + V_i}{U}\right).$$

Furthermore, the dimension of the tangent space is

$$\dim T_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) = \sum_{i \in I} k'_i(\dim V_i - k_i).$$

Example 1.5.10. Suppose we have four subspaces of V with the following Hasse diagram:

$$\begin{tikzcd}[column sep=small] & V \arrow[d, no head] & \\ & V_4 \arrow[dl, no head] \arrow[dr, no head] & \\ V_2 \arrow[dr, no head] & & V_3 \arrow[dl, no head] \\ & V_1 & \end{tikzcd}$$

We can use the corollary to give the tangent space of $\mathcal{V}_n(V;\{V_i\}_i;\{k_i\}_i)$ at U as

$$\begin{gathered}\operatorname{Hom}\left(\frac{U}{U\cap V_4},\frac{V}{U}\right)\\ \oplus\operatorname{Hom}\left(\frac{U\cap V_4}{U\cap V_2+U\cap V_3},\frac{U+V_4}{U}\right)\\ \oplus\operatorname{Hom}\left(\frac{U\cap V_2}{U\cap V_1},\frac{U+V_2}{U}\right)\oplus\operatorname{Hom}\left(\frac{U\cap V_3}{U\cap V_1},\frac{U+V_3}{U}\right)\\ \oplus\operatorname{Hom}\left(U\cap V_1,\frac{U+V_1}{U}\right)\end{gathered}$$

Notice that the codomains are all ordered by inclusion in the same way the subspaces are. Furthermore, the direct sum of the domains gives a decomposition of U. The dimension of the tangent space at U is

$$k_1'(\dim V_1-k_1)+k_2'(\dim V_2-k_2)+k_3'(\dim V_3-k_3)+k_4'(\dim V_4-k_4)+n'(\dim V-n)$$

where

$$k_1'=k_1\qquad k_2'=k_2-k_1\qquad k_3'=k_3-k_1\qquad k_4'=k_4-k_2-k_3+k_1\qquad n'=n-k_1$$

which are obtained by inclusion-exclusion. $\Diamond$

Lemma 1.5.11. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections, let $n\in\mathbb{N}$, and for all $i\in I$ let $k_i\in\mathbb{N}$. For $U\in\mathcal{V}_n(V;\{V_i\}_i;\{k_i\}_i)$, then we have a canonical isomorphism*

$$T_U\mathcal{V}_n(V;\{V_i\}_i;\{k_i\}_i)\cong\bigcap_{i\in I}\{f:\operatorname{Hom}(U,V/U)\mid f(U\cap V_i)\subseteq\tfrac{U+V_i}{U}\}$$

and it commutes with the canonical isomorphism $T_U\operatorname{Gr}_n(V)\cong\operatorname{Hom}(U,V/U)$ by the natural inclusions.

Proof. Since it is a submanifold, we have the inclusion $T_U\mathcal{V}_n(V;\{V_i\}_i;\{k_i\}_i)\subseteq T_U\operatorname{Gr}_n(V)$, which we compose with the isomorphism $T_U\operatorname{Gr}_n(V)\cong\operatorname{Hom}(U,V/U)$. Now that we have established that $\mathcal{V}_n(V;\{V_i\}_i;\{k_i\}_i)$ is a submanifold, we can determine what the tangent space is inside $\operatorname{Hom}(U,V/U)$ more directly.

Let $i\in I$ and consider $\mathcal{V}_n(V;V_i;k_i)$. This has a map to $\operatorname{Gr}_{k_i}(V_i)$ by $U\mapsto U\cap V_i$ and induces a map

$$T_U\mathcal{V}_n(V;V_i;k_i)\to T_{U\cap V_i}\operatorname{Gr}_{k_i}(V_i)\cong\operatorname{Hom}(U\cap V_i,\tfrac{V_i}{U\cap V_i}).$$

The preimage of this hom set in $\operatorname{Hom}(U,V/U)$ is the set of all f such that $f(U\cap V_i)\subseteq\frac{U+V_i}{U}$. Since $\mathcal{V}_n(V;V_i;k_i)$ is precisely those elements of $\operatorname{Gr}_n(V)$ that can restrict to $\operatorname{Gr}_{k_i}(V_i)$, this characterizes the tangent space.

By intersecting these conditions for all $i\in I$, we get the desired result. $\square$

Remark 1.5.12. If we do not require that the set of subspaces be closed under intersection, then, if it is locally a submanifold at all, $\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$ might not have constant dimension. For example, consider the following subspaces of $\mathbb{R}^6$ with each $e_i \in \mathbb{R}^6$ a standard basis vector:

$$V_1 = \langle e_1, e_2, e_4, e_5 \rangle \qquad\qquad V_2 = \langle e_2, e_3, e_4, e_5 \rangle$$

The subspaces $U = \langle e_1, e_2, e_3 \rangle$ and $U' = \langle e_4, e_5, e_6 \rangle$ are in $\mathcal{V}_3(V; V_1, V_2; 2, 2)$, and using the characterization of the tangent space from Lemma 1.5.11, whose proof computes the tangent space directly as an intersection of tangent spaces, one can calculate that the dimension at U would be 6 and the dimension at U' would be 5, with the difference being that $V_1 \cap V_2 \cap U = \langle e_2 \rangle$ and $V_1 \cap V_2 \cap U' = \langle e_4, e_5 \rangle$ have different dimensions. What we can say, however, is that we have a stratification

$$\mathcal{V}_3(V; V_1, V_2; 2, 2) = \bigcup_{k \in \mathbb{N}} \mathcal{V}_3(V; V_1, V_2, V_1 \cap V_2; 2, 2, k)$$

by submanifolds. These submanifolds are nonempty exactly when $k \in \{1, 2\}$, and using Corollary 1.5.9 we calculate that their dimensions are given by $5 + 2k - k^2$. A warning: the dimension formula is positive for all $0 \leq k \leq 3$, however $k = 0$ and $k = 3$ are non-viable since these give negative-dimensional direct summands. ∎

Now that we have the Lemma 1.5.11 formulation of the tangent bundle of the space $\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$, we can work out a useful description of the normal bundle. Assume the hypotheses of that lemma, that $\{V_i\}_i$ is closed under intersections, but also suppose V is included among the subspaces. Regarding I as a poset category, the indexed family may be regarded as a functor $V_\bullet : I \to \mathsf{Vect}$, sending i to V_i and $i \leq j$ to $V_i \hookrightarrow V_j$. Given $U \in \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$, there are also functors $U \cap V_\bullet : I \to \mathsf{Vect}$ and $(U + V_\bullet)/U : I \to \mathsf{Vect}$ sending i respectively to $U \cap V_i$ and $(U + V_i)/U$. An element $f \in T_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$ of the tangent space describes a natural transformation from $U \cap V_\bullet$ to $(U + V_\bullet)/U$, where f_i is the restriction of f to $\mathrm{Hom}(U \cap V_i, (U + V_i)/U)$. Naturality is that for $i \leq j$ there is a commutative diagram:

$$\begin{tikzcd} U \cap V_i \arrow[r, hook] \arrow[d, "f_i"] & U \cap V_j \arrow[d, "f_j"] \\ \frac{U+V_i}{U} \arrow[r, hook] & \frac{U+V_j}{U} \end{tikzcd}$$

We summarize this in the following lemma.

Lemma 1.5.13. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections and contains V, let $n \in \mathbb{N}$, and for all $i \in I$ let $k_i \in \mathbb{N}$. Then for $U \in \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$, the tangent space at U is*

$$T_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \cong \mathrm{Hom}(U \cap V_\bullet, (U + V_\bullet)/U)$$

where $V_\bullet$ is the indexed family as a functor $I \to \mathsf{Vect}$ and the homomorphism set is in the category of natural transformations.[7]

Proof. As discussed, every tangent vector gives a natural transformation like this using the respresentation of the tangent space from Lemma 1.5.11. Conversely, given a natural transformation $\eta : U \cap V_\bullet \to (U + V_\bullet)/U$, since V is one of the subspaces, say V_m, then η_m is a map $U \to V/U$ that we can take to be the tangent vector. We see η_i agrees with the natural transformation for η_m since $V_i \subseteq V$ and so there is a commutative diagram from naturality that implies η_i is determined by η_m. □

Lemma 1.5.14. *Suppose $\{V_i\}_i$ is a finite I-indexed family of distinct subspaces that is closed under intersections and contains V, let $n \in \mathbb{N}$, and for all $i \in I$ let $k_i \in \mathbb{N}$. Then for $U \in \mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$, the normal space in $\operatorname{Gr}_U(V)$ at U is*

$$N_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \cong \operatorname{Hom}\left(U \cap V_\bullet, \frac{V}{U + V_\bullet}\right)$$

where $V_\bullet$ is the indexed family as a functor $I \to \mathsf{Vect}$ and the homomorphism set is in the category of natural transformations.

Proof. The normal space is defined by the short exact sequence

$$0 \to T_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \to T_U \operatorname{Gr}_U(V) \to N_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \to 0$$

and substituting the spaces we know, this short exact sequence is isomorphic to

$$0 \to \operatorname{Hom}(U \cap V_\bullet, (U + V_\bullet)/U) \to \operatorname{Hom}(U, V/U) \to N_U\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i) \to 0$$

where the first map sends a natural transformation to its map on V. For $f \in \operatorname{Hom}(U, V/U)$ and $i \in I$, by restricting and projecting we get a map $f_i : U \cap V_i \to V/(U + V_i)$. This extends to a map $\operatorname{Hom}(U, V/U) \to \operatorname{Hom}(U \cap V_\bullet, V/(U + V_\bullet))$ since for $i \leq j$, we have commutative diagrams

$$\begin{tikzcd} U \cap V_i \arrow[r, hook] \arrow[d, "f_i"] & U \cap V_j \arrow[d, "f_j"] \\ \frac{V}{U+V_i} \arrow[r, two heads] & \frac{V}{U+V_j} \end{tikzcd}$$

(Note that $V/(U + V_\bullet) : I \to \mathsf{Vect}$ sends morphisms to surjections rather than injections.) We claim this map is surjective by the following algorithm. Suppose we have a natural transformation $f : U \cap V_\bullet \to V/(U + V_\bullet)$ with projections f_i for all $i \in I$. We inductively define a linear map $f_S : \sum_{i \in S} U \cap V_i \to V/U$ on downward sets $S \subseteq I$. Suppose $j \in I$ is a minimal element not in the downward set S. Consider $U \cap V_j \cap \sum_{i \in S} U \cap V_i$, which is

[7] We use "Hom" rather than "Nat" to underline that these are linear maps, albeit ones that preserve additional internal structure.

the subspace of $U \cap V_j$ in the domain of definition of S. Choose a complementary subspace $V_j' \subseteq U \cap V_j$ to this. Restricting f_j to $V_j' \to V/(U+V_j)$, choose an arbitrary lift of it to $f_j' : V_j' \to V/U$. We can add f_j' to f_S using a splitting of $U \cap V_j$ to get the map $f_{S \cup \{j\}}$.

We claim that this map fits into a short exact sequence:

$$0 \to \operatorname{Hom}(U \cap V_\bullet, (U+V_\bullet)/U) \to \operatorname{Hom}(U, V/U) \to \operatorname{Hom}(U \cap V_\bullet, V/(U+V_\bullet)) \to 0$$

For $f \in \operatorname{Hom}(U \cap V_\bullet, (U+V_\bullet)/U)$, the image in $\operatorname{Hom}(U \cap V_\bullet, V/(U+V_\bullet))$ at $i \in I$ is $U \cap V_i \hookrightarrow U \xrightarrow{f} V/U \twoheadrightarrow V/(U+V_i)$, and since f restricted to $U \cap V_i$ is f_i, we know its image lies in $(U+V_i)/U$, which is being quotiented out. Therefore the image is 0. Next, for $f \in \operatorname{Hom}(U, V/U)$ mapping to 0, this means for all i the restriction to $U \cap V_\bullet$ has an image that, when quotiented by $U + V_\bullet$, is 0, hence $f(U \cap V_\bullet) \subseteq (U+V_\bullet)/U$. Therefore f lies in the image of $\operatorname{Hom}(U \cap V_\bullet, (U+V_\bullet)/U)$.

Since this is a short exact sequence, we obtain the identification of the normal bundle with this set of homomorphisms of natural transformations. □

We now return to the setup for Equation (1.5.1). Let $\Gamma : \operatorname{Hom}(V, W) \to \operatorname{Gr}_{\dim V}(V \times W)$ be the inverse coordinate chart centered at $V \times 0 \in \operatorname{Gr}_{\dim V}(V \times W)$ using the canonical splitting $V \times W = V \times 0 \oplus 0 \times W$; the graph of $a \in \operatorname{Hom}(V, W)$ is given by $\Gamma(a) = \{(v, av) \mid v \in V\}$.

Theorem 1.5.15. *Let I be a finite index set, and for each $i \in I$ let $V_i \subseteq V$ and $W_i \subseteq W$ be subspaces and let $r_i \in \mathbb{N}$. Suppose the family $\{V_i \times W_i\}_i$ consists of distinct subspaces, is closed under intersections, and contains $V \times W$. Let $\operatorname{Hom}(V, W; \{r_i\}_i) = \bigcap_{i \in I}\{a \in \operatorname{Hom}(V, W) \mid \operatorname{drk} a_i = r_i\}$ be as in Equation (1.5.1) with $a_i : V_i \to W/W_i$ the restriction and projection of a, and let $\Gamma : \operatorname{Hom}(V, W) \to \operatorname{Gr}_{\dim V}(V \times W)$ be as immediately above. Let $k_i = r_i + \max(0, \dim V_i + \dim W_i - \dim W)$ for all $i \in I$ (the kernel dimensions associated to the dropped ranks). Then*

$$\operatorname{Hom}(V, W; \{r_i\}_i) = \Gamma^{-1}(\mathcal{V}_{\dim V}(V \times W; \{V_i \times W_i\}_i, \{k_i\}_i),$$

and thus $\operatorname{Hom}(V, W; \{r_i\}_i)$ is a submanifold of $\operatorname{Hom}(V, W)$. For $a \in \operatorname{Hom}(V, W; \{r_i\}_i)$, we have the following natural isomorphism:

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \bigcap_{i \in I}\{f \in \operatorname{Hom}(V, W) \mid f(\ker a_i) \subseteq \operatorname{im} a_i + W_i\}$$

where $\operatorname{im} a_i + W_i$ means $q^{-1}(\operatorname{im} a_i)$ for the quotient map $q : W \twoheadrightarrow W/W_i$. Furthermore, we have the following natural isomorphisms:

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \operatorname{Hom}(\ker a_\bullet, \operatorname{im} a_\bullet + W_\bullet)$$
$$N_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \operatorname{Hom}(\ker a_\bullet, \operatorname{coker} a_\bullet)$$

Here, $a_\bullet \in \operatorname{Hom}(V_\bullet, W/W_\bullet)$ is the natural transformation associated to $\{a_i\}_i$ for a given a, and whose kernels, cokernels, and images define functors $I \to \mathsf{Vect}$.

Proof. For $a \in \operatorname{Hom}(V, W)$ and $i \in I$,

$$\begin{aligned}\Gamma(a) \cap (V_i \times W_i) &= \{(v, av) \mid v \in V_i \text{ and } av \in W_i\} \\ &= \{(v, av) \mid v \in \ker a_i\}.\end{aligned}$$

This is isomorphic to $\ker a_i$ via the projection onto the first component. Hence, $\dim \ker a_i = k_i$ if and only if $\dim(\Gamma(a) \cap (V_i \times W_i)) = k_i$.

For the tangent space, by Lemma 1.5.11 we have

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \bigcap_{i \in I} \{f \in \operatorname{Hom}(\Gamma(a), V \times W/\Gamma(a)) \mid f(\Gamma(a) \cap V_i \times W_i) \subseteq \tfrac{\Gamma(a)+V_i \times W_i}{\Gamma(a)}\}.$$

The projection $\pi_1 : \Gamma(a) \to V$ defined by $\pi_1(v, av) = v$ is an isomorphism, as is the map $\varphi : V \times W/\Gamma(a) \to W$ defined by $\varphi((v, w) + \Gamma(a)) = w - av$. Thus, we have an isomorphism $\operatorname{Hom}(V, W) \to \operatorname{Hom}(\Gamma(a), V \times W/\Gamma(a))$ defined by $f \mapsto \psi^{-1} \circ f \circ \varphi$ that we can use to get an isomorphism

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \bigcap_{i \in I} \{f \in \operatorname{Hom}(V, W) \mid f(\pi_1(\Gamma(a) \cap V_i \times W_i)) \subseteq \varphi(\tfrac{\Gamma(a)+V_i \times W_i}{\Gamma(a)})\}.$$

We know that $\pi_1(\Gamma(a) \cap V_i \times W_i) = \ker a_i$, and we can see that $\varphi((\Gamma(a) + V_i \times W_i)/\Gamma(a)) = \operatorname{im}(a|_{V_i}) + W_i$. Hence, the condition in each set is $f(\ker a_i) \subseteq \operatorname{im}(a|_{V_i}) + W_i$.

For the next isomorphism, by Lemma 1.5.13 we have

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \operatorname{Hom}(\Gamma(a) \cap (V_\bullet \times W_\bullet), (\Gamma(a) + V_\bullet \times W_\bullet)/\Gamma(a)),$$

and by the same maps above this is $\operatorname{Hom}(\ker a_\bullet, \operatorname{im} a_\bullet + W_\bullet)$. By Lemma 1.5.14,

$$N_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \operatorname{Hom}(\Gamma(a) \cap (V_\bullet \times W_\bullet), V \times W/(\Gamma(a) + V_\bullet \times W_\bullet)).$$

Since $\Gamma(a) \subseteq V \times W$ and

$$\Gamma(a) + V_i \times W_i = \Gamma(a) + V_i \times 0 + 0 \times W_i = \Gamma(a) + 0 \times \operatorname{im}(a|_{V_i}) + 0 \times W_i,$$

we have

$$\frac{V \times W}{\Gamma(a) + V_i \times W_i} \cong \frac{W}{\operatorname{im}(a|_{V_i}) + W_i} \cong \frac{W/W_i}{\operatorname{im}(a_i)} = \operatorname{coker}(a_i).$$

Therefore we have the identification $N_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \operatorname{Hom}(\ker a_\bullet, \operatorname{coker} a_\bullet)$. □

Remark 1.5.16. For convenience, here is an unpacked version of the natural isomorphism for the tangent space at $a \in \operatorname{Hom}(V, W; \{r_i\}_i)$:

$$T_a \operatorname{Hom}(V, W; \{r_i\}_i) \cong \bigcap_{i \in I} \{f \in \operatorname{Hom}(V, W) \mid f(V_i \cap a^{-1}(W_i)) \subseteq f(V_i) + W_i\}.$$

This natural isomorphism respects the inclusion $T_a \operatorname{Hom}(V, W; \{r_i\}_i) \subseteq T_a \operatorname{Hom}(V, W)$ and the natural isomorphism $T_a \operatorname{Hom}(V, W) = \operatorname{Hom}(V, W)$. ■

$$N_a H \cong \operatorname{Hom}\left(\begin{tikzcd} \ker a_{11} \arrow[r, hook] \arrow[d, hook] & \ker a_{12} \arrow[d, hook] \\ \ker a_{21} \arrow[r, hook] \arrow[d, hook] & \ker a_{22} \arrow[d, hook] \\ \ker a_{31} \arrow[r, hook] & \ker a_{32} \end{tikzcd} , \begin{tikzcd} \operatorname{coker} a_{11} \arrow[r, two heads] \arrow[d, two heads] & \operatorname{coker} a_{12} \arrow[d, two heads] \\ \operatorname{coker} a_{21} \arrow[r, two heads] \arrow[d, two heads] & \operatorname{coker} a_{22} \arrow[d, two heads] \\ \operatorname{coker} a_{31} \arrow[r, two heads] & \operatorname{coker} a_{32} \end{tikzcd} \right)$$

Figure 1.2: The normal space at the homomorphism described in Example 1.5.19

Example 1.5.17. The normal space at $a \in \operatorname{Hom}^r(V, W)$ is

$$N_a \operatorname{Hom}^r(V, W) \cong \operatorname{Hom}(\ker a, \operatorname{coker} a).$$

The dimension of this space is $(\dim V - \operatorname{rk} a)(\dim W - \operatorname{rk} a)$, which in terms of dropped rank is $(\dim V - \min(\dim V, \dim W) + \operatorname{drk} a)(\dim W - \min(\dim V, \dim W) + \operatorname{drk} a)$. Thus,

$$\dim N_a \operatorname{Hom}^r(V, W) = r(r + |\dim V - \dim W|). \qquad \Diamond$$

Example 1.5.18. Suppose $V_1 \subseteq V$ and $H = \{a \in \operatorname{Hom}(V, W) \mid \operatorname{drk}(a|_{V_1}) = r\}$. Then the normal space at $a \in H$ is

$$NH_a \cong \operatorname{Hom}(\ker(a|_{V_1}), \operatorname{coker}(a|_{V_1})).$$

The calculation of dimension in terms of r is essentially the same as in the previous example:

$$\dim NH_a = r(r + |\dim V_1 - \dim W|). \qquad \Diamond$$

Example 1.5.19. Suppose we have subspaces $V_{ij} \subseteq V$ and $W_{ij} \subseteq W$ with $1 \le i \le 3$ and $1 \le j \le 2$ that are bifiltered (if $i \le i'$ and $j \le j'$ then $V_{ij} \subseteq V_{i'j'}$ and $W_{ij} \subseteq W_{i'j'}$). Also suppose we have dropped ranks $r_{ij} \in \mathbb{N}$, and let $H = \operatorname{Hom}(V, W; \{r_{ij}\}_{ij})$. If $a \in H$, then the normal bundle of H at U can be represented as in Figure 1.2, where $a_{ij} : V_{ij} \to W/W_{ij}$ is the restriction and projection for a, and the homomorphism set is understood to be the set of natural transformations between the functors represented by these diagrams. $\Diamond$

Corollary 1.5.20. *Let I be a finite index set, for each $i \in I$ let $V_i \subseteq V$ and $W_i \subseteq W$ be subspaces and let $r_i \in \mathbb{N}$, and suppose the family $\{V_i \times W_i\}_i$ is closed under intersections. Let $H = \operatorname{Hom}(V, W; \{r_i\}_i)$. Consider the trivial bundle over H with fiber $\operatorname{Hom}(V_\bullet, W/W_\bullet)$, and let $A : H \to \operatorname{Hom}(V_\bullet, W/W_\bullet)$ be the section sending each $a \in H$ to its associated natural transformation. Then the normal bundle of H is naturally $NH \cong \operatorname{Hom}(\ker A, \operatorname{coker} A)$.*

Proof. We have $TH \subseteq T\operatorname{Hom}(V,W) \cong \operatorname{Hom}(V,W)$ naturally, and all isomorphisms from Theorem 1.5.15 were natural. In particular, the isomorphism $NH \cong \operatorname{Hom}(\ker A, \operatorname{coker} A)$ is from restricting and projecting a representative of a given vector in the normal bundle (a homomorphism) to its kernel and cokernel. □

1.5.2 The intrinsic derivative

We now resume our discussion of the intrinsic derivative. Suppose V and W are smooth vector bundles over a smooth manifold M and $f : M \to \operatorname{Hom}(V,W)$ is a smooth section. Suppose that $f_p \in \operatorname{Hom}^r(V_p, W_p)$ at $p \in M$, where V_p and W_p denote the fibers at p. The *intrinsic derivative* Df_p of Porteous [Por71] is the following composition of maps:

$$T_pM \xrightarrow{df_p} T_{f_p}\operatorname{Hom}(V,W) \twoheadrightarrow N_{f_p}\operatorname{Hom}^r(V,W) \cong \operatorname{Hom}(\ker f_p, \operatorname{coker} f_p).$$

The map Df_p depends only on the 1-jet $j^1 f_P$, and so the intrinsic deriative of a germ $[f]_p$ at p is well-defined.

Remark 1.5.21. Recall that if $\pi : \operatorname{Hom}(V,W) \to M$ is the bundle projection, then there is an induced surjective map of vector bundles $d\pi : T\operatorname{Hom}(V,W) \to TM$ whose kernel is the *vertical tangent bundle*, which is canonically isomorphic to $\pi^*\operatorname{Hom}(V,W)$. This fits into a short exact sequence of vector bundles over $\operatorname{Hom}(V,W)$:

$$0 \to \pi^*\operatorname{Hom}(V,W) \to T\operatorname{Hom}(V,W) \to \pi^*TM \to 0$$

The short exact sequence splits, but not canonically (splittings are equivalently connections on π). If $f : M \to \operatorname{Hom}(V,W)$ is a smooth section such that $f_p \in \operatorname{Hom}^r(V_p, W_p)$, then for $v \in T_pM$ we have $df_p(v) \in T_{f_p}(V,W) \approx T_pM \oplus \operatorname{Hom}(V_p,W_p)$, and, conceptually, the intrinsic derivative is extracting the portion of the $\operatorname{Hom}(V_p,W_p)$ component of $df_p(v)$ that is independent of the choice of splitting. ■

Remark 1.5.22. This is explained in [Boa67] in the following way. If $p \in M$ and $w \in W_p$ is in the zero section, then one has a canonical splitting of the sequence

$$0 \to W_w \to T_wW \to T_pM \to 0$$

Furthermore, if $s : M \to W$ is a section with s_p in the zero section, then the differential ds_p yields a linear map $T_pM \to W_p$ by using the splitting. Hence, whenever $\chi : M \to V$ is a section such that $f_p \circ \chi_p = 0$, then from the section $f \circ \chi$ we obtain a linear map $T_pM \to W_p$. If $\chi_p = 0$, then we can write $\chi = \sum_i \alpha_i\rho_i$ with each $\alpha_i : M \to \mathbb{R}$ a smooth function that vanishes at p and each $\rho_i : M \to V$ any section, and we can calculate that the $T_pM \to W_p$ map associated to $f \circ \alpha_i\rho_i$ is $v \mapsto (d\alpha_i)_p(v)(f \circ \rho_i)_p$, whose image lies in $\operatorname{coker} f_p$. This means that we can define a linear map $\ker f_p \to \operatorname{Hom}(T_pM, \operatorname{coker} f_p)$ by extending a kernel vector to a section χ, obtaining the map associated to $f \circ \chi$, and then composing it with the quotient $W_p \twoheadrightarrow \operatorname{coker} f_p$ to remove the dependence on the choice of χ. ■

Remark 1.5.23. Here is a description of how Df_p is computed in a trivialization. Supposing $V = M \times \mathbb{R}^k$ and $W = M \times \mathbb{R}^\ell$, then $T_{f_p} \operatorname{Hom}(V, W) = T_pM \oplus \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^\ell)$ and $T_{f_p} \operatorname{Hom}^r(V, W) = T_pM \oplus T_p \operatorname{Hom}^r(\mathbb{R}^k, \mathbb{R}^\ell)$. The quotient of these is $N_{f_p} \operatorname{Hom}^r(\mathbb{R}^k, \mathbb{R}^\ell)$, and as in Theorem 1.5.15 (and in particular Example 1.5.17), the normal bundle can be identified with the subspace $\operatorname{Hom}(\ker f_p, \operatorname{coker} f_p)$. For $v \in T_pM$, the intrinsic derivative $Df_p(v) \in \operatorname{Hom}(\ker f_p, \operatorname{coker} f_p)$ is the result of restricting $df_p(v)$ to $\ker f_p$ and composing with $W_p \twoheadrightarrow \operatorname{coker} f_p$. That is to say, after choosing splittings $V = \ker f_p \oplus \operatorname{coim} f_p$ and $W = \operatorname{im} f_p \oplus \operatorname{coker} f_p$, the block matrix for $df_p(v)$ has the following form:

$$\begin{array}{cc} & \begin{array}{cc} \ker f_p & \operatorname{coim} f_p \end{array} \\ \begin{array}{r} \operatorname{im} f_p \\ \operatorname{coker} f_p \end{array} & \begin{bmatrix} * & * \\ Df_p(v) & * \end{bmatrix} \end{array}$$

$\blacksquare$

Remark 1.5.24. The intrinsic derivative is merely a linear map defined over a single point of M and not immediately a section of a vector bundle. If $M' = f^{-1}(\operatorname{Hom}^r(V, W))$ is a submanifold of M, then the intrinsic derivative may be regarded as a section of a vector bundle over M' in the following way. Let K be the vector bundle over M' of kernels of f and L the vector bundle over M' of cokernels of f (these form valid vector bundles because f has constant rank on M'). Then the intrinsic derivatives assemble to form a section

$$Df : M' \to \operatorname{Hom}(TM|_{M'}, \operatorname{Hom}(K, L))$$

This means that in the ideal case when $f \pitchfork \operatorname{Hom}^r(V, W)$ on M, the intrinsic derivative forms a section over the submanifold $f^{-1}(\operatorname{Hom}^r(V, W))$. The following Lemma 1.5.25 gives a necessary and sufficient condition for this to occur. $\blacksquare$

Lemma 1.5.25 ([GG73, VI.3.7])**.** *Suppose V and W are smooth vector bundles over a smooth manifold M, $p \in M$, and $f : M \to \operatorname{Hom}(V, W)$ is a smooth section such that $f_p \in \operatorname{Hom}^r(V_p, W_p)$. The following are equivalent:*

1. *$Df_p : T_pM \to \operatorname{Hom}(\ker f_p, \operatorname{coker} f_p)$ is surjective.*
2. *$f \pitchfork \operatorname{Hom}^r(V, W)$ at p.*

Proof. The normal bundle to $\operatorname{Hom}^r(V, W)$ at f_p is isomorphic to the normal bundle to $\operatorname{Hom}^r(V_p, W_p)$ at f_p, so the intrinsic derivative may be regarded as taking values in the normal bundle to $\operatorname{Hom}^r(V, W)$ at f_p, and $f \pitchfork \operatorname{Hom}^r(V, W)$ at p is equivalently stated as that $T_pM \xrightarrow{df_p} T_{f_p} \operatorname{Hom}(V, W) \twoheadrightarrow N_{f_p} \operatorname{Hom}^r(V, W)$ is surjective. $\square$

All of this generalizes to the situation from the previous section. Suppose $\{V_i\}_i$ and $\{W_i\}_i$ are I-indexed families of subbundles of vector bundles V and W, respectively, such that $\{V_i \times W_i\}_i$ is closed under intersections and contains $V \times W$, and suppose $r_i \in \mathbb{N}$ for all $i \in I$. If $\operatorname{drk}(f_p) = r_i$ for all $i \in I$ (that is, if we have $f_p \in \operatorname{Hom}(V_p, W_p; \{r_i\})$, then we define an intrinsic derivative Df_p by the composition

$$T_p M \xrightarrow{df_p} T_{f_p} \operatorname{Hom}(V, W; \{r_i\}) \twoheadrightarrow N_{f_p} \operatorname{Hom}(V, W; \{r_i\}) \cong \operatorname{Hom}(\ker(f_p)_\bullet, \operatorname{coker}(f_p)_\bullet)$$

using the isomorphism from Theorem 1.5.15, where by $(f_p)_\bullet$ we mean the natural transformation in $\operatorname{Hom}((V_p)_\bullet, W_p/(W_p)_\bullet)$ associated to f_p. It has the same sort of description for how it is computed in a trivialization, though coming up with a matrix form involves finding bases that are compatible with each of the subspaces in the families.

Lemma 1.5.26. *Let $H = \operatorname{Hom}(V, W; \{r_i\}_i)$ using the preceding hypotheses, which is a smooth fiber bundle over M. Suppose $p \in M$ and that $f : M \to \operatorname{Hom}(V, W)$ is a smooth section such that $f_p \in H_p$. The following are equivalent:*

1. *$Df_p : T_p M \to \operatorname{Hom}(\ker(f_p)_\bullet, \operatorname{coker}(f_p)_\bullet)$ is surjective.*

2. *$f \pitchfork H$ at p.*

If $f \pitchfork H$, then the intrinsic derivative forms a section $Df : TM \to \operatorname{Hom}(\ker f_\bullet, \operatorname{coker} f_\bullet)$ over the submanifold $f^{-1}(H)$.

Remark 1.5.27. Since we can consider multiple different families of subspaces, there is no longer "the" intrinsic derivative, and we will often work with multiple families simultaneously. This can give us multiple submanifolds of M to work with, which can yield additional subspaces that we can use for rank constraints. ∎

1.5.3 Symmetric products and more submanifolds of linear maps

The definition of the singularity types involves taking iterated intrinsic derivatives of the differential. In coordinates, the kth intrinsic derivative is some restriction and projection of the kth iterated Jacobian of a smooth map that can be given in an invariant way, where the *kth iterated Jacobian* of a smooth function $f : U \to \mathbb{R}^n$ with $U \subseteq \mathbb{R}^m$ open is defined to be, using $(x_1, \ldots, x_m)$ for the standard coordinates for $\mathbb{R}^m$ and $(y_1, \ldots, y_n)$ for $\mathbb{R}^n$,

$$\begin{aligned} J^k f &: U \to \operatorname{Hom}((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n) \\ &x \mapsto v_1 \otimes \cdots \otimes v_k \mapsto \left[\sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} v_{1j_1} \cdots v_{kj_k} \frac{\partial^k f^* y_i}{\partial x_{j_1} \cdots \partial x_{j_k}}(x) \right]_{i=1}^{n} \end{aligned}$$

The *Jacobian* $Jf = J^1 f$ is $df : U \to \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ in coordinates, and the second iterated Jacobian $J^2 f$ is the *Hessian* $Hf : U \to \operatorname{Hom}(\mathbb{R}^m \otimes \mathbb{R}^m, \mathbb{R}^n)$. These are called iterated Jacobians since the differential of $J^k f$ is a section

$$d(J^k f) : U \to \operatorname{Hom}(TU, T \operatorname{Hom}((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n))$$

and this is induced by a map $U \to \mathrm{Hom}(\mathbb{R}^m, \mathrm{Hom}((\mathbb{R}^m)^{\otimes k}, \mathbb{R}^n))$ since the tangent bundles are trivial. Applying the tensor-hom adjunction to this yields $J^{k+1}f$.

In general, $J^k f_p \in \mathrm{Hom}((\mathbb{R}^n)^{\otimes k}, \mathbb{R}^m)$ is a symmetric function due to the symmetry of taking partial derivatives. We take a diversion through the generalization of symmetric products introduced by [Boa67, Por71] to handle symmetries in the higher intrinsic derivatives.

Definition 1.5.28 ([Boa67])**.** Let V be a vector space and $n \in \mathbb{N}$. The *nth symmetric power* $V^{\odot n}$ of V is the quotient of the nth tensor power $V^{\otimes n}$ by the permutation action of the symmetric group S_n, where $\sigma \in S_n$ has the right action $(x_1 \otimes \cdots \otimes x_n) \cdot \sigma = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$.

For subspaces $W_1, \ldots, W_n \subseteq V$, the *symmetric product* $W_1 \odot \cdots \odot W_n$ is the image of the composition $W_1 \otimes \cdots \otimes W_n \hookrightarrow V^{\otimes n} \twoheadrightarrow V^{\odot n}$. $\diamondsuit$

Remark 1.5.29. The iterated Jacobians define functions $J^k f : U \to \mathrm{Hom}((\mathbb{R}^m)^{\odot k}, \mathbb{R}^n)$. It should be pointed out that the function

$$C_p^\infty(\mathbb{R}^m, \mathbb{R}^n) \to \prod_{k \in \mathbb{N}} \mathrm{Hom}((\mathbb{R}^m)^{\odot k}, \mathbb{R}^n)$$

defined by $[f]_p \mapsto (J^k f_p)_{k \in \mathbb{N}}$ is surjective, since this is giving the Taylor series of the germ with respect to the chosen coordinates (see Remark 1.1.10). In particular, this product is non-canonically isomorphic to the infinite jet space $J_p(\mathbb{R}^m, \mathbb{R}^n)$. The grading induced by the product is not preserved by changes of coordinates, with the change in $J^k f_p$ depending in some complicated way on $J^i f_p$ for all $i \leq k$. Instead, there is a filtration on the infinite product, where subspace $i \in \mathbb{N}$ is given by those element $a \in \prod_k \mathrm{Hom}((\mathbb{R}^m)^{\odot k}, \mathbb{R}^n)$ for which $a_j = 0$ for all $1 \leq j < i$. From the point of view of the infinite jet space, for each $i \in \mathbb{N}$ there is a projection $J_p(\mathbb{R}^m, \mathbb{R}^n) \to J_p^i(\mathbb{R}^m, \mathbb{R}^n)$, and the preimage of the set of jets of constant functions gives subspace i of the filtration. ■

Lemma 1.5.30. *Suppose $W_1, \ldots, W_n \subseteq V$ are subspaces of a vector space V, and let U be a vector space. The map $W_1 \otimes \cdots \otimes W_n \twoheadrightarrow W_1 \odot \cdots \odot W_n$ induces an injection*

$$\mathrm{Hom}(W_1 \odot \cdots \odot W_n, U) \hookrightarrow \mathrm{Hom}(W_1 \otimes \cdots \otimes W_n, U).$$

If f is in the image, then for all $w_1 \otimes \cdots \otimes w_n \in W_1 \otimes \cdots \otimes W_n$ and $\sigma \in S_n$ such that $w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \in W_1 \otimes \cdots \otimes W_n$ then

$$f(w_1 \otimes \cdots \otimes w_n) = f(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}).$$

In particular, f is in the image if and only if there exists an $\overline{f} : V^{\odot n} \to U$ such that the following diagram commutes:

$$\begin{tikzcd} V^{\otimes n} \arrow[r, two heads] & V^{\odot n} \arrow[d, "\overline{f}"] \\ W_1 \otimes \cdots \otimes W_n \arrow[u, hook] \arrow[r, "f"] & U \end{tikzcd}$$

Proof. By exactness, we have the following commutative diagram of injections and surjections:

$$\begin{tikzcd}
\mathrm{Hom}(V^{\otimes n}, U) \arrow[d, two heads] & \mathrm{Hom}(V^{\odot n}, U) \arrow[l, hook'] \arrow[d, two heads] \\
\mathrm{Hom}(W_1 \otimes \cdots \otimes W_n, U) & \mathrm{Hom}(W_1 \odot \cdots \odot W_n, U) \arrow[l, hook']
\end{tikzcd}$$

In particular, the bottom arrow is an injection. If f is in the image of this map, then certainly tensor factors can be reordered in the described way since these have the same image in $V^{\odot n}$. Furthermore, the characterization of elements of the image in terms of the existence of an extension to $V^{\odot n}$ follows from surjectivity of $\mathrm{Hom}(V^{\odot n}, U) \twoheadrightarrow \mathrm{Hom}(W_1 \odot \cdots \odot W_n, U)$. □

Corollary 1.5.31. *Suppose $W_1, \ldots, W_n \subseteq V$ are subspaces such that there exists a basis for V for which each W_i is a span of some subset of the basis vectors. Letting U be a vector space, then $\mathrm{Hom}(W_1 \odot \cdots \odot W_n, U)$ can be identified with the set of linear maps $f : W_1 \otimes \cdots \otimes W_n \to U$ that satisisfy the property that for all $w_1 \otimes \cdots \otimes w_n \in W_1 \otimes \cdots \otimes W_n$ and $\sigma \in S_n$ such that $w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \in W_1 \otimes \cdots \otimes W_n$ then*

$$f(w_1 \otimes \cdots \otimes w_n) = f(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}).$$

Proof. By the previous lemma, it suffices to show that there is a symmetric function $\overline{f} : V^{\otimes n} \to U$ such that f is the restriction of $\overline{f}$. Let $\{e_j\}_j$ be a basis for V such that each W_i is a span of basis vectors. We define $\overline{f}$ on a basis by the following. For $e_{j_1} \otimes \cdots \otimes e_{j_n} \in V^{\otimes n}$ a simple tensor of basis vectors, if there is a permutation $\sigma \in S_n$ such that

$$e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_n)} \in W_1 \otimes \cdots \otimes W_n$$

then set $\overline{f}(e_{j_1} \odot \cdots \odot e_{j_n}) = f(e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_n)})$, and otherwise set $\overline{f}(e_{j_1} \otimes \cdots \otimes e_{j_n}) = 0$. This rule does not depend on the choice of σ because if $\sigma' \in S_n$ were another permutation with the same property, then using $\sigma'\sigma^{-1}$ in the hypothesis gives

$$f(e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_n)}) = f(e_{\sigma'(j_1)} \otimes \cdots \otimes e_{\sigma'(j_n)}).$$

We also see that $\overline{f}$ is symmetric since (1) for a simple tensor with a permutation that carries it into $W_1 \otimes \cdots \otimes W_n$, we just established that every permutation of that simple tensor yields the same value through $\overline{f}$, and (2) for simple tensors without such a permutation, no permutation of it has one either, so all permutations take the value 0 through $\overline{f}$. Lastly, $\overline{f}(w_1 \odot \cdots \odot w_n) = f(w_1 \otimes \cdots \otimes w_n)$ for all $w_1 \otimes \cdots \otimes w_n \in W_1 \otimes \cdots \otimes W_n$ since we may write this as a linear combination of tensors of basis elements, and this identity holds for such elements by construction. □

Remark 1.5.32. It is tempting to reduce this to the hypotheses that for all $w_1 \otimes \cdots \otimes w_n \in W_1 \otimes \cdots \otimes W_n$ and $1 \leq j \leq k \leq n$ such that $w_j, w_k \in W_j \cap W_k$, then

$$f(w_1 \otimes \cdots \otimes w_j \otimes \cdots \otimes w_k \otimes \cdots \otimes w_n) = f(w_1 \otimes \cdots \otimes w_k \otimes \cdots \otimes w_j \otimes \cdots \otimes w_n).$$

However, not every permutation can be factored into transpositions of this type. For example, consider $\mathbb{R}^3$ as the span of $\langle e_1, e_2, e_3\rangle$, and let

$$W_1 = \langle e_1, e_2\rangle \qquad W_2 = \langle e_2, e_3\rangle \qquad W_3 = \langle e_3, e_1\rangle$$

Then $e_1 \otimes e_2 \otimes e_3$ and $e_2 \otimes e_3 \otimes e_1$ are both in $W_1 \otimes W_2 \otimes W_3$ (and have the same image in $W_1 \odot W_2 \odot W_3$), and indeed their indices are related by a 3-cycle, but every transposition of indices yields a vector outside $W_1 \otimes W_2 \otimes W_3$. ■

Lemma 1.5.33. *Let $W_1, \ldots, W_n \subseteq V$ be subspaces of a vector space V, and let $W_1' \subseteq V$ be another subspace such that $W_1 \subseteq W_1'$. Then there is an induced injective map*

$$W_1 \odot \cdots \odot W_n \hookrightarrow W_1' \odot \cdots \odot W_n.$$

Proof. By definition of the symmetric product, $W_1 \odot \cdots \odot W_n \subseteq W_1' \odot \cdots \odot W_n$. □

1.5.4 The singularity types

Using what we have developed, we will now describe a generalization to the Thom–Boardman singularity types. Unlike the classical types, these types are more of a general framework; it is a suggestion for how to incorporate as many subspaces as you can get your hands on. We do not prove here that we can carry out a procedure to define jet submanifolds of all orders, partly because we do not need them in low dimensions, and partly because it is not yet clear what is the "correct" general procedure. We also do not give general formulae for normal bundles of higher-order types here, since the ones that arose for our classification of singularities were simple enough to compute as needed.

Suppose M and N are foliated manifolds (as in Section 1.3). By virtue of being foliated, they each have a corresponding family of integrable distributions. Let I and J be finite index sets, and for each $i \in I$ and $j \in J$ let $V_i \subseteq TM$ and $W_j \subseteq TN$ each be one of these integral distributions. We assume the families $\{V_i\}$ and $\{W_j\}$ are closed under intersections and contain no repetitions, but we do not assume they contain TM or TN. These define an $(I \times J)$-indexed family $\{V_i \times W_j\}$ that is closed under intersections.

Recall from Remark 1.1.8 the isomorphism $J^1(M, N) \cong \mathrm{Hom}(TM, TN)$ as bundles over $M \times N$. For each $(I \times J)$-indexed family $\{r_{ij}\}$ of natural numbers, we define the first-order types $S[\{r_{ij}\}] \subseteq J^1(M, N)$ by taking the preimage of the submanifold $\mathrm{Hom}(TM, TN; \{r_{ij}\})$ through this isomorphism:

$$\begin{tikzcd} J^1(M,N) \arrow[r, dash, "\cong"] & \mathrm{Hom}(TM,TN) \\ S[\{r_{ij}\}] \arrow[u, hook] \arrow[r, dash, "\cong"] & \mathrm{Hom}(TM,TN;\{r_{ij}\}) \arrow[u, hook] \end{tikzcd}$$

Definition 1.5.34. For $S \subseteq J^k(M, N)$ a submanifold and $f : M \to N$ a smooth map, we write $S(f)$ for the set of S-*type singularity points* of f, which consists of those $p \in M$ for which $j^k f_p \in S$. That is to say, $S(f) = (j^k f)^{-1}(S)$. ◊

Remark 1.5.35. Theorem 1.2.9 (the Thom Transversality Theorem) implies that the set of $f \in C^\infty(M,N)$ for which f is transverse to each $S[\{r_{ij}\}]$ is dense. Hence, for f in this dense subset the $S[\{r_{ij}\}](f)$ sets are submanifolds of M. ■

Now that we have the first-order types, we next want a way to determine whether a jet extension is transverse to the $S[\{r_{ij}\}]$ manifolds. For $f \in C^\infty(M,N)$ such that $j^1 f_p \in S[\{r_{ij}\}]$, then there is a *second intrinsic derivative*[8] $d^2 f_p := D(df)_p$

$$d^2 f_p : T_pM \to \operatorname{Hom}(\ker{(df_p)_\bullet}, \operatorname{coker}{(df_p)_\bullet})$$

that by Lemma 1.5.25 is surjective if and only if $j^1 f$ intersects $S[\{r_{ij}\}]$ transversely at p. The second intrinsic derivative $d^2 f_p$ depends only on the 2-jet $j^2 f_p$, in the sense that by Lemma 1.1.9 there is a smooth map $J^2(M,N) \to \operatorname{Hom}(TM, T\operatorname{Hom}(TM,TN))$ defined by $j^2 f_p \mapsto d(df)_p$. Hence, we can classify the 2-jets of generic maps according to their $S[\{r_{ij}\}]$ types.

We now work our way up to second-order types. Let $S[\{r_{ij}\}]' \subseteq J^2(M,N)$ be the preimage of $S[\{r_{ij}\}]$ through the projection $J^2(M,N) \to J^1(M,N)$. We define the following bundles over $S[\{r_{ij}\}]$:

- $K[\{r_{ij}\}]$ is such that over $j^1 f_p$ the fiber is the functor $\ker{(df)_\bullet} : I \times J \to \mathsf{Vect}$.
- $L[\{r_{ij}\}]$ is such that over $j^1 f_p$ the fiber is the functor $\operatorname{coker}{(df)_\bullet} : I \times J \to \mathsf{Vect}$.

The sense in which these are bundles is that these functors are describing an $I \times J$-indexed family of vector spaces, albeit with the additional structure of having smoothly varying morphisms between them. The second intrinsic derivatives of the elements of $S[\{r_{ij}\}]'$ assemble into a map of bundles over $S[\{r_{ij}\}]$:

$$\begin{tikzcd} S[\{r_{ij}\}]' \arrow[r, "d^2"] \arrow[d, two heads] & \operatorname{Hom}(TM, \operatorname{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}])) \arrow[d, two heads] \\ S[\{r_{ij}\}] \arrow[r, equal] & S[\{r_{ij}\}] \end{tikzcd}$$

The interpretation of $\operatorname{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}])$ is the vector bundle such that over $j^1 f_p$ the fiber is the vector space of natural transformations between the two functors.

The idea now is to use d^2 to further stratify $S[\{r_{ij}\}]'$ by rank conditions. A technical complication is that, while we have the machinery to show that various rank conditions define submanifolds of $\operatorname{Hom}(TM, \operatorname{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}]))$, the map d^2 is not in general a surjective submersion, so *a priori* we do not know whether we can take the preimage of these submanifolds through d^2 to further stratify $S[\{r_{ij}\}]'$.

By adjunction, we have the isomorphism

$$\operatorname{Hom}(TM, \operatorname{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}])) \cong \operatorname{Hom}(TM \otimes K[\{r_{ij}\}], L[\{r_{ij}\}]).$$

[8] We will try to consistently call this the "second intrinsic derivative" rather than "intrinsic derivative."

Thus, for $j^2 f_p \in S[\{r_{ij}\}]'$, the $(i,j) \in I \times J$ projection of $d^2 f_p$ may be seen as an element of

$$\mathrm{Hom}(T_pM \otimes \ker(V_i \xrightarrow{df_p} T_{f(p)}N/W_j), \mathrm{coker}(V_i \xrightarrow{df_p} T_{f(p)}N/W_j)).$$

This element may be regarded as a restriction and projection of the iterated Jacobian $J^2 f_p$, so it is symmetric. Using the symmetric product from Section 1.5.3, we may view the $(i,j) \in I \times J$ projection of $d^2 f_p$ as an element of

$$\mathrm{Hom}(T_pM \odot \ker(V_i \xrightarrow{df_p} T_{f(p)}N/W_j), \mathrm{coker}(V_i \xrightarrow{df_p} T_{f(p)}N/W_j))$$

and in fact we may view d^2 as a map

$$S[\{r_{ij}\}]' \xrightarrow{d^2} \mathrm{Hom}(TM \odot K[\{r_{ij}\}], L[\{r_{ij}\}]),$$

where the meaning of $TM \odot K[\{r_{ij}\}]$ is to take the symmetric product componentwise (giving another functor $I \times J \to \mathsf{Vect}$). By the discussion about iterated Jacobians, d^2 is surjective. It is a submersion, too, by the following argument. Since the distributions are integrable, there are trivializations of the tangent bundles such that the distributions $\{V_i\}$ and $\{W_j\}$ take the form of trivial bundles in the trivializations (see Lemma 1.3.21). This homogeneity implies that it suffices to show that d^2 is a submersion on fibers. The fiber above $j^1 f_p$ in $S[\{r_{ij}\}]'$ is, in coordinates, the second-order Taylor polynomials of f at p that coincide with $j^1 f_p$ to first order, hence all of $J^2 f_p$ is realized in the fiber. In coordinates, d^2 is computing a restriction and projection of $J^2 f_p$, and restrictions and projections are submersions.

Before describing the second-order types, we mention some properties of the subspaces of the bundle TM over $j^1 f_p \in S[\{r_{ij}\}]$ that we have at our disposal: the distributions $\{V_i\}$ at p, the kernels $K[\{r_{ij}\}]$ at $j^1 f_p$, and also T_pM itself. For $i, i' \in I$ and $j, j' \in J$, we have a commutative diagram of kernels and cokernels, where recall that the meet $\wedge$ is defined to be such that $V_{i \wedge i'} = V_i \cap V_{i'}$ and $W_{j \wedge j'} = W_j \cap W_{j'}$, and for notational compactness we write $K_{i,j}$ for $K[\{r_{ij}\}]_{ij}$ and $L_{i,j}$ for $L[\{r_{ij}\}]_{ij}$:

$$\begin{tikzcd}
K_{i,j} \arrow[d, hook] & K_{i\wedge i',j\wedge j'} \arrow[l, hook'] \arrow[r, hook] \arrow[d, hook] & K_{i',j'} \arrow[d, hook] \\
V_i \arrow[d, "df_p"] & V_{i\wedge i'} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "df_p"] & V_{i'} \arrow[d, "df_p"] \\
T_pM/W_j \arrow[d, two heads] & T_pM/W_{j\wedge j'} \arrow[l, two heads] \arrow[r, two heads] \arrow[d, two heads] & T_pM/W_{j'} \arrow[d, two heads] \\
L_{i,j} & L_{i\wedge i',j\wedge j'} \arrow[l, two heads] \arrow[r, two heads] & L_{i',j'}
\end{tikzcd}$$

When $j = j'$, we can see that $K_{i\wedge i',j} = K_{i,j} \cap K_{i',j}$, and when $i = i'$, we can see that $K_{i,j\wedge j'} = K_{i,j} \cap K_{i,j'}$. Hence, the kernels are closed under intersections. Furthermore, the intersection of $(V_i)_p$ and $K_{i',j}$ is $K_{i\wedge i',j}$, hence the collection of all V_i, $K[\{r_{ij}\}]_{ij}$, and TM over $j^1 f_p$ is closed under intersections.

We should also mention subspaces of $\mathrm{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}])$. To be able to apply our normal bundle machinery, we want these subspaces to be closed under intersections. As this is a vector space of natural transformations, we have projections to $\mathrm{Hom}(K[\{r_{ij}\}]_{ab}, L[\{r_{ij}\}]_{ab})$ for each $(a, b) \in I \times J$, and the kernels of these projections give subspaces. However, they are not necessarily closed under intersections. Instead, for every $A \subseteq I$ and $B \subseteq J$ whose corresponding subfamilies are closed under intersections, we can restrict the natural transformations to the category $A \times B$. The kernels associated to these restriction maps then give a family of subspaces (and in fact subbundles) that are closed under intersection.

Now let I' and J' be finite index sets, where for each $i \in I'$ and $j \in J'$ we let V'_i be a subbundle of the TM bundle and $W'_{j'}$ be a subbundle of the $\mathrm{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}])$ bundle, both only of the types mentioned above, such that $\{V'_{i'}\}$ and $\{W'_{j'}\}$ are closed under intersections and contain no repetitions. We assume TM is among the V'_i and the zero subspace is among the W'_j.

For each $(I' \times J')$-indexed family $\{r'_{i'j'}\}$ of natural numbers, we have a submanifold $\mathrm{Hom}(TM, \mathrm{Hom}(K[\{r_{ij}\}], L[\{r_{ij}\}]); \{r'_{i'j'}\})$. Assuming that its intersection with $\mathrm{Hom}(TM \odot K[\{r_{ij}\}], L[\{r_{ij}\}])$ is also a submanifold, then we define the second-order type $S[\{r_{ij}\}][\{r'_{i'j'}\}] \subseteq S[\{r_{ij}\}]'$ to be the preimage of this submanifold through d^2.

Remark 1.5.36. The reason we included TM in the $\{V'_i\}$ family and the zero subspace in $\{W'_j\}$ is that this lets us encode surjectivity of the second intrinsic derivative as a rank condition. The second-order types that encode non-surjective second intrinsic derivatives do not generically occur — thus we do not further stratify such types.

We can generically assume we are only considering the case where on TM the second intrinsic derivative is surjective. We can also generically assume that for each $j' \in J'$ that the projection $TM \to \mathrm{Hom}(K[\{r_{ij}\}]_{j'}, L[\{r_{ij}\}]_{j'})$ of the second intrinsic derivative is surjective. ∎

The next step is working out the condition on 3-jets for which a map's jet extension intersects $S[\{r_{ij}\}][\{r'_{i'j'}\}]$ transversely. Suppose $j^2 f_p \in S[\{r_{ij}\}][\{r'_{i'j'}\}]$ with f generic. In the basic form, we calculate the normal bundle for the rank conditions in $\mathrm{Hom}(TM \odot K[\{r_{ij}\}], L[\{r_{ij}\}])$, calculate the third intrinsic derivative, and then check that it is surjective. There is of course the complication that this normal bundle is for a more complicated space than before, but we handle this on a case-by-case basis.

For higher types, there is more information we may make use of. As discussed, for each $j' \in J'$ we have a projection of the second intrinsic derivative

$$d^2 f_p : T_pM \to \mathrm{Hom}((K[\{r_{ij}\}]_{j'})_p, (L[\{r_{ij}\}]_{j'})_p),$$

and since we assumed this is surjective, there is a smooth submanifold $S[\{r_{ij}\}]_{j'}(f)$ of "j'" points of f whose tangent space at p is the kernel of this j' projection of $d^2 f_p$. These second intrinsic derivatives assemble into a bundle map over this submanifold. Assuming the rank conditions $\{r'_{i'j'}\}$ define a submanifold of $\mathrm{Hom}(T_pM \odot (K[\{r_{ij}\}]_{j'})_p, (L[\{r_{ij}\}]_{j'})_p)$ and that we can calculate a normal bundle $P[\{r_{ij}\}]_{j'}[\{r'_{i'j'}\}]_p$ for it, the j' projection bundle map has a *third intrinsic derivative*,

$$d^3 f_p : T_p S[\{r_{ij}\}]_{j'}(f) \to P[\{r_{ij}\}]_{j'}[\{r'_{i'j'}\}]_p.$$

This third intrinsic derivative depends on no more than the 3-jet of f. For all $j', j'' \in J'$ such that $j' \leq j''$, we have a commutative diagram:

$$\begin{tikzcd} T_p S[\{r_{ij}\}]_{j'}(f) \arrow[r, "d^3 f_p"] \arrow[d, hook] & P[\{r_{ij}\}]_{j'}[\{r'_{i'j'}\}]_p \arrow[d, two heads] \\ T_p S[\{r_{ij}\}]_{j''}(f) \arrow[r, "d^3 f_p"] & P[\{r_{ij}\}]_{j''}[\{r'_{i'j'}\}]_p \end{tikzcd}$$

This means $d^3 f_p$ can be seen as a natural transformation $T_p S[\{r_{ij}\}]_\bullet(f) \to P[\{r_{ij}\}]_\bullet[\{r'_{i'j'}\}]_p$ of functors $J' \to \mathsf{Vect}$. Care must be taken with this since it's a bundle map only over $S[\{r_{ij}\}][\{r'_{i'j'}\}](f)$, which limits the potential for higher derivatives since the intrinsic derivative depends on having bundles defined on an open neighborhood of p.

Let $S[\{r_{ij}\}][\{r'_{i'j'}\}]' \subseteq J^3(M,N)$ be the preimage of the submanifold $S[\{r_{ij}\}][\{r'_{i'j'}\}] \subseteq J^2(M,N)$. Making use of the $J^3(M,N) \to \mathrm{Hom}(TM, TJ^2(M,N))$ map from Lemma 1.1.9, we may obtain a bundle homomorphism over $S[\{r_{ij}\}][\{r'_{i'j'}\}]$:

$$S[\{r_{ij}\}][\{r'_{i'j'}\}]' \xrightarrow{d^3} \mathrm{Hom}(TS[\{r_{ij}\}]_\bullet, P[\{r_{ij}\}]_\bullet[\{r'_{i'j'}\}])$$

Being loose with notation, $TS[\{r_{ij}\}]_\bullet$ is the bundle whose fiber above $j^2 f_p$ is the functor $(j' \in J') \mapsto TS[\{r_{ij}\}]_{j'}(f)$, and similarly for $P[\{r_{ij}\}]_\bullet[\{r'_{i'j'}\}]$. Then, in a similar way as before, we may look for rank conditions that define submanifolds in the image of d^3 and pull them back, yielding third-order types. We must take care in choosing our rank conditions. For $j'' > j'$ we have that $S[\{r_{ij}\}]_{j'} \subseteq S[\{r_{ij}\}]_{j''}$, so bundles associated to j' cannot necessarily be used in rank conditions associated to j'' — intrinsic derivatives depend on having a bundle defined in a neighborhood of p.

Now it is time to use these ideas to compute singularity types. It should be said that it is easier to execute it in practice than to explain it in the abstract.

Chapter 2

Classification of singularities

Table 2.1: Classification of Morse singularities.

Type	Codim.	Representative germ	See:
$S[0]$	0	$f^*y = x_1$	Lemma 2.1.2
$S[1]$	m	$f^*y = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2$	Lemma 2.1.3

Table 2.2: Classification of Cerf singularities.

Type	Codim.	Representative germ	See:	Suspends
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]$	0	$f^*y = x_1$	Lemma 2.1.8	$S[0]$
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$	m	$f^*y = -x_1^2 - \cdots - x_k^2$ $+ x_{k+1}^2 + \cdots + x_m^2$	Lemma 2.1.9	$S[1]$
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$	$m+1$	$f^*y = -x_1^2 - \cdots - x_k^2$ $+ x_{k+1}^2 + \cdots + x_{m-1}^2$ $+ x_m(t + x_m^2)$	Lemma 2.1.10	
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]\|[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$	$m\|m$	$f_p^*y = -x_1^2 - \cdots - x_k^2$ $+ x_{k+1}^2 + \cdots + x_m^2 + t$ $f_{p'}^*y = -(x_1')^2 - \cdots - (x_{k'}')^2$ $+ (x')_{k'+1}^2 + \cdots + (x_m')^2$	Lemma 2.1.12	

In this chapter we work through the classification of singularities for a number of situations. We start in Section 2.1 by showing how to use our version of Thom–Boardman singularities to derive the singularity types for Morse and Cerf theory. In Section 2.3 we derive singularity types for curves and graphs in the plane along with their Cerf theory when the plane is given the standard $\mathbb{R} \subseteq \mathbb{R}^2$ foliation, which is the model useful for decomposing diagrams for monoidal categories. Then, in Sections 2.4 to 2.6 we classify the singularities for the Morse and Cerf theory of maps $\Sigma \to (\mathbb{R} \subseteq \mathbb{R}^2)$ for Σ a surface, reprising [SP09] but with exact normal forms. Finally, Sections 2.7 to 2.9 is a classification of singularities for the Morse and Cerf theory for maps $\Sigma \to (\mathbb{R} \subseteq \mathbb{R}^2)$ when the surface Σ contains a 1-manifold. Since most of the singularities we come across are in some way Morin singularities due to the low dimensions involved, Section 2.2 is a general overview.

In tables of singularity types, we will list a family of polynomial representatives (what we call "normal forms," which are *not* generally canonical to be clear!) per Thom–Boardman symbol. Our spaces of singularities have no moduli, which is to say each terminal Thom–Boardman class corresponds to only a finite set of equivalence classes of germs, and we can generally represent these in tables by giving schemata for the plus and minus signs. In higher dimensions one will not be so lucky.

2.1 Morse and Cerf theory

In this section, as an exercise we apply the machinery to rederiving the descriptions of Morse and Cerf functions. It is well-known that the Thom–Boardman classification is sufficient for density of Morse functions [GG73], and it is also sufficient for Cerf functions when allowing a more lax notion of equivalence [SP09], but we will demonstrate that the classification can be used to derive the notion of a Cerf function with the strict notion of equivalence. It is also worth reviewing this since we will lean heavily on the techniques demonstrated in the Morse Lemma and applications of Lemma 1.1.69 in later classifications. The resulting representative jets for Morse and Cerf singularities are respectively summarized in Tables 2.1 and 2.2, and note that these results coincide with the classical theory.

2.1.1 Morse singularities

First, we take on Morse functions, which are a certain kind of generic function in $C^\infty(M, \mathbb{R})$ for M a smooth m-manifold. The first-order singularity types are $S[r] \subseteq J^1(M, \mathbb{R})$, where $r \in \mathbb{N}$ is the dropped rank for the differential — the only possibilities of course are $r \in \{0, 1\}$. The codimension of $S[r]$ in $J^1(M, \mathbb{R})$ (and thus the codimension of this type of singularity in M) is given in the following table:

Type	$S[0]$	$S[1]$
Codim.	0	m

The second intrinsic derivative $d^2f_p : T_pM \to \mathrm{Hom}(K[0]_p, L[0]_p)$ for the $S[0]$ type is the zero function since $L[0]_p := \operatorname{coker} df_p = 0$ for $j^1f_p \in S[0]$, and so it is automatically surjective. If $j^1f_p \in S[1]$, its second intrinsic derivative is a function

$$d^2f_p : T_pM \to \mathrm{Hom}(K[1]_p, L[1]_p)$$

with $K[1]_p = T_pM$ and $L[1]_p = T_q\mathbb{R}$, following the convention that $q = f(p)$. For this to be surjective, then, by dimension, the restriction $d^2f_p|_{K[1]_p}$ is a bijection, so the third intrinsic derivative will be the zero function, and thus there is nothing more to classify.

Hence, $S[0]$ and $S[1]$ enumerate the viable Thom–Boardman singularity types, and with those determined we next consider multijet singularities. Letting $\Delta^2_{\mathbb{R}} = \{(y, y) \mid y \in \mathbb{R}\}$ be the diagonal, we may take its preimage $\beta^{-1}(\Delta^2_{\mathbb{R}}) \subseteq J^{(1,1)}(M, \mathbb{R})$, which is a submanifold of codimension 1. We can take the preimage of $S[r_1] \times S[r_2] \subseteq J^1(M, \mathbb{R}) \times J^1(M, \mathbb{R})$ in $J^{(1,1)}(M, \mathbb{R})$, which is a submanifold of codimension $(r_1 + r_2)m$. One can check that its intersection with $\beta^{-1}(\Delta^2_{\mathbb{R}})$ is a submanifold of codimension $(r_1 + r_2)m + 1$, which we call $S[r_1]||[r_2]$, and the set $S[r_1]||[r_2](f)$ is pairs of distinct points $(p_1, p_2) \in M \times M$ such that $f(p_1) = f(p_2)$, $p_1 \in S[r_1](f)$, and $p_2 \in S[r_2](f)$. Thus, applying Theorem 1.2.12 (the Multijet Transversality Theorem), there is a dense subset of maps $M \to \mathbb{R}$ whose multijet extension $j^{(1,1)} : M^2 \setminus \mathbf{\Delta}^2_M \to J^{(1,1)}(M, \mathbb{R})$ is transverse to $S[r_1]||[r_2]$, and for such a map f, $S[r_1]||[r_2](f)$ has codimension $(r_1 + r_2)m + 1$ in $M^2 \setminus \mathbf{\Delta}^2_M$. The types where $r_1 = 0$ or $r_2 = 0$ are uninteresting since $S[0]$ is the type for a submersion, and for the remaining case we see that $S[1]||[1](f)$ is empty since the type has too high of codimension.

By Theorems 1.2.9 and 1.2.12, the subset of $C^\infty(M, \mathbb{R})$ whose jet extensions are transverse to $S[r]$ and $S[r_1]||[r_2]$ for all $r, r_1, r_2 \in \mathbb{N}$ is a residual subset, and thus dense — functions in this dense subset are called *generic*. We now classify the singularities in coordinates, starting from these types.

Remark 2.1.1. At this point we can infer that generic functions have the property that the set of critical points is $S[1](f)$, this set is discrete, and all the critical values are distinct since $S[1]||[1](f)$ is empty. ■

Recall from Definition 1.4.2 that two germs $\sigma, \tau \in C^\infty_p(\mathbb{R}^m, \mathbb{R})$ are *equivalent* if there is a diffeomorphism germ $\varphi \in \mathrm{Diff}_p(\mathbb{R}^m)$ and a diffeomorphism germ $\psi : \mathrm{Diff}_{\tau(p)}(\mathbb{R})_{\sigma(p)}$ such that $\sigma = \psi \circ \tau \circ \varphi$. We will be using the convention that $p = 0 \in \mathbb{R}^m$. By equivalence, we may also assume that our germs are all in $C^\infty_p(\mathbb{R}^m, \mathbb{R})_0$.

Lemma 2.1.2. *If $f \in C^\infty_p(\mathbb{R}^m, \mathbb{R})$ is a generic germ that is an $S[0]$ singularity (i.e., $j^1f_p \in S[0]$), then, up to equivalence, it is given by*

$$f(x) = x_1.$$

Proof. After a linear change of variables, we may assume $df = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$. By the constant rank theorem, there is a change of coordinates for $\mathbb{R}^m$ and $\mathbb{R}$ such that on a neighborhood of p, $f(x) = Jf_p(x)$, thus we get the desired from $f(x) = x_1$. □

Lemma 2.1.3 (Morse Lemma)**.** *If $f \in C^\infty_p(\mathbb{R}^m, \mathbb{R})$ is a generic germ that is an $S[1]$ singularity, then, up to equivalence, it is given by*

$$f(x) = a_1 x_1^2 + \cdots + a_m x_m^2$$

where $a_1, \ldots, a_m \in \{-1, 1\}$.

Proof. By assumption, $df_p = 0$, and we can calculate that the second intrinsic derivative is

$$\begin{aligned} d^2 f_p : \mathbb{R}^m &\to \operatorname{Hom}(\mathbb{R}^m, \mathbb{R}) \\ v &\mapsto \big(w \mapsto w^T H f(p) v\big) \end{aligned}$$

where $Hf(p)$ is the Hessian matrix of f at p. Since $j^1 f$ is transverse to the $S[1]$ submanifold, $d^2 f_p$ is surjective, which implies $Hf(p)$ is nonsingular. Symmetric matrices can be orthogonally diagonalized, so there is a linear reparameterization such that $Hf(p)$ is diagonal. Scaling variables, we may assume the diagonal entries are ± 2.

Applying the Hadamard lemma (Lemma 1.1.69), there are functions $h_{ij} \in C^\infty(\mathbb{R}^m)$ for all $1 \leq i \leq j \leq n$ such that

$$f(x) = \sum_{1 \leq i \leq j \leq n} h_{ij}(x) x_i x_j$$

where $h_{ii}(p) = \pm 1$ and $h_{ij}(p) = 0$ if $i < j$. Let $a_i = h_{ii}(p)$ for all $1 \leq i \leq n$.

We show by induction on $0 \leq k \leq n$ that there is a reparameterization of the domain such that (1) for all $1 \leq i \leq k$, $h_{ii} = a_i$ and (2) for all $1 \leq i < k$ and $i < j \leq n$, $h_{ij} = 0$. The base case $k = 0$ is vacuously satisfied. Assume the induction hypothesis is true for some k, which gives the representation

$$f(x) = a_1 x_1^2 + \cdots + a_k x_k^2 + \sum_{k \leq i \leq j \leq n} h_{ij}(x) x_i x_j.$$

Extracting those terms involving x_{k+1}, we may rewrite this as

$$f(x) = a_1 x_1^2 + \cdots + a_k x_k^2 + h_{k+1,k+1}(x) x_k^2 + x_{k+1} \sum_{k+1 < j \leq n} h_{k+1,j}(x) x_j + \sum_{k+1 \leq i \leq j \leq n} h_{ij}(x) x_i x_j.$$

Consider the reparameterization with $\overline{x}_i = x_i$ for all $i \neq k$ and

$$\overline{x}_k = \sqrt{|h_{kk}(x)|} \left(x_i - \frac{x_{k+1}}{2\sqrt{|h_{kk}(x)|}} \sum_{k+1 < j \leq n} h_{k+1,j}(x) x_j \right).$$

The Jacobian of this reparameterization at p is the identity matrix (hence invertible) thus it is a valid reparameterization. It is completing the square for x_k, so applying it (and going back to having $x_1, \dots, x_m$ denote the coordinates in place of $\overline{x}_1, \dots, \overline{x}_m$), the result is that we may assume $h_{kk} = a_k$ and furthermore

$$f(x) = a_1 x_1^2 + \cdots + a_k x_k^2 + a_{k+1} x_{k+1}^2 + \sum_{k+1 \le i \le j \le n} h_{ij}(x) x_i x_j.$$

This completes the induction. □

The results are summarized in Table 2.1, which records representative germs for each singularity type.

2.1.2 Cerf singularities

We now move on to the Cerf singularities. A Cerf function is a path through $C^\infty(M, \mathbb{R})$ of functions that are generically (in a suitable sense) Morse functions. With the appropriate topologies, a smooth such path is equivalently an element of $C^\infty(M \times [0,1], \mathbb{R})$. The more classic way to study these is to consider instead functions $f \in C^\infty(M \times [0,1], \mathbb{R} \times [0,1])$ with the property that $f(x,t)_2 = t$ for all $(x,t) \in M \times [0,1]$. This constraint is inconvenient to incorporate into the Thom–Boardman framework, so instead what we will do is make things generic with respect to the product foliation of $M \times [0,1]$. An intuition behind this is that a function $f \in C^\infty(M \times [0,1], \mathbb{R})$ defines the function $F : M \times [0,1] \to \mathbb{R} \times [0,1]$ given by

$$\begin{aligned} F_1(x,t) &= f(x,t) \\ F_2(x,t) &= t \end{aligned}$$

Using the product foliation for $\mathbb{R} \times [0,1]$ (since, after all, we are wanting to do a 2-categorical decomposition of smooth homotopies between Morse functions), then the second intrinsic derivative of this with respect to just the foliation on the codomain is a function

$$d^2 F_p : T_p(M \times [0,1]) \to \operatorname{Hom}\left(\ker dF_p \hookrightarrow \ker d(F_2)_p, \operatorname{coker} dF_p \hookrightarrow \operatorname{coker} d(F_2)_p\right),$$

and we have $\ker dF_p = T_pM \cap \ker df_p$, $\ker d(F_2)_p = T_pM$, and $\operatorname{coker} d(F_2)_p = 0$. After a foliation-preserving linear change of coordinates in the codomain, $\operatorname{coker} dF_p = \operatorname{coker} df_p$. Thus, the second intrinsic derivative reduces to being a map

$$d^2 F_p : T_p(M \times [0,1]) \to \operatorname{Hom}\big(\ker(df_p|_{T_pM}), \operatorname{coker} df_p\big)$$

In particular, we get T_pM in an intrinsic way from the foliation of the codomain, and T_pM forms a distribution for the domain that foliates it. From this point of view, we may as well start with this foliation. The only catch is that when we consider equivalence of germs $C_p^\infty(M \times [0,1], \mathbb{R})$, we apply "affine" diffeomorphisms to the codomain. In other words, we consider equivalence of germs as if they were germs in the F form:

Definition 2.1.4. Let M and N be smooth manifolds with structure, and let $p \in M$ and $t_0 \in \mathbb{R}$. Germs $f, f' \in C^\infty_{(p,t_0)}(M \times \mathbb{R}, N)$ are *Cerf equivalent* if the germs $F, F' \in C^\infty_{(p,t_0)}(M \times \mathbb{R}, N \times \mathbb{R})$ defined by

$$\begin{aligned} F_1(x,t) &= f(x,t) & \qquad F_1'(x,t) &= f'(x,t) \\ F_2(x,t) &= t & \qquad F_2'(x,t) &= t \end{aligned}$$

are equivalent with respect to $\mathrm{Diff}_{(p,t_0)}(M \times \mathbb{R})$ and $\mathrm{Diff}(N \times 0 \subseteq N \times \mathbb{R})$. $\Diamond$

For example, adding an element of $C^\infty_{p_2}([0,1], \mathbb{R})$ to a germ or negating t gives a Cerf equivalent germ — up to Cerf equivalence we may assume for $f \in C^\infty_p(M \times [0,1], \mathbb{R})$ that $f(p_1, t) = 0$ for all t.

Remark 2.1.5. This is the same notion as equivalence of 1-dimensional unfoldings; see [Was75] for an introduction to this theory of Thom and some generalizations to unfoldings with, essentially, foliated additional parameters. ■

Another way to justify this type of equivalence is that translations of a Morse function through time are "uninteresting," so we may as well ignore translations in our classification. Supporting this is the following general lemma about suspensions of germs (see Definition 2.1.14):

Lemma 2.1.6. *Let $m, n, s \in \mathbb{N}$ and $f \in C^\infty_0(\mathbb{R}^m \times \mathbb{R}^s, \mathbb{R}^n)$ be such that $df|_{T_0\mathbb{R}^s} = 0$. Suppose f is a generic Thom–Boardman singularity with respect to $\{T_0\mathbb{R}^m\}$ for the domain and $\{T_0\mathbb{R}^n\}$ for the codomain. Then $F \in C^\infty_0(\mathbb{R}^m \times \mathbb{R}^s, \mathbb{R}^n \times \mathbb{R}^s)$ defined by $F(x,t) = (f(x,t), t)$ is a generic Thom–Boardman singularity with respect to $\mathbb{R}^m \times \mathbb{R}^s$ and the product foliation on $\mathbb{R}^n \times \mathbb{R}^s$. This also holds when $\mathbb{R}^n \times \mathbb{R}^s$ is not given the product foliation.*

Proof. The differential of F has the form

$$dF_0 = \left[\begin{array}{c|c} \multicolumn{2}{c}{df_0} \\ \hline 0 & I_s \end{array}\right],$$

and hence the second intrinsic derivative is a function

$$d^2F_0 : T_0(\mathbb{R}^m \times \mathbb{R}^s) \to \mathrm{Hom}\left(\begin{array}{c} T_0\mathbb{R}^m \cap \ker df_0 \\ \downarrow \\ T_0\mathbb{R}^m \end{array}, \begin{array}{c} \operatorname{coker} df_0 \\ \downarrow \\ 0 \end{array} \right).$$

By the assumption that $df|_{T_0\mathbb{R}^s} = 0$ we have $\operatorname{coker} df_0 = \operatorname{coker} (df_0)|_{T_0\mathbb{R}^m}$. We also have that $T_0\mathbb{R}^m \cap \ker df_0 = \ker (df_0)|_{T_0\mathbb{R}^m}$, thus $d^2F_0|_{T_0\mathbb{R}^m} = d^2 f_0$ (which holds even if $\mathbb{R}^n \times \mathbb{R}^s$ is not given the product foliation). Hence, d^2F_0 is surjective because $d^2 f_0$ is. For the higher intrinsic derivatives, the pattern is (1) that we look at restrictions to the same subspaces of $T_0\mathbb{R}^m$ (starting with $\ker df_0$) as we would if we had started with f and (2) that $\ker d^k f_0 \subseteq \ker d^k F_0$, so each $d^{k+1}F_0$ is surjective since $d^{k+1} f_0$ is. $\square$

With that said, we will now analyze the Thom–Boardman singularities up to Cerf equivalence for $C^\infty(M \times [0,1], \mathbb{R})$ where the domain is given the product foliation, but we only consider the TM distribution and we never look at rank on $T(M \times [0,1])$ (this is not invariant under Cerf equivalence). With a nod toward the motivation for these singularities, we will write the first-order types as $S\left[\begin{smallmatrix}r\\0\end{smallmatrix}\right]$, which helps distinguish them from their $S[r]$ Morse counterparts.

The $S\left[\begin{smallmatrix}r\\0\end{smallmatrix}\right] \subseteq J^1(M \times [0,1], \mathbb{R})$ type is given by those $j^1 f_p$ such that $df_p|_{T_pM}$ has dropped rank equal to r. The codimensions from before still apply, and the only possibilities are $r \in \{0,1\}$. The $S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ type has no subtypes, so we turn our attention to $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. The second intrinsic derivative for $j^2 f_p \in S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ is a function

$$d^2 f_p : T_p(M \times [0,1]) \to \mathrm{Hom}(K\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_p, L\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_p)$$

with $K\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_p = \ker(df_p|_{T_pM}) = T_pM$ and $L[1]_p = \mathrm{coker}\,(df_p|_{T_pM}) = T_q\mathbb{R} = \mathbb{R}$. In particular it is given by

$$\begin{aligned} d^2 f_p : T_p(M \times [0,1]) &\to \mathrm{Hom}(T_pM, \mathbb{R}) \\ v &\mapsto (w \mapsto Hf_p(v \otimes w)) \end{aligned}$$

where we use the inclusion $T_pM \hookrightarrow T_p(M) \oplus T_p([0,1]) \cong T_p(M \times [0,1])$ to evaluate the Hessian. Recall that the kernel of $d^2 f_p$ is the tangent space to $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](f)$ at p, which we see is 1-dimensional when $d^2 f_p$ is surjective.

The second-order types are $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][r] \subseteq J^2(M \times [0,1], \mathbb{R})$ of those jets $j^2 f_p$ such that $d^2 f_p|_{T_pM}$ has dropped rank equal to r (recall: this corresponds to the dropped rank for $d(f|_{S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](f)})_p$). By consideration of the 1-dimensionality of $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](f)$, the possibilities are $r \in \{0,1\}$. The type $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][0]$ has no subtypes, but subtypes of $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]$ cannot yet be excluded. Restricted to T_pM, the second intrinsic derivative $d^2 f_p$ is the Hessian restricted to $T_pM \otimes T_pM$. For this type, there is a one-dimensional kernel $K\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]_p \subseteq T_pM$ of the Hessian, and the third intrinsic derivative can be calculated to be a function

$$d^3 f_p : T_p S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right](f) \to \mathrm{Hom}(K\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]_p \odot K\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]_p, \mathbb{R}).$$

Both the domain and codomain of this function are one-dimensional, so surjectivity reduces to $d^3 f_p$ being nonzero and thus a bijection. Thus, $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]$ has no subtypes.

We thus have that the viable Thom–Boardman singularity types are $S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][0]$, and $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]$. We defer discussion of multijet singularities to Section 2.1.2.

By Theorem 1.2.9 (the Thom Transversality Theorem), there is a dense subset of $C^\infty(M \times [0,1], \mathbb{R})$ of *generic* functions that have the property that their $S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$, $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][0]$, and $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]$ points stratify $M \times [0,1]$. We now derive normal forms. We again use the convention that $p = 0 \in \mathbb{R}^m \times \mathbb{R}$, and we use the coordinates $(x_1, \ldots, x_m, t)$ for $\mathbb{R}^m \times \mathbb{R}$. It is relatively easy to derive a normal form for the k-jet of a kth order type, and then using stability results from Section 1.4 we can exactly identify equivalence classes.

Remark 2.1.7. Even before obtaining normal forms, what we know so far is that for a generic $f \in C^\infty(M \times [0,1], \mathbb{R})$, there is a one-dimensional submanifold $S[\begin{smallmatrix}1\\0\end{smallmatrix}](f)$ of those points $(x,t) \in M \times [0,1]$ where f_t has a critical point, and furthermore the one-dimensional submanifold $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ is where $S[\begin{smallmatrix}1\\0\end{smallmatrix}](f)$ meets the leaves of the foliation transversely, and the zero-dimensional submanifold $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1](f)$ is where it temporarily becomes tangent to the leaves. ■

Lemma 2.1.8. *If $f \in C^\infty_p(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$ is a generic germ that is an $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularity, then, up to Cerf equivalence, it is given by*

$$f(x,t) = x_1$$

Proof. After a linear change of variables from $\mathrm{GL}(\mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R})$, we may assume $df = \begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}$. By the implicit function theorem, there is a function $\varphi : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ such that

$$f(\varphi(\overline{x}_1, \dots, \overline{x}_m, \overline{t}), \overline{x}_2, \dots, \overline{x}_m, \overline{t}) = \overline{x}_1$$

for all $(\overline{x}_1, \dots, \overline{x}_m, \overline{t}) \in \mathbb{R}^m \times \mathbb{R}$. The following is a valid coordinate change for $F(\mathbb{R}^m \subseteq \mathbb{R}^m \times \mathbb{R})$:

$$x_1 = \varphi(\overline{x}_1, \dots, \overline{x}_m, \overline{t}) \qquad x_2 = \overline{x}_2 \qquad \cdots \qquad x_m = \overline{x}_m \qquad t = \overline{t}$$

With it,

$$f^* y = f(x_1, x_2, \dots, x_m, t) = f(\varphi(\overline{x}_1, \dots, \overline{x}_m, \overline{t}), \overline{x}_2, \dots, \overline{x}_m, \overline{t}) = \overline{x}_1,$$

which has the desired form. □

Lemma 2.1.9. *If $f \in C^\infty_p(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$ is a generic germ that is an $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ singularity, then, up to Cerf equivalence, it is given by*

$$f(x,t) = a_1 x_1^2 + \cdots + a_m x_m^2$$

where $a_1, \dots, a_m \in \{-1, 1\}$.

Proof. Since it is an $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ singularity, $df_p|_{\mathbb{R}^m} = 0$, and by a Cerf change of coordinates for the codomain we may assume $df_p = 0$. As we determined earlier, the second intrinsic derivative is

$$\begin{aligned} d^2 f_p : T_p(\mathbb{R}^m \times \mathbb{R}) &\to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}) \\ v &\mapsto (w \mapsto Hf_p(v \otimes w)). \end{aligned}$$

Since we are considering type $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$, then $d^2 f_p|_{\mathbb{R}^m}$ is surjective, and hence the Hessian restricted to $\mathbb{R}^m \otimes \mathbb{R}^m$ is nondegenerate. We have $\ker d^2 f_p = T_p S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$, hence $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ is one-dimensional, and also we see $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ intersects $\mathbb{R}^m \times 0$ transversely. Thus, there is a local parameterization $\gamma : \mathbb{R} \hookrightarrow S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ with $\gamma(t)_2 = t$ for all $t \in \operatorname{dom} \gamma$. With this we construct the change of coordinates $x = \overline{x} + \gamma(t)_1$ and $t = \overline{t}$, after which we may assume $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ coincides with $0 \times \mathbb{R}$ in a neighborhood of p. Replacing $f(x,t)$ with $f(x,t) - f(0,t)$ gives a Cerf equivalent germ, so we may further assume that $f(0,t) = 0$ for all t. Thus, we may apply the Parameterized Hadamard Lemma (Lemma 1.1.69) to get germs $f_{ij} \in C_p^\infty(\mathbb{R}^m \times \mathbb{R})$ such that

$$f(x,t) = \sum_{1 \le i \le j \le n} f_{ij}(x,t) x_i x_j.$$

Like in the Morse Lemma (Lemma 2.1.3), after a linear change of coordinates we may assume Hf_p is diagonal when restricted to $T_p\mathbb{R}^m \otimes T_p\mathbb{R}^m$. The structure of the proof of the Morse Lemma carries through to this parameterized situation since there is an $\varepsilon > 0$ such that for all $|t| < \varepsilon$, $f_{ii}(0,t) \neq 0$, and furthermore the coordinate changes always preserve t. Thus, there is a coordinate change such that f can be written in the desired form. $\square$

Lemma 2.1.10. *If $f \in C_p^\infty(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$ is a generic germ that is an $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ singularity, then, up to Cerf equivalence, it is given by*

$$f(x,t) = a_1 x_1^2 + \cdots + a_{m-1} x_{m-1}^2 + x_m(t + x_m^2)$$

where $a_1, \ldots, a_{m-1} \in \{-1, 1\}$. Such a germ is locally stable with respect to Cerf equivalence and is 3-determined.

Proof. Unlike in Lemma 2.1.9, $\ker d^2 f_p$ is now a one-dimensional subspace of T_pM. After a Cerf change of coordinates and a linear change of coordinates for $\mathbb{R}^m$, we may assume that $df_p = 0$, that $\ker d^2 f_p$ is $\mathbb{R}_m = \{x \in \mathbb{R}^m \mid x_1 = \cdots = x_{m-1} = 0\}$, and that in coordinates $d^2 f_p$ is in the form

$$\begin{bmatrix} \pm 2 & 0 & \cdots & 0 & 0 & * \\ 0 & \pm 2 & \cdots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \pm 2 & 0 & * \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Running through the proof for Lemma 2.1.3 using the Parameterized Hadamard Lemma like in Lemma 2.1.9 and making sure to also clear out $x_i t$ terms, there is a change of coordinates such that

$$f(x,t) = a_1 x_1^2 + \cdots + a_{m-1} x_{m-1}^2 + h(x,t) x_m^2 + x_m t$$

for $a_1, \ldots, a_{m-1} \in \{-1, 1\}$ and $h \in C_p^\infty(\mathbb{R}^m \times \mathbb{R}, \mathbb{R})$. Since $h(p) = 0$, by the Hadamard Lemma there are functions $h_0, \ldots, h_m$ such that $h = th_0 + x_1 h_1 + \cdots + x_m h_m$, and by a Morse-lemma-style changes of coordinates, we may assume $h_1, \ldots, h_{m-1} = 0$. Hence,

$$f(x,t) = a_1 x_1^2 + \cdots + a_{m-1} x_{m-1}^2 + h_m(x,t) x_m^3 + h_0(x,t) x_m^2 t + x_m t.$$

The third intrinsic derivative computes $\frac{\partial^3 f}{\partial x_m^3}(p) = 6 h_m(p)$, which must be nonzero for $j^2 f$ to be transverse to $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$. By scaling x_m and t as needed, we may assume that $h_m(p) = 1$.

The change of coordinates with $x_m = \overline{x}_m - h_0(x,t)\overline{x}_m^2$ lets us assume $h_0(p) = 0$, and therefore we have found a normal form for the 3-jet of this singularity:

$$f(x,t) = a_1x_1^2 + \cdots + a_{m-1}x_{m-1}^2 + x_m^3 + x_m t + r(x,t),$$

where $r \in \mathfrak{m}_p^4$. We may assume $r(0,t) = 0$ by subtracting $r(0,t)$ from f via a Cerf change of coordinates. By the Parameterized Hadamard Lemma (Lemma 1.1.69), there are functions $r_I \in C_p^\infty(\mathbb{R}^m \times \mathbb{R})$ such that

$$r(x,t) = \sum_{|I|=4} r_I(x,t)x^I.$$

Using $x_1^2, \ldots, x_{m-1}^2$ to clear out terms like in the Morse Lemma, we may assume that

$$r(x,t) = r'(x,t)x_m^4$$

for some $r' \in C_p^\infty(\mathbb{R}^m \times \mathbb{R})$.

For $c > 0$, consider the substitution $x_i = c^{1/2}\overline{x_i}$ for $1 \le i < m$, $x_m = c^{1/3}\overline{x_m}$, $t = c^{2/3}\overline{t}$, and $y = c\overline{y}$. This has

$$f^*\overline{y} = a_1\overline{x}_1^2 + \cdots + a_{m-1}\overline{x}_{m-1}^2 + \overline{x}_m^3 + \overline{x}_m\overline{t} + c^{4/3}\overline{x}_m^4 r'(c^{1/2}\overline{x}_1, \ldots, c^{1/2}\overline{x}_{m-1}, c^{1/3}\overline{x}_m, c^{2/3}\overline{t}).$$

Thus, for arbitrarily small c we can make all higher derivatives become arbitrarily small. Hence, within every neighborhood of $a_1x_1^2 + \cdots + a_{m-1}x_{m-1}^2 + x_m^3 + x_m t$ in the C^∞ topology, there exists a germ that is equivalent to the given $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ germ. It suffices now to show that f is locally stable since then there exists a change of coordinates that can eliminate r', and this furthermore implies that f is 3-determined.

We can show f is locally stable by showing that the map $F(x,t) = (f(x,t),t)$ is locally stable using Corollary 1.4.11. Let $R = C_p^\infty(\mathbb{R}^m \times \mathbb{R})$ and let

$$A = \sum_{i=1}^m R\tfrac{\partial F}{\partial x_i} + R\tfrac{\partial F}{\partial t} + (f^2, t)C_p^\infty(F^*T\mathbb{R}^2) + \mathfrak{m}_p(\mathbb{R}^m \times \mathbb{R})^4 C_p^\infty(F^*T\mathbb{R}^2),$$

which is an R-submodule of $C_p^\infty(F^*T\mathbb{R}^2)$. What we need to show is that

$$C_p^\infty(F^*T\mathbb{R}^2) = A + \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R}f\tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2}.$$

We have

$$\begin{aligned}
\tfrac{\partial F}{\partial x_i} &= (2a_ix_i + x_m^4\tfrac{\partial r'}{\partial x_i})\tfrac{\partial}{\partial y_1} &&\equiv_A 2a_ix_i\tfrac{\partial}{\partial y_1} &&\text{for } 1 \le i < m\\
\tfrac{\partial F}{\partial x_m} &= (3x_m^2 + t + 4r'x_m^3 + x_m^4\tfrac{\partial r'}{\partial x_m})\tfrac{\partial}{\partial y_1} &&\equiv_A x_m^2(3 + 4r'x_m)\tfrac{\partial}{\partial y_1}\\
\tfrac{\partial F}{\partial t} &= x_m(1 + x_m^3\tfrac{\partial r'}{\partial t})\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_2} &&\equiv_A x_m\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_2},
\end{aligned}$$

where we have reduced these calculations modulo $(t)C_p^\infty(F^*T\mathbb{R}^2) + \mathfrak{m}_p(\mathbb{R}^m \times \mathbb{R})^4 C_p^\infty(F^*T\mathbb{R}^2)$ in particular. Since we can divide by $3 + 4r'x_m$ and $2a_i$ for all $1 \le i < m$, we see that

A contains $(x_1, \ldots, x_{m-1}, x_m^2, t)\frac{\partial}{\partial y_1}$. Using $x_m \frac{\partial}{\partial y_1} \equiv_A -\frac{\partial}{\partial y_2}$, we also see that A contains $(x_1, \ldots, x_m, t)\frac{\partial}{\partial y_2}$. Again using $\frac{\partial}{\partial y_2} \equiv_A -x_m \frac{\partial}{\partial y_1}$, then with the Hadamard lemma we deduce that

$$A + \mathbb{R}\frac{\partial}{\partial y_1} + \mathbb{R}\frac{\partial}{\partial y_2} = C_p^\infty(F^*T\mathbb{R}^2).$$

This completes the proof that f is locally stable. □

The results are summarized in Table 2.2.

Multijet Cerf singularities

For multijet singularities, like before we can take products of singularity types in the multijet bundle then take an intersection with an appropriate submanifold. We are only interested in when two singularities take the same value at the same time. Let $\mathbf{k} \in \mathbb{N}^k$ be a multiindex. The submanifold that represents taking the same value is $\beta^{-1}(\Delta^k_{\mathbb{R}}) \subseteq J^{\mathbf{k}}(M \times [0,1], \mathbb{R})$. For the submanifold that represents occuring at the same time, recall the source map $\alpha : J^{\mathbf{k}}(M \times [0,1], \mathbb{R}) \to (M \times [0,1])^k \setminus \mathbf{\Delta}^k_{M\times[0,1]}$, so, letting $\pi_t : M \times [0,1] \to [0,1]$ be the projection to time, we have a map $\pi_t^{\times k} \circ \alpha : J^{\mathbf{k}}(M \times [0,1], \mathbb{R}) \to [0,1]^k$, and thus $(\pi_t^{\times k} \circ \alpha)^{-1}(\Delta^k_{[0,1]})$ is the submanifold. Thus, we can intersect products of singularities in the multijet bundle with the submanifold

$$\beta^{-1}(\Delta^k_{\mathbb{R}}) \cap (\pi_t^{\times k} \circ \alpha)^{-1}(\Delta^k_{[0,1]}),$$

which has codimension $2k - 2$.

We ignore $S[{}^0_0]$ since such points are submersions. With $S[{}^1_0][r_1]|[{}^1_0][r_2]$ denoting the intersection of $S[{}^1_0][r_1] \times [{}^1_0][r_2]$ with the above submanifold in $J^{(2,2)}(M \times [0,1], \mathbb{R})$, we calculate the following codimensions in consultation with Table 2.2:

Type	$S[{}^1_0][0]\vert[{}^1_0][0]$	$S[{}^1_0][0]\vert[{}^1_0][1]$	$S[{}^1_0][1]\vert[{}^1_0][1]$
Codim.	$2m+2$	$2m+3$	$2m+4$

Since the domain for the multijet transversality theorem is $(M \times [0,1])^2 \setminus \mathbf{\Delta}^2_{M\times[0,1]}$, which has dimension $2m + 2$, this means only the first of the three generically occurs. Then we consider triple coincidences of this remaining type:

Type	$S[{}^1_0][0]\vert[{}^1_0][0]\vert[{}^1_0][0]$
Codim.	$3m+4$

This time $(M \times [0,1])^2 \setminus \mathbf{\Delta}^2_{M\times[0,1]}$ has dimension $3m + 3$, so they do not generically occur.

The $S[{}^1_0][0]|[{}^1_0][0]$ type singularity has dimension 0, so there are no further subtypes. Applying Theorem 1.2.12 (the Multijet Transversality Theorem) to all the multijet singularity types mentioned yields a residual subset of $C^\infty(M \times [0,1], \mathbb{R})$, which we can intersect with the residual subset we obtained from the last section to get a residual (and thus dense) subset of *generic* functions. These are the Cerf functions.

Definition 2.1.11. Let M and N be smooth manifolds with structure. Let $S \subseteq M$ be a finite set and let $q \in N$. We call $C^\infty_S(M,N)_q$ the collection of *multigerms*, and it is equivalently the disjoint union $\coprod_{p\in S} C^\infty_p(M,N)_q$. The group $(\prod_{p\in S}\mathrm{Diff}_p(M)) \times \mathrm{Diff}_q(N)$ acts on multigerms, and multigerms in the same orbit are called *equivalent multigerms*.[1]

Similarly, we say elements of $C^\infty_{S\times t_0}(M\times\mathbb{R},N)$ are *Cerf equivalent multigerms* if the corresponding multigerms $C^\infty_{S\times t_0}(M\times\mathbb{R},N\times\mathbb{R})$ with the second component being the identity on $\mathbb{R}$ are equivalent (like in Definition 2.1.4). $\diamondsuit$

Exact normal forms for multigerms are more difficult to come by since each germ in the multigerm has the same codomain, so there is less freedom to reparameterize. In this case, however, since Lemma 2.1.9 mostly applied domain transformations, we can still get an exact normal form. Let $p, \overline{p} \in \mathbb{R}^m$ be distinct points and let $q = 0 \in \mathbb{R}$.

Lemma 2.1.12. *If $f \in C^\infty_{(p,0)}(\mathbb{R}^m\times\mathbb{R},\mathbb{R})_q$ and $f' \in C^\infty_{(p',0)}(\mathbb{R}^m\times\mathbb{R},\mathbb{R})_q$ define a generic multigerm that is an $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$\begin{aligned} f(x,t) &= a_1x_1^2 + \cdots + a_mx_m^2 \\ f'(x',t) &= a_1'(x_1')^2 + \cdots + a_m'(x_m')^2 + t, \end{aligned}$$

where $(x_1,\ldots,x_m)$ are coordinates centered at p, $(x_1',\ldots,x_m')$ are coordinates centered at p', and where $a_1,\ldots,a_m,a_1',\ldots,a_m' \in \{-1,1\}$.

Proof. The germs together define a germ $f\times f' \in C^\infty_{\mathbf{p}}(\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R},\mathbb{R}\times\mathbb{R})$, where $\mathbf{p} = (p,0,p',0)$. Recall that we are considering the multijet submanifold

$$\beta^{-1}(\Delta^2_{\mathbb{R}}) \cap (\pi_t^{\times 2}\circ\alpha)^{-1}(\Delta^2_{\mathbb{R}}) \cap S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0],$$

and we need the multijet extension $j^{2,2}(f\times f')$ to be transverse to this submanifold at $j^{2,2}(f\times f')_{\mathbf{p}}$. The transversality condition associated to $(\pi_t^{\times 2}\circ\alpha)^{-1}(\Delta^2_{\mathbb{R}})$ results in us only needing to check transversality when both time variables are equal. Since $T\Delta^2_{\mathbb{R}}$ is spanned by $\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}$, the quotient of $T\mathbb{R}^2$ by this can be represented by differences of coefficients, so the $\beta^{-1}(\Delta^2_{\mathbb{R}})$ transversality condition is that the differences in the differentials of the two functions is surjective. Thus, the function we need to consider for this multisingularity is $F \in C^\infty_{\mathbf{p}}(\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R})$ defined by $F(x,x',t) = f(x,t) - f'(x',t)$.

Each germ is a $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$-type singularity, so their differentials are zero. Hence,

$$dF_{\mathbf{p}} = \begin{bmatrix} 0_{1\times m} & 0_{1\times m} & \partial_{01}f(p,0) - \partial_{01}f'(p',0)\end{bmatrix}.$$

By Cerf equivalence, we may assume that $\partial_{01}f(p,0) = 0$. This must be surjective to meet the $\beta^{-1}(\Delta^2_{\mathbb{R}})$ transversality condition, and thus $\partial_{01}f'(p',0) \neq 0$.

[1]Note that $\mathrm{Diff}_q(N)$ acts on the germs simultaneously.

Then we can essentially apply the argument of Lemma 2.1.9 to reparameterize the two germs as

$$f(x,t) = a_1 x_1^2 + \cdots + a_m x_m^2 + \varphi(t)$$
$$f(x,t) = a_1'(x_1')^2 + \cdots + a_m'(x_m')^2 + \varphi'(t).$$

By Cerf equivalence, we may assume that $\varphi(t) = 0$, and then since we established that the first derivative of φ' is nonzero, by the inverse function theorem we may reparameterize t to let us assume that $\varphi'(t) = t$. □

We have now handled multigerm singularities such that the germs are mapping to the same point, but there is still an additional type of multigerm singularity. To be able to generalize to foliations, what we have not done is consider germs that map to the same point after being projected to maps $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}/\mathbb{R}$. These functions are trivial in this case, but they do serve an important function: by making our Cerf functions generic with respect to this, what happens is that $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ singularities cannot happen at the same time since the codimension is $2m+3$ in a $(2m+2)$-dimensional space.

This completes the classification of Cerf functions. We summarize most of the structure of Cerf graphics here, and what would remain is to analyze the singularity types to see exactly how they fit together, as we will illustrate in Chapter 3 for cobordism categories.

Theorem 2.1.13. *Let f be a Cerf function in $C^\infty(M \times [0,1], \mathbb{R})$ (that is, f is in the dense subset of generic functions with respect to the jet and multijet submanifolds from this and the last section). Define $F : M \times [0,1] \to \mathbb{R} \times [0,1]$ by $F(x,t) = (f(x,t), t)$ and $\pi : \mathbb{R} \times [0,1] \to [0,1]$ by $\pi(y,t) = t$. Then:*

- *$M \times [0,1]$ is stratified by $S[\begin{smallmatrix}0\\0\end{smallmatrix}](f)$, $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$, and $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1](f)$ points.*
- *The $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ points are further stratified by whether or not they appear in one of the $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ pairs.*
- *The images of all these strata (except $S[\begin{smallmatrix}0\\0\end{smallmatrix}](f)$) through F are disjoint, and wherever $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ points have the same image they come in pairs as $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0](f)$ points.*
- *The images of the $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ points through $\pi \circ F$ are disjoint.*

The Cerf singularities are enumerated in Table 2.2. The meaning of the "suspends" column is whether the singularity is a suspension of another:

Definition 2.1.14. Let $m, n, s \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)$. The *suspension* of the germ f is the smooth map $F \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^s, \mathbb{R}^n \times \mathbb{R}^s)$ defined by $F(x,t) = (f(x), t)$.[2] ♢

[2] In singularity theory literature, this is the simplest kind of *unfolding* of a germ.

Remark 2.1.15. We usually imagine the codomain of a suspension is given the product foliation. If Morse functions are the objects of a category and Cerf functions are the morphisms, then suspension is the operation that gives identity morphisms. In the context of higher categories, the $s = 1$ suspension corresponds to taking an m-morphism and constructing its identity $(m+1)$-morphism. ■

Lemma 2.1.16. *Let $m, n, s \in \mathbb{N}$ and $f \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)$. Suppose f is a generic Thom–Boardman singularity. Then the suspension $F \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^s, \mathbb{R}^n \times \mathbb{R}^s)$ of f is a generic Thom–Boardman singularity, even when the codomain (but not the domain) is given the product foliation.*

Proof. This is a specialization of Lemma 2.1.6. □

Lastly, a word about the existence of Cerf functions between any two Morse functions. Given Morse functions $f_0, f_1 \in C^\infty(M, \mathbb{R})$, then we can see they can be turned into Cerf functions $f_0' \in C^\infty(M \times [0, \varepsilon], \mathbb{R})$ and $f_1' \in C^\infty(M \times [1-\varepsilon, 1], \mathbb{R})$ for $0 < \varepsilon < \frac{1}{2}$ by $f_0'(x, t) = f_1(x)$. The reason our versions of the transversality theorems (Theorems 1.2.9 and 1.2.12) also accept a closed subset is that they allow us to show that there exists a Cerf function $f \in C^\infty(M \times [0, 1], \mathbb{R})$ such that $f_0(x) = f(x, 0)$ and $f_1(x) = f(x, 1)$. All we need to do is take f_0' and f_1' together and fill in values for $M \times [-\varepsilon, 1-\varepsilon]$ in some arbitrary smooth way, which is possible by using a straight-line homotopy that we smooth at the endpoints using bump functions. Thus, since the set of Cerf functions is dense, every open neighborhood of this filled-in function contains a Cerf function.

If we assert that Cerf functions are equivalences in the category of Morse functions on a given smooth manifold, then all objects are isomorphic. In this sense, we can say that the choice of a Morse function can be made arbitrarily.

2.2 The Morin singularities

The Morin singularities [Mor65] are certain germs $f \in C_p^\infty(M, N)$ with $\dim M \leq \dim N$ such that $C_p^\infty(M)/C_p^\infty(M) f^* \mathfrak{m}_{f(p)}$ is isomorphic to $\mathbb{R}[x]/(x^{k+1})$ for some $k \geq 0$. Equivalently, they are the Thom–Boardman singularities with $j^k f_p \in S[1][1] \cdots [1][0]$ (and $j^k f$ transverse to this jet submanifold). The case of $\dim M = \dim N$ is explained in [GG73].

All the singularities we will meet are essentially Morin singularities, at least after forgetting additional structures such as foliations in the codomain — and our stratifications are generally refinements of the Thom–Boardman stratification. This is because of the low dimensions we consider: if $j^1 f_p \in S[r]$, then the second intrinsic derivative is a function

$$d^2 f_p : T_p M \to \mathrm{Hom}(K[r]_p, L[r]_p)$$

where $\dim K[r]_p = r$ and $\dim L[r]_p = \dim N - \dim M + r$, hence the codimension of $S[r](f)$ in M is $r(\dim N - \dim M + r)$. As Table 2.3 illustrates, when $\dim M \leq 3$ then, for the dense subset of germs with jet extensions transverse to $S[r]$ for all $r \in \mathbb{N}$, the only types that occur

Table 2.3: Generically occurring types $S[r]$ for maps $M \to N$ in low dimensions; sets indicate values of r. By definition, Morin singularities occur only outside the shaded region.

	dim N							
dim M	1	2	3	4	5	6	7	8
1	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0\}$	$\{0\}$	$\{0\}$
4	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0\}$
5	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
6	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1\}$
7	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
8	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Table 2.4: Generically occurring Morin singularity types for maps $M \to N$ in low dimensions. Each cell gives the minimal k for which $S[0]$, $S[1][0]$, …, $S[1]^k[0]$ are the only generically occurring Morin singularities. Shaded cells indicate possibility for non-Morin singularities.

	dim N							
dim M	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0
2		2	1	0	0	0	0	0
3			3	1	1	0	0	0
4				4	2	1	1	0
5					5	2	1	1
6						6	3	2
7							7	3
8								8

are $S[0]$ and $S[1]$. The large region where only $S[0]$ singularities occur can be summarized in the following classical theorem:

Theorem 2.2.1 (Weak Whitney Immersion Theorem)**.** *Let M and N be smooth manifolds with $2\dim M \le \dim N$. The dense subset of $C^\infty(M,N)$ from applying Theorem 1.2.9 to $\{S[r] \mid r \in \mathbb{N}\}$ consists only of immersions. Thus, every smooth map $f : M \to N$ can be Whitney C^∞ approximated by an immersion*

Proof. Let $X \subseteq C^\infty(M,N)$ be the described subset. By the previous discussion, since $2\dim M \le \dim N$, if $f \in X$ and $j^1 f \in S[r]$ then $r = 0$, and $j^1 f_p \in S[0]$ is exactly the condition that f is an immersion at p. $\square$

With essentially the same proof, we can give an analogous theorem for approximating smooth maps by ones with only Morin singularities. The bound clarifies the importance of Morin singularities in low dimensions:

Theorem 2.2.2. *Let M and N be smooth manifolds with $\dim M \leq \frac{2}{3}\dim N + 1$. The dense subset of $C^\infty(M,N)$ from applying Theorem 1.2.9 to the set of all Thom–Boardman singularity types consists only of maps whose germs are all Morin singularities.*

For sake of demonstration, we go through the analysis of Morin singularities within the Thom–Boardman framework. We do not derive the normal forms, and instead we reproduce the result from [Mor65] in Theorem 2.2.5.

Suppose that $m = \dim M$ and $n = \dim N$, that $m \leq n$, and that $f \in C^\infty_p(M,N)$ is generic and an $S[1]$ singularity. Let $K = \ker df_p$ and $L = \operatorname{coker} df_p$, the dimensions of which are $\dim K = 1$ and $\dim L = n - m + 1$. The second intrinsic derivative is a surjective map

$$d^2 f_p : T_pM \to \operatorname{Hom}(K, L)$$

Since K is one-dimensional, $T_pM \odot K = T_pM \otimes K$, so we do not need to worry about symmetry. The $S[1][r]$ type is for $\operatorname{drk} d^2 f_p|_K = r$, and if $r = 0$ then $d^3 f_p = 0$. Otherwise, for $S[1][1]$ we have $\ker d^2 f_p|_K = K$ and $\operatorname{coker} d^2 f_p|_K = \operatorname{Hom}(K, L)$. Thus, with $T_pS[1](f) = \ker d^2 f_p$, the third intrinsic derivative is a surjective map

$$d^3 f_p : T_pS[1](f) \to \operatorname{Hom}(K, \operatorname{Hom}(K, L)),$$

whose codomain we may write as $\operatorname{Hom}(K^{\otimes 2}, L)$. The $S[1][1][r]$ type is for $\operatorname{drk} d^3 f_p|_K = r$, and the cycle repeats; more formally:

Lemma 2.2.3. *Let $m = \dim M$, $n = \dim N$, $m \leq n$, and $f \in C^\infty_p(M,N)$. For $i \geq 1$, define $S[1]^i$ by $S[1]^1 = S[1]$ and $S[1]^{i+1} = S[1]^i[1]$. Suppose $k \geq 1$, $j^k f_p \in S[1]^k$, and $j^i f$ is transverse to $S[1]^i$ for all i. Then, with $K = \ker df_p$ and $L = \operatorname{coker} df_p$, the $(k+1)$th intrinsic derivative is a surjective map*

$$d^{k+1} f_p : T_pS[1]^k \to \operatorname{Hom}(K^{\otimes k}, L)$$

that in coordinates is given by restricting the iterated Jacobian $J^{k+1} f_p$ to $T_pS[1]^k \otimes K^{\otimes k}$ and taking the image in the quotient L. Thus if f is a $S[1]^k[0]$ singularity, in coordinates $J^{k+1} f_p$ restricted to $K^{\otimes(k+1)}$ is nonzero.

Furthermore, the dimension of $T_pS[1]^k$ is $m - k(n - m + 1)$. Hence, if $\mu = \lfloor \frac{m}{n-m+1} \rfloor$, the only Morin singularity types that generically occur are

$$S[0],\ S[1][0],\ S[1]^2[0],\ \ldots,\ S[1]^{\mu-1}[0],\ \text{and } S[1]^\mu[0].$$

Proof. Through our discussion we already showed all of this for $k = 1$ except for noting that $\dim T_pS[1]$ is $m - (n - m + 1)$ since $\dim \operatorname{Hom}(K, L) = n - m + 1$. We proceed by induction and assume k is such that all the hypotheses hold and that f is a $S[1]^{k+1}$ singularity. Then the restriction $d^{k+1}f_p|_K$ has rank 0, so the kernel is K and the cokernel is $\mathrm{Hom}(K^{\otimes k}, L)$. Since $\dim K^{\otimes k} = 1$ and $K \subseteq T_pS[1]^k$, the normal bundle calculation does not have to address symmetry of $d^{k+1}f_p$, hence the next intrinsic derivative is a map

$$d^{k+2}f_p : T_pS[1]^{k+1} \to \mathrm{Hom}(K, \mathrm{Hom}(K^{\otimes k}, L))$$

which by the tensor-hom adjunction is of the desired form. Lastly, we have $\dim T_pS[1]^{k+1} = \dim T_pS[1]^k - \dim L = m - (k+1)(n-m+1)$. □

Corollary 2.2.4. *Suppose* $\dim M \leq \frac{2}{3}\dim N + 1$ *and let "generic" refer to maps in the dense subset from Theorem 2.2.2. The only possible Morin singularity types that generically occur are*

$$S[0],\ S[1][0],\ \textit{and}\ S[1][1][0]$$

with the exception of $\dim M = \dim N = 3$*, in which case* $S[1][1][1][0]$ *is an additional possibility. Furthermore, if* $2\dim M \leq \dim N + 1$*, only* $S[0]$ *and* $S[1][0]$ *singularities generically occur. (See Table 2.4.)*

The following theorem is due to Morin in [Mor65]. Some examples are given in Table 2.5, Table 2.6, and Table 2.7.

Theorem 2.2.5. *Let* $m \leq n$ *and let* $p = 0 \in \mathbb{R}^m$*. For* $k > 0$*, suppose* $f \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)$ *is a* $S[1]^k[0]$ *singularity and generic (its jet extensions are transverse to* $S[1]^i$ *for all* $i \geq 0$*). Let* $q = n - m + 1$*. Then, up to equivalence,* f *is given by*

$$\text{for } 1 \leq j \leq m-1, \qquad f^*y_j = x_j$$

$$\text{for } 1 \leq j \leq q-1, \qquad f^*y_{m-1+j} = \sum_{i=1}^{k} x_{(j-1)k+i}x_m^i$$

$$f^*y_n = x_m^{k+1} + \sum_{i=1}^{k-1} x_{(q-1)k+i}x_m^i.$$

Furthermore, f *is locally stable and* $(k+1)$*-determined (Definition 1.4.6).*

Table 2.5: The generic germs $C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ are equivalent to one of the following. Note that the illustrations use x_2 for the third dimension to make the structure of each singularity type clearer. The $S[0]$ and $S[1][0]$ types are suspensions of the same types for $C_0^\infty(\mathbb{R}, \mathbb{R})$.

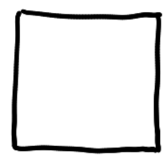
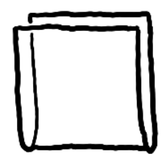
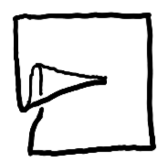
Type	Codim.	Representative germ	Illustration	Name
$S[0]$	0	$f^*y_1 = x_1$ $f^*y_2 = x_2$		local homeomorphism
$S[1][0]$	1	$f^*y_1 = x_1$ $f^*y_2 = x_2^2$		fold
$S[1][1][0]$	2	$f^*y_1 = x_1$ $f^*y_2 = x_1x_2 + x_2^3$		cusp

Table 2.6: The generic germs $C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ are equivalent to one of the following. This local model for the $S[1][0]$ singularity is known as a Whitney umbrella (also illustrated in Figure 2.1).

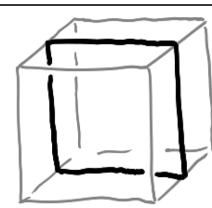
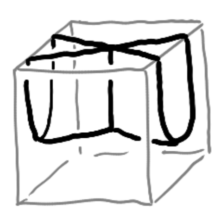
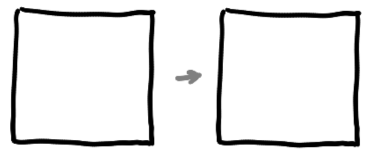
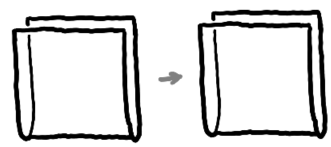
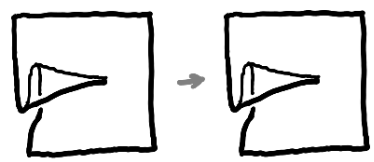
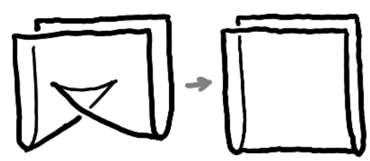
Type	Codim.	Representative germ	Illustration	Name
$S[0]$	0	$f^*y_1 = x_1$ $f^*y_2 = 0$ $f^*y_3 = x_2$		immersion
$S[1][0]$	2	$f^*y_1 = x_1$ $f^*y_2 = x_1x_2$ $f^*y_3 = x_2^2$		pinch point

Table 2.7: The generic germs $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ are equivalent to one of the following. The illustrations use x_2 as a time parameter, showing the $x_2 = -1$ and $x_2 = 1$ cross-sections. The swallowtail is named for the shape of the image of the $S[1][0]$ and $S[1][1][0]$ points in $\mathbb{R}^3$. The first three types are suspensions of their $C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$ counterparts.

Type	Codim.	Representative germ	Illustration	Name
$S[0]$	0	$f^*y_1 = x_1$ $f^*y_2 = x_2$ $f^*y_3 = x_3$		local homeomorphism
$S[1][0]$	1	$f^*y_1 = x_1$ $f^*y_2 = x_2$ $f^*y_3 = x_3^2$		fold
$S[1][1][0]$	2	$f^*y_1 = x_1$ $f^*y_2 = x_2$ $f^*y_3 = x_1x_3 + x^3$		cusp
$S[1][1][1][0]$	3	$f^*y_1 = x_1$ $f^*y_2 = x_2$ $f^*y_3 = x_1x_3 + x_2x_3^2 + x_3^4$		swallowtail[a]

[a] The illustration only shows one possible time parameterization, and other choices of time axes lead to very different illustrations.

2.3 Curves and graphs in the plane

In this section, we consider generic maps of 1-manifolds and graphs in the plane. Generic maps of closed 1-manifolds give the theory of *knot shadows*, which are diagrams of knots and links without over/under crossing information. We then extend this to be able to handle graphs by introducing vertices along the 1-manifolds. We also do all of this but with respect to the product foliation on the plane, which is a way to get a monoidal category description of curve and graph diagrams.

One of our motivations for working out the singularities for curves and graphs is that these correspond to surface graphs minus the surfaces. Surface graphs are strictly more difficult, and it is good to have a complete list of moves for the "2-categorical" decomposition of graphs to double check things later. Also, graphs are a case where we actually carry out the idea of doing an "$(n+1)$-categorical" decomposition for n-dimensional objects to get an n-category with a monoidal structure.

The principal complication with 1-manifolds in $\mathbb{R}^2$ is that their Cerf theory involves pinch

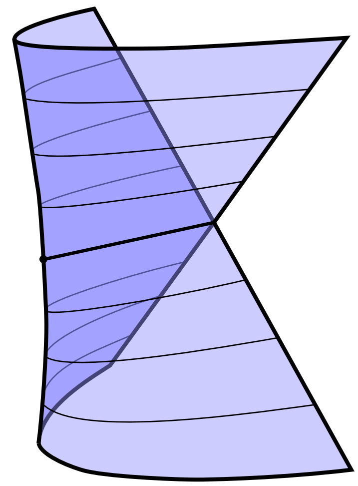


Figure 2.1: A 2D Whitney umbrella in $\mathbb{R}^3$ parameterized by $(s,t) \mapsto (st, s^2, t)$. The locus of double points ends in a pinch point singularity at $(0,0,0)$.

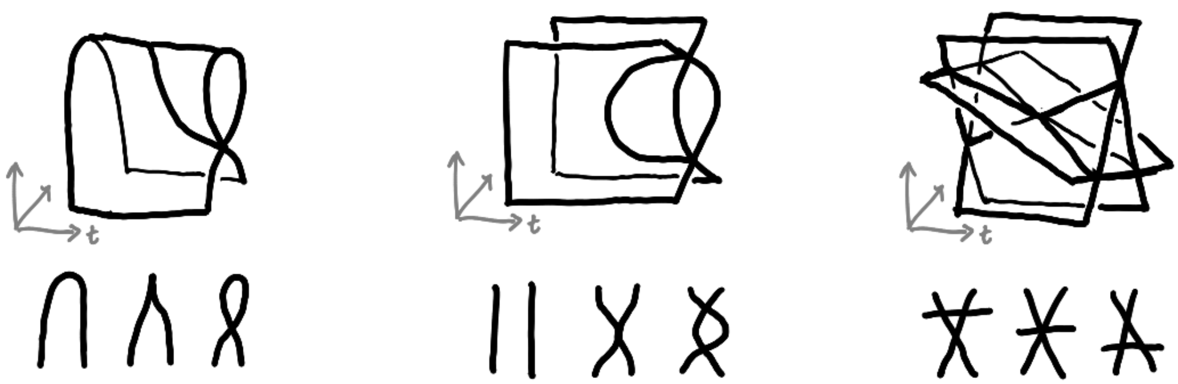


Figure 2.2: The three types of Cerf singularities for generic maps $C^\infty(C, \mathbb{R}^2)$, where C is a 1-manifold. For $f \in C^\infty(C \times \mathbb{R}, \mathbb{R}^2)$, the top illustrations are of $(x,t) \mapsto (f(x,t), t)$, and the bottom are of $t \in \{-1, 0, 1\}$ time slices. The knot diagram analogues of these moves are respectively known as the Reidemeister I, II, and III moves. In Table 2.9 they are called $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$, $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0][0]$, and $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$.

point singularities through time. Considered as a map $\mathbb{R}^2 \to \mathbb{R}^3$, these are type $S[1][0]$ Morin singularities, which are locally modeled by the Whitney umbrella; see Figure 2.1. Pinch point singularities correspond to the Reidemeister I move (see Figure 2.2), but they cause some trouble for the basic Thom–Boardman classification since the only submanifold that arises through the process is the pinch point itself, so there are no higher intrinsic derivatives to consider. This means that there is nothing to ensure that the move does not happen "all at once." That is to say, the locus of double points intersecting the leaves of the $\mathbb{R}^2 \times \mathbb{R}$ product foliation might not do so transversely at the pinch point.

This is certainly not the first time singularities of smooth homotopies of plane curves have been worked out. Dufour in [Duf83] found stable families of plane curves using generic diagrams $\mathbb{R} \xleftarrow{f} \mathbb{R}^2 \xrightarrow{\gamma} \mathbb{R}^2$ where f is a submersion. One side effect of this setup is that generic families curves are almost always in constant motion, a feature that our method does not share.

2.3.1 Curves

Letting C be a 1-manifold, we consider singularities for maps in $C^\infty(C, \mathbb{R}^2)$. Singularities for germs are very straightforward. As we discussed in Section 2.2, type $S[r] \subseteq J^1(C, \mathbb{R}^2)$ has codimension $1 + r$, thus generically only $S[0]$ points occur (in other words, Theorem 2.2.1 applies).

Multijets are also straightforward. The codimension of $\beta^{-1}(\Delta^2_{\mathbb{R}^2}) \subseteq J^{1,1}(\mathbb{R}, \mathbb{R}^2)$ in the multijet bundle is 2, and so the type $S[0]||[0]$ of pairs of $S[0]$ germs with the same image has codimension 2 in $\mathbb{R} \times \mathbb{R} \setminus \boldsymbol{\Delta}^2_{\mathbb{R}}$. Triple intersections do not generically occur because $S[0]||[0]||[0]$ ends up having codimension 4 in the three-dimensional space $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \setminus \boldsymbol{\Delta}^2_{\mathbb{R}}$.

As the following lemma shows, the normal form for the double point singularity is a pair of transversely intersecting lines, which we give a quick proof of. See, for example [GG73, III.3] details about the general case; multijet transversality for immersions gives normal crossings, which are locally modeled as intersecting vector spaces.

Lemma 2.3.1. *Let C be a smooth 1-manifold and $f : C^\infty(C, \mathbb{R}^2)$. Suppose $p, p' \in C$ are distinct points such that the multijet for $[f]_p$ and $[f]_{p'}$ is an $S[0]||[0]$ singularity. Then there are local coordinates x centered at p, x' centered at p', and (y_1, y_2) centered at $q = f(p) = f(p')$ such that*

$$\begin{aligned} f_p^* y_1 &= x & \qquad f_{p'}^* y_1 &= 0 \\ f_p^* y_2 &= 0 & \qquad f_{p'}^* y_2 &= x' \end{aligned}$$

(We write f_p for $[f]_p$ here and in tables.)

Proof. It is easy to check that being transverse to $\beta^{-1}(\Delta^2_{\mathbb{R}^2})$ implies that $T_pC + T_{p'}C = T_q\mathbb{R}^2$. Thus $C^\infty_{(p,p')}(C \times C, \mathbb{R}^2)$ defined by $(x, x') \mapsto f(x) + f(x')$ is locally a homeomorphism. □

Table 2.8: Classification of plane curve singularities.

Type	Codim.	Representative germ	See:
$S[0]$	0	$f^*y_1 = x$ $f^*y_2 = 0$	Theorem 2.2.5
$S[0]\|[0]$	$1\|1$	$f_p^*y_1 = x \quad f_{p'}^*y_1 = 0$ $f_p^*y_2 = 0 \quad f_{p'}^*y_2 = x'$	Lemma 2.3.1

Table 2.9: Classification of plane curve Cerf singularities.

Type	Codim.	Representative germ	See:	Suspends
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]$	0	$f^*y_1 = x$ $f^*y_2 = 0$	Lemma 2.3.7	$S[0]$
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$	2	$f^*y_1 = xt + x^3$ $f^*y_2 = x^2$	Lemma 2.3.6	
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}];[0\ 0]$	$1\|1$	$f_p^*y_1 = x \quad f_{p'}^*y_1 = 0$ $f_p^*y_2 = 0 \quad f_{p'}^*y_2 = x'$	Lemma 2.3.8	$S[0]\|[0]$
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0][0]$	$2\|2$	$f_p^*y_1 = x \quad f_{p'}^*y_1 = x'$ $f_p^*y_2 = 0 \quad f_{p'}^*y_2 = (x')^2 + t + O(x',t)^3$	Lemma 2.3.9	
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]$ [a]	$2\|2\|2$	$f_p^*y_1 = x \quad f_{p'}^*y_1 = 0 \quad f_{p''}^*y_1 = x''$ $f_p^*y_2 = 0 \quad f_{p'}^*y_2 = x' \quad f_{p''}^*y_2 = x'' + t$	Lemma 2.3.10	

[a] Representative germ is to first order.

Like usual, we define a *generic* map in $C^\infty(C, \mathbb{R}^2)$ to be one in the dense subset from applying Theorems 1.2.9 and 1.2.12 with all the $S[r]$ types for all $r \in \mathbb{N}$ and the multijet types $S[0]|[0]$ and $S[0]|[0]|[0]$. We summarize this in the following theorem:

Theorem 2.3.2. *Let $f \in C^\infty(C, \mathbb{R}^2)$ be generic. Then*

1. *f is an immersion;*
2. *no triple points occur (that is, $|f^{-1}(y)| \leq 2$ for all $y \in \mathbb{R}^2$); and*
3. *double points are locally modeled as transversely intersecting lines.*

Cerf singularities for curves

Now we consider Cerf singularities like in Section 2.1.2. We are considering $C_p^\infty(C \times \mathbb{R}, \mathbb{R}^2)$ where $C \times \mathbb{R}$ has the product foliation, and we use erf equivalence (Definition 2.1.4). Recall that the Cerf equivalence class of $f \in C_p^\infty(C \times \mathbb{R}, \mathbb{R}^2)$ corresponds to the equivalence class of the map $F : C \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ defined by $F(x,t) = (f(x,t), t)$ with respect to the codomain foliation.

The type $S[\begin{smallmatrix} r \\ 0 \end{smallmatrix}] \subseteq J^1(C \times \mathbb{R}, \mathbb{R}^2)$ denotes when the restriction of the differential to TC has dropped rank equal to r, with $r \in \{0, 1\}$. For $j^1 f_p \in S[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$, since $\ker df_p = 0$, the second intrinsic derivative is zero so there are no subtypes. For $j^1 f_p \in S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$, the second intrinsic derivative is a function

$$d^2 f_p : T_p(C \times \mathbb{R}) \to \mathrm{Hom}(K[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_p, L[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_p)$$

with $K[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_p = T_p C$ and $L[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]_p = T_{f(p)}\mathbb{R}^2$, hence $S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}](f)$ has codimension 2 in $C \times \mathbb{R}$. There are no more subtypes by continuing in this way.

At this point, it is not clear that the $S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ type is not sufficient — there do exist germs of this type that are not locally stable. We will now calculate a 2-jet normal form for this type, using the $p = 0 \in \mathbb{R} \times \mathbb{R}$ convention.

Lemma 2.3.3. *Suppose $f \in C_p^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ is such that $j^1 f$ is transverse to $S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ at p. There exists some $r_1, r_2 \in \mathfrak{m}_p^3$ such that f is Cerf equivalent to*

$$f(x,t) = (xt + r_1(x,t), x^2 + r_2(x,t)).$$

Proof. After a Cerf equivalence, we may assume $f(0,t) = 0$, and after a linear change of domain coordinates we may assume that

$$df_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the convention that $\partial_{ij}\varphi = \frac{\partial^{i+j}\varphi(x_1,x_2)}{\partial x_1^i \partial x_2^j}$, the second intrinsic derivative is, in coordinates,

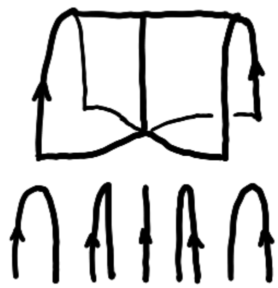
$$\begin{aligned} d^2 f_p : \mathbb{R} \times \mathbb{R} &\to \mathrm{Hom}(\mathbb{R}, \mathbb{R}^2) \\ v &\mapsto \begin{bmatrix} \partial_{20} f_1(p) & \partial_{11} f_1(p) \\ \partial_{20} f_2(p) & \partial_{11} f_2(p) \end{bmatrix} v \end{aligned}$$

where we identify $\mathrm{Hom}(\mathbb{R}, \mathbb{R}^2)$ with $\mathbb{R}^2$. Since $d^2 f_p$ is surjective by the transversality hypothesis, this matrix has rank 2. After a linear change of coordinates in the codomain, we may assume that $\partial_{20} f_2(p) = 2$ and $\partial_{20} f_1(p) = 0$. There is then a linear change of coordinates in the domain such that

$$\begin{bmatrix} \partial_{20} f_1(p) & \partial_{11} f_1(p) \\ \partial_{20} f_2(p) & \partial_{11} f_2(p) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}.$$

Since we assumed $\partial_{02} f_1(p) = 0$ and $\partial_{02} f_2(p) = 0$, we have shown that f is Cerf equivalent to a function with the desired 2-jet. □

If we look at the behavior of the second-order approximation $f(x,t) = (xt, x^2)$ as t varies, we see that there is a problem:



The top part of the figure is showing the image of $\mathbb{R} \times \mathbb{R}$ in $\mathbb{R}^2 \times \mathbb{R}$, and the bottom part is showing f_t for a handful of values of t. This qualitatively cannot be locally stable — how can all the double points happen all at once at $t = 0$? The Thom–Boardman stratification does not promise local stability, so it is not surprising that this might happen, but it is somewhat surprising that it fails in such low dimensions.

There a few ways we might solve this. A first is to appeal to the fact that, after we apply the multijet transversality theorem, we can assume the locus of double points away from this singularity will be transverse to the leaves of $\mathbb{R}^2 \times \mathbb{R}$ at all but finitely many points, so we can locally correct the problem after the fact since we know how it supposed to look. This is not satisfactory, however, since a purpose of this toy example is to show that we can derive singularities in a systematic way.

A second is to characterize the locus of the nearby double points. Forgetting the foliation on the domain, since this is a Morin $S[1][0]$ singularity (Table 2.6) it is locally stable, hence the topology of the double points is determined by just the 2-jet. With the parameterization immediately above, we see the double point locus is $(0,u)$ for $u \in \mathbb{R} \setminus 0$, and thus the closure is a smooth submanifold containing the pinch point $(0,0)$ itself. With this, we can justify speaking of the double point locus *at* the pinch point. Then we can try to construct a jet submanifold whose transversality condition ensures the double point locus is suitably generic (one should think about wanting to have it be generic with respect to the $\mathbb{R} \times \mathbb{R}$ foliation, being tangent to the leaf $\mathbb{R} \times \{0\}$ at a single point). The complication here is that this requires being able to calculate such a solution and determining how it varies with r_1 and r_2, and then finding a coordinate-invariant characterization of the transversality condition. To demonstrate the first half, we have the following lemma, which suggests that generically a function of the given form has $\partial_{30} r_1(0,0) \neq 0$:

Lemma 2.3.4. *Let $f \in C_p^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ be defined by $f(x,t) = (xt + r_1(x,t), x^2 + r_2(x,t))$ for some $r_1, r_2 \in \mathfrak{m}_p^3$. Define $F \in C_p^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2 \times \mathbb{R})$ by $F(x,t) = (f(x,t), t)$ Then if $[\gamma]_0 \in C_0^\infty(\mathbb{R}, \mathbb{R} \times \mathbb{R})_p$ satisfies $F(\gamma(u)) = F(\gamma(-u))$ for all $u \in \operatorname{dom}\gamma$ and $J\gamma_0 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, then*

$$F(\gamma(u)) = u^2 \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{6}\partial_{30} r_1(0,0) \end{bmatrix} + O(u)^3.$$

Proof. Calculate using third-order Taylor polynomials. □

A third (and related) method is to consider some kind of "probe," like in [Por83] (and explored further in [KÍ0]). Porteous considered ideas that we will recount here for motivation, but we do not use them directly. The first is that for germs $f \in C_0^\infty(\mathbb{R}^2, \mathbb{R})_0$, one can take a *probe* $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)_0$, compose this probe with derivatives of f,

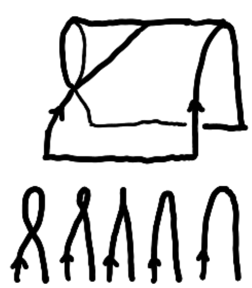
$$\mathbb{R} \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{d^m} \mathrm{Hom}((\mathbb{R}^2)^{\odot m}, \mathbb{R}),$$

and then take derivatives $d^{n+1}(d^m f \circ \varphi)_0$ of this composition evaluated at zero. For a given probe, one gets a set of all pairs (m, n) for which $d^{n+1}(d^m f \circ \varphi)_0 = 0$, and then one can consider the set of all sets of these pairs for all probes.

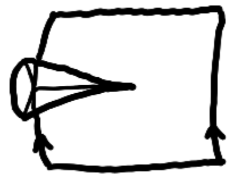
Remark 2.3.5. An invariant of germs up to equivalence is the collection of all these sets of pairs, and Porteous worked out "the best" sets to characterize many different singularity types, including the Thom–Boardman types (not just the generic ones, of which there are only two for maps $\mathbb{R}^2 \to \mathbb{R}$). Through this, Porteous discovered another description for the Thom–Boardman singularity types.

A *Boardman index* is a sequence $(r_1, \dots, r_k)$ of non-increasing natural numbers, and these sequences are ordered lexicographically. For a germ $f \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^n)$, a *probe* associated to the Boardman index consists of a family of germs $\varphi_i \in C_0^\infty(\mathbb{R}^{r_i}, \mathbb{R}^{r_{i-1}})$ with $d(\varphi_i)_0$ injective for all $1 \leq i \leq k$ (setting $r_0 = n$) such that $d(f\varphi_1)_0 = 0$, $d(d(f\varphi_1)\varphi_2)_0 = 0$, $d(d(d(f\varphi_1)\varphi_2)\varphi_3) = 0$, and so on. If such a probe exists and there exists no probe for a greater Boardman index with the same length k, then f is said to have type $S[r_1]\dots[r_k]$. An interesting property about probes of both types is that, for a fixed probe, the conditions (except for maximality) are linear in the germ f. ■

For exploratory purposes, let us see what the above Lemma 2.3.4 might give us. The foliation-respecting projection onto $(\mathbb{R}^2 \times \mathbb{R})/\mathbb{R}^2$ gives the map $u \mapsto -\frac{1}{6}u^2\partial_{30}r_1(0,0)$, which generically ought to be nonzero. If we then look at $f(x,t) = (xt + x^3, x^2)$ through time, we see that the problem is indeed resolved:



If we look at the map $(x,t) \mapsto (xt + x^3, x^2, t)$ projected onto the (y_1, y_3) plane, this turns out to be a cusp singularity ($S[1][1][0]$ in Table 2.5), which we illustrate with a bit of perspective here by looking at the previous illustration "from above":



The rough idea for to proceed is that we want to find an invariant way to define such a projection (a "coprobe") from a germ with a $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ singularity and then show that we can make the germ be generic with respect to this projection.

Consider $f \in C_p^\infty(C \times \mathbb{R}, \mathbb{R}^2)_q$ generic of type $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$, and recall that the second intrinsic derivative is a surjective function

$$d^2 f_p : T_p(C \times \mathbb{R}) \to \operatorname{Hom}(T_pC, T_q\mathbb{R}^2)$$

that is nonzero when restricted to T_pC. Thinking of $d^2f_p|_{T_pC}$ as a function $T_pC \odot T_pC \to T_q\mathbb{R}^2$, then we have a one-dimensional subspace $\operatorname{im} d^2 f_p|_{T_pC} \subseteq T_q\mathbb{R}^2$. Let $\varphi \in C_q^\infty(\mathbb{R}^2, \mathbb{R})_0$ be any function such that we have a short exact sequence

$$0 \to \operatorname{im} d^2 f_p|_{T_pC} \hookrightarrow T_q\mathbb{R}^2 \xrightarrow{d\varphi_q} T_0\mathbb{R} \to 0.$$

This φ serves as a projection $\mathbb{R}^2 \to \mathbb{R}$ near q, locally defining a codimension-1 foliation. We claim we can ask for f to be generic with respect to φ in a way that has no dependence on the choice of φ. Supposing we have chosen coordinates to put f into the form of Lemma 2.3.3, then, using only the assumptions that $\varphi(q) = 0$, $\partial_{1,0}\varphi(q) = 1$, and $\partial_{0,1}\varphi(q) = 0$, one can calculate that

$$(\varphi \circ f)(x,t) = xt + \tfrac{1}{6}\partial_{3,0}r_1(p)x^3 + \tfrac{1}{2}\partial_{2,1}r_1(p)x^2t + \tfrac{1}{2}\partial_{1,2}r_1(p)xt^2 + \tfrac{1}{6}\partial_{0,3}r_1(p)t^3 + O(x,t)^4.$$

(In fact, the dependence on the choice of φ only begins to appear in the fourth-order terms.) We temporarily let $S_\varphi\begin{bmatrix}a_1\\a_2\\0\end{bmatrix}(f) \subseteq C \times \mathbb{R}$ denote points for singularity types with respect to φ, where $p' \in S_\varphi\begin{bmatrix}a_1\\a_2\\0\end{bmatrix}(f)$ means that $a_1 = \operatorname{drk} df_{p'}|_{T_{p'}C}$ and $a_2 = \operatorname{drk} d(\varphi \circ f)_{p'}|_{T_{p'}C}$. We can see that, up to second-order, $d(\varphi \circ f)|_{TC}$ is t and $df|_{TC}$ is $[\begin{smallmatrix}t\\x\end{smallmatrix}]$, so it follows that $p \in S_\varphi\begin{bmatrix}1\\1\\0\end{bmatrix}(f)$ is the unique point of that type in a small enough neighborhood of p and, using "$*$" to denote "no constraint," that $S_\varphi\begin{bmatrix}1\\*\\0\end{bmatrix}(f)$ is a 1-dimensional submanifold germ of $C \times \mathbb{R}$ such that

$$T_pS_\varphi\begin{bmatrix}1\\*\\0\end{bmatrix}(f) = T_pC.$$

The second intrinsic derivatives with respect to φ for $S_\varphi\begin{bmatrix}1\\*\\0\end{bmatrix}$-type points form a bundle over $S_\varphi\begin{bmatrix}1\\*\\0\end{bmatrix}(f)$, where for $p' \in S_\varphi\begin{bmatrix}1\\*\\0\end{bmatrix}(f)$ it is a map

$$d^2 f_{p'} : T_{p'}(C \times \mathbb{R}) \to \operatorname{Hom}(T_{p'}C, T_{f(p')}\mathbb{R}^2 / \operatorname{im}(d(\varphi \circ f)_{p'})).$$

We define $S_\varphi\begin{bmatrix}1\\ *\\ 0\end{bmatrix}[b](f)$ to be those points $p' \in S_\varphi\begin{bmatrix}1\\ *\\ 0\end{bmatrix}(f)$ such that $\operatorname{drk}(d^2f_{p'}|T_{p'}C) = b$. Evidently, $S_\varphi\begin{bmatrix}1\\ *\\ 0\end{bmatrix}[1](f) = S[\begin{smallmatrix}1\\0\end{smallmatrix}](f)$. Thus, we can extract a third intrinsic derivative for f as a $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ singularity by using the fact that $p \in S_\varphi\begin{bmatrix}1\\ *\\ 0\end{bmatrix}[1](f)$. Using the fact that $T_pS_\varphi\begin{bmatrix}1\\ *\\ 0\end{bmatrix}(f) = T_pC$, the third intrinsic derivative is a linear map

$$d^3f_p : T_pC \to \operatorname{Hom}(T_pC, \operatorname{Hom}(T_pC, T_q\mathbb{R}^2/\operatorname{im} d^2f_p|_{T_pC})).$$

Both the domain and codomain of this function are 1-dimensional, and with some identification of the spaces with $\mathbb{R}$, we have $d^3f_p = \partial_{3,0}r_1(p)$, which does not depend on φ. The transversality condition for this singularity is that $\partial_{3,0}r_1(p) \neq 0$, matching up with expectations from the prior exploration.

How do we construct this jet submanifold? We start by lifting $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ to $S[\begin{smallmatrix}1\\0\end{smallmatrix}]' \subseteq J^3(C \times \mathbb{R}, \mathbb{R}^2)$. The second intrinsic derivatives define a bundle map

$$d^2 : S[\begin{smallmatrix}1\\0\end{smallmatrix}]' \to \operatorname{Hom}(T(C \times \mathbb{R}), \operatorname{Hom}(TC, T\mathbb{R}^2))$$

of bundles over $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$. We let $S[\begin{smallmatrix}1\\0\end{smallmatrix}]'' \subseteq S[\begin{smallmatrix}1\\0\end{smallmatrix}]'$ be those germs for which d^2 is surjective, which defines an open submanifold. Each $j^2f_p \in S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ has a corresponding one-dimensional subspace $\operatorname{im}(d^2f_p|_{T_pC})$, and by considering $T\mathbb{R}^2$ as a bundle over $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ (via the pullback over $\beta : S[\begin{smallmatrix}1\\0\end{smallmatrix}][1] \to \mathbb{R}^2$), these subspaces assemble into a subbundle B of $T\mathbb{R}^2$. For each jet in $j^3f_p \in S[\begin{smallmatrix}1\\0\end{smallmatrix}]''$, for each $j^3f_p \in S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]'$, as we saw there is a well-defined map $d^3f_p : T_pC \to \operatorname{Hom}(T_pC, \operatorname{Hom}(T_pC, T_q\mathbb{R}^2/\operatorname{im} d^2f_p|_{T_pC}))$ where $\varphi \in C^\infty_{f(p)}(\mathbb{R}^2, \mathbb{R})$, and by using B these assemble into a bundle map from $S[\begin{smallmatrix}1\\0\end{smallmatrix}]''$. Then define $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][r]$ for $0 \le r \le 1$ is those jets $j^3f_p \in S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]''$ for which $\operatorname{drk} d^3f_p = r$, where the "[1]" in the name comes from thinking of $d^2f_p|_{T_pC}$ as having dropped rank equal to one. For codimension reasons, only the $r = 0$ occurs for generic functions.

Lemma 2.3.6. *If $f \in C^\infty_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ is a generic germ that is an $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$ singularity, then, up to Cerf equivalence, it is given by*

$$f^*y_1 = xt + x^3$$
$$f^*y_2 = x^2$$

Proof. Continuing from where we left off in Lemma 2.3.3, we found the second intrinsic derivative could be put into the form

$$\begin{aligned} d^2f_p : \mathbb{R} \times \mathbb{R} &\to \operatorname{Hom}(\mathbb{R}, \mathbb{R}^2) \\ v &\mapsto \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} v. \end{aligned}$$

Then the third intrinsic derivative, justified above, is

$$\begin{aligned} d^3f_p : \mathbb{R} &\to \operatorname{Hom}(\mathbb{R}, \operatorname{Hom}(\mathbb{R}, \mathbb{R}^2/0 \times \mathbb{R})) \\ v &\mapsto \partial_{30}f_1(p)v \end{aligned}$$

This does not need to be surjective, but since $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ has codimension-2 already, we can conclude $\partial_{30} f_1(p) \neq 0$.

Using techniques from the parameterized version of the Morse Lemma (Lemma 2.1.9), we can reparameterize x such that $f^* y_2 = x^2$ exactly, and this does not change the above analysis. After scaling x, t, and y_2, we may assume that $\partial_{30} f_1(p) = 6$, and by reparameterizing (as a Cerf equivalence) with

$$\overline{y}_1 = y_1 - \tfrac{1}{2}\partial_{21}h(p)y_2y_3 - \tfrac{1}{2}\partial_{12}h(p)y_1y_3 - \tfrac{1}{6}\partial_{03}h(p)y_3^3 \qquad \overline{y}_2 = y_2 \qquad \overline{y}_3 = y_3$$

we may assume that we have, for some $h \in \mathfrak{m}_p^4$,

$$\begin{aligned} f^* y_1 &= xt + x^3 + h(x,t) \\ f^* y_2 &= x^2. \end{aligned}$$

Continuing in this way (using the lower order terms of f^*y_1, f^*y_2, and f^*y_3), we can apply another Cerf reparameterization to let us assume that $h \in \mathfrak{m}_p^5$, and again to let us assume that $h \in (x^5) + \mathfrak{m}_p^6$, where we cannot eliminate the x^5 by a Cerf reparameterization. We will show that for all $k \in \mathbb{N}$ we are able to reparameterize the germ to put it into the form

$$\begin{aligned} f^* y_1 &= xt + x^3 + cx^{5+2k} + h(x,t) \\ f^* y_2 &= x^2. \end{aligned}$$

with $c \in \mathbb{R}$ and $h \in \mathfrak{m}_p^{6+2k}$, where we have already shown the base case. We see that $f^*y_1 = xt + x^3(1 + cx^{2+2k}))$. Consider the reparameterization

$$\begin{aligned} \overline{x} &= \frac{x}{1 + cx^{2+2k}} & \overline{t} &= t & & \\ \overline{y}_1 &= \frac{y_1}{1 + cy_2^{1+k}} & \overline{y}_2 &= \frac{y_2}{(1 + cy_2^{1+k})^2} & \overline{y}_3 &= y_3. \end{aligned}$$

Since $\frac{x}{1+cx^{2+2k}} = x - cx^{3+2k} + \dots$, we have that $x = \overline{x} + g(\overline{x})$ for some $g \in \mathfrak{m}_0(\mathbb{R})^{3+2k}$. Then,

$$\begin{aligned} f^*\overline{y}_1 &= \frac{1}{1 + cx^{2+2k}}(xt + x^3(1 + cx^{2+2k})) = \overline{x}\overline{t} + x^3 = \overline{x}\overline{t} + \overline{x}^3 + h'(\overline{x}, \overline{t}) \\ f^*\overline{y}_2 &= \frac{x^2}{(1 + c(x^2)^{1+k})^2} = \overline{x}^2 \\ f^*\overline{y}_3 &= \overline{t}. \end{aligned}$$

for some $h' \in \mathfrak{m}_p^{5+2k}$. We can then clear the degree-$(5 + 2k)$ terms from h' by a Cerf reparameterization like before, letting us assume that $h' \in \mathfrak{m}_p^{6+2k}$, and then we can clear every degree-$(6 + 2k)$ term expect for, possibly, x^{6+2k}, which lets us assume that $f^*y_1 = xt + x^3 + cx^{7+2k} + h(x,t)$ for some $c \in \mathbb{R}$ and $h \in \mathfrak{m}_p^{8+2k}$, as desired.

Having proved this, we have a sequence of equivalent germs for all k that converge in the Whitney C^∞ topology to the germ given by $(x,t) \mapsto (xt + x^3, x^2)$, so it suffices to prove that this germ is locally stable. Suppose f is this germ, thought of as a germ in $C_p^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^3)_0$.

We use Corollary 1.4.13. Letting $I = (f_1, f_2)^2 + (f_3)$ be an ideal of $C_p^\infty(\mathbb{R} \times \mathbb{R})$, we calculate that $I = (x^4, t)$. It suffices to show that the quotient

$$A = C_p^\infty(f^*\mathbb{R}^3)/(f_*C_p^\infty(T(\mathbb{R} \times \mathbb{R})) + IC_p^\infty(f^*T\mathbb{R}^3) + \mathfrak{m}_p(M)^6 C_p^\infty(f^*T\mathbb{R}^3))$$

is a vector space generated by the set

$$S = \left\{ \tfrac{\partial}{\partial y_1}, \tfrac{\partial}{\partial y_2}, \tfrac{\partial}{\partial y_3}, f_1\tfrac{\partial}{\partial y_1}, f_1\tfrac{\partial}{\partial y_2}, f_2\tfrac{\partial}{\partial y_1}, f_2\tfrac{\partial}{\partial y_2} \right\}.$$

We calculate that

$$\begin{aligned} f_*\tfrac{\partial}{\partial x_1} &= (t + 3x^2)\tfrac{\partial}{\partial y_1} + 2x\tfrac{\partial}{\partial y_2} \equiv_A 3x^2\tfrac{\partial}{\partial y_1} + 2x\tfrac{\partial}{\partial y_2} \\ f_*\tfrac{\partial}{\partial x_2} &= x\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_3}. \end{aligned}$$

Thus, for all $k \in \mathbb{N}$ we have the reduction rules

$$\begin{aligned} x^{k+1}\tfrac{\partial}{\partial y_2} &\equiv_A -\tfrac{3}{2}x^{k+2}\tfrac{\partial}{\partial y_1} \\ x^k\tfrac{\partial}{\partial y_3} &\equiv_A -x^{k+1}\tfrac{\partial}{\partial y_1}, \end{aligned}$$

and so we can see that A is generated as a vector space by the set $\{\frac{\partial}{\partial y_1}, x\frac{\partial}{\partial y_1}, x^2\frac{\partial}{\partial y_1}, x^3\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$. We calculate

$$\begin{aligned} \tfrac{\partial}{\partial y_1} &= \tfrac{\partial}{\partial y_1} & f_1\tfrac{\partial}{\partial y_1} &\equiv_A x^3\tfrac{\partial}{\partial y_1} \\ \tfrac{\partial}{\partial y_2} &= \tfrac{\partial}{\partial y_2} & f_1\tfrac{\partial}{\partial y_2} &\equiv_A x^3\tfrac{\partial}{\partial y_2} \equiv_A -\tfrac{3}{2}x^4\tfrac{\partial}{\partial y_1} \equiv_A 0 \\ \tfrac{\partial}{\partial y_3} &\equiv_A -x\tfrac{\partial}{\partial y_1} & f_2\tfrac{\partial}{\partial y_1} &= x^2\tfrac{\partial}{\partial y_1} \\ & & f_2\tfrac{\partial}{\partial y_2} &\equiv_A -\tfrac{3}{2}x^4\tfrac{\partial}{\partial y_1} \equiv_A 0. \end{aligned}$$

Therefore A is indeed generated as a vector space by the set S. □

Multijet Cerf singularities for curves

Now that we have placed our Whitney umbrellas in the umbrella stand, let us return to the general analysis. We found all the germ Cerf singularities (which we list in Table 2.9), so next are multijet singularities. The process is similar to Section 2.1.2, but we are looking at intersections with the submanifold

$$\beta^{-1}(\Delta^k_{\mathbb{R}^2}) \cap (\pi_t^{\times k} \circ \alpha)^{-1}(\Delta^k_{\mathbb{R}}) \subseteq J^{\mathbf{k}}(C \times \mathbb{R}, \mathbb{R}^2),$$

where $\mathbf{k} \in \mathbb{N}^k$ is a multiindex. The codimension of this submanifold is $3k-3$. Since the codimensions of $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ and $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$ are respectively 0 and 2, we get the following codimensions for multijet singularities between two singularity types:

Type	$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]$	$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$	$S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]\|[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$
Codim.	3	5	7

Since the domain for the multijet transversality theorem is $(C \times \mathbb{R})^2 \setminus \boldsymbol{\Delta}^2_{C\times\mathbb{R}}$, which has dimension 4, only $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$ can generically occur. For the next table, we look at codimensions for more combinations of the $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularity:

Type	$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]$	$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]$	$S[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]\|[\begin{smallmatrix}0\\0\end{smallmatrix}]$
Codim.	3	6	9
Ambient dim.	4	6	8

Thus, the only additional possibility is triple intersections.

An additional wrinkle compared to the analysis of double points earlier in Lemma 2.3.1 is that curves are now allowed to become momentarily tangent, which requires higher multijet singularity types.

We start with $f \in C^\infty_{\{(p,t_0),(p',t_0)\}}(C \times \mathbb{R}, \mathbb{R}^2)_q$ with $p \neq p'$, which we assume is a generic multigerm of type $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$. To deal with transversality for $\beta^{-1}(\Delta^2_{\mathbb{R}^2})$, we immediately pass to the quotient space $(T\mathbb{R}^2)^2/T\Delta^2_{\mathbb{R}^2}$ by subtracting corresponding codomain coordinates. And for transversality to $(\pi^{\times 2} \circ \alpha)^{-1}(\Delta^2_{\mathbb{R}})$ we identify t coordinates. The function we thus need to consider is $F(x, x', t) = f(x, t) - f(x', t)$, whose differential is

$$\begin{aligned} dF_{(p,t_0),(p',t_0)} &: T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R} \to T_q\mathbb{R}^2 \\ & v \mapsto \begin{bmatrix} \partial_{10} f_1(p) & -\partial_{10} f_1(p') & \partial_{01} f_1(p) - \partial_{01} f_1(p') \\ \partial_{10} f_2(p) & -\partial_{10} f_2(p') & \partial_{01} f_2(p) - \partial_{01} f_2(p') \end{bmatrix} v. \end{aligned}$$

Since both germs are $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularities, restricted to TC they are both nonzero. After a change of coordinates, we may assume the differential has a matrix of the form

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -a & b \end{bmatrix}$$

for some constants a and b. For subtypes, we have the subspaces $T_{(p,t_0)}C$, $T_{(p',t_0)}C$, their direct sum, and $T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R}$. The differential is injective when restricted to either TC individually, so we define $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[r_1\ r_2]$ to be the type where the dropped rank on $T_{(p,t_0)}C \oplus T_{(p',t_0)}C$ is r_1 and the dropped rank on $T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R}$ is r_2. If $r_2 > 0$, then its second intrinsic derivative brings the total codimension to 5, which does not generically occur. The type $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[0\ 0]$ has no subtypes, so we look toward $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0]$. We get a second intrinsic derivative

$$d^2F_{(p,p',t_0)} : T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R} \to \mathrm{Hom}(K[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0], L[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0])$$

where $K[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0]$ is spanned by $(1, 1, 0)$ and $L[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0]$ is $T_q\mathbb{R}^2/T_q\mathbb{R}$, so the total codimension for this singularity type is 4, and thus still generically possible. Remembering that there are *two* spatial variables, so derivatives of one function with respect to the other spatial variable is zero, as a matrix we can write d^2F as

$$v \mapsto \left[\partial_{20} f_2(p,t_0) \quad -\partial_{20} f_2(p',t_0) \quad \partial_{11} f_2(p,t_0) - \partial_{11} f_2(p',t_0)\right] v$$

We can restrict to $K[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$;[1 0] to get further subtypes, and for codimension reasons generically we need the function to not be zero, hence $\partial_{20} f_2(p,t_0) \neq \partial_{20} f_2(p',t_0)$. We call this type $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$;[1 0][0].

Lemma 2.3.7. *If $f \in C^\infty_p(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ is a generic germ that is an $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularity, then, up to Cerf equivalence, it is given by*

$$f^* y_1 = x$$
$$f^* y_2 = 0$$

Proof. This is essentially the same as Lemma 2.1.8. □

Lemma 2.3.8. *If $f \in C^\infty_{\{(p,0),(q,0)\}}(C\times\mathbb{R},\mathbb{R}^2)_q$ defines a generic multigerm that is an $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$;[0 0] multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$f^*_{(p,0)} y_1 = x \qquad f^*_{(p',0)} y_1 = 0$$
$$f^*_{(p,0)} y_2 = 0 \qquad f^*_{(p',0)} y_2 = x'$$

Proof. This is similar to Lemma 2.3.1, but we use the local homeomorphism $(x,x',t) \mapsto (f(x,t)+f(x',t),t)$ to reparameterize. □

Lemma 2.3.9. *If $f \in C^\infty_{\{(p,0),(q,0)\}}(C\times\mathbb{R},\mathbb{R}^2)_q$ defines a generic multigerm that is an $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$;[1 0][0] multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$f^*_{(p,0)} y_1 = x \qquad f^*_{(p',0)} y_1 = x'$$
$$f^*_{(p,0)} y_2 = 0 \qquad f^*_{(p',0)} y_2 = (x')^2 + O(x',t)^3$$

Proof. There is a reparameterization of the codomain to get $f_1(x,t) = x$ and $f_2(x,t) = 0$, and by the inverse function theorem we may assume $f_1(x',t) = x'$. By assumption about the multisingularity type, then $f_2(x',t) = a(x')^2 + bx't + ct + dt^2 + h(x,t)$ for some constants with $a \neq 0$ and $c \neq 0$ and $h \in \mathfrak{m}^3_{(p',0)}$. Using the inverse function theorem, we can reparameterize t so that $c = 1$ and $d = 0$, and we may reparameterize such that $a = 1$. Then we can complete the square for x' to eliminate the $x't$ term, which introduces a t term in $f_1(x',t)$, which we can cancel out using a Cerf reparameterization, but this introduces a t term in $f_1(x,t)$, which we can cancel out by reparameterizing x. Hence, we have

$$f^*_{(p,0)} y_1 = x \qquad f^*_{(p',0)} y_1 = x'$$
$$f^*_{(p,0)} y_2 = 0 \qquad f^*_{(p',0)} y_2 = (x')^2 + t + h(x,t)$$

for some $h \in \mathfrak{m}^3_{(p',0)}$. □

Lemma 2.3.10. *If $f \in C^\infty(C \times \mathbb{R}, \mathbb{R}^2)$ is such that for three distinct points $p, q, r \in \mathbb{R}$ that $f(p,0) = f(q,0) = f(r,0)$ and the germs of f at these three points define a generic multigerm of type $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$, then, up to Cerf equivalence, the multigerm is, to first order, given by*

$$\begin{aligned} f_p^* y_1 &= x & f_q^* y_1 &= 0 & f_r^* y_1 &= x'' \\ f_p^* y_2 &= 0 & f_q^* y_2 &= x' & f_r^* y_2 &= x'' + t \end{aligned}$$

Proof. To avoid tedium, we appeal to the fact that the codimension begins at 6 for triple singularities, so nontrivial higher intrinsic derivatives to not occur generically. Hence restricted to each germ we have immersions through time, and restricted to pairs they intersect transversely. Also, this can be given as a zeroth-order type, so everything must be expressable as linear functions. □

While we handled multijet singularities for germs that map to the same point, we still need to consider multijet singularities for when germs are projected to be maps $C \times \mathbb{R} \to \mathbb{R}^2/\mathbb{R}^2$. Such functions are trivial still (and it will get more interesting once the codomain has a foliation) but there is still a consequence, which is that none of the following singularities generically occur at the same time: $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1][0]$, $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}];[1\ 0][0]$, and $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$. This handles multijet singularities.

2.3.2 2D Morse and Cerf theory for curves

For a monoidal category decomposition of 1-manifolds, we can consider generic maps $C \to \mathbb{R}^2$ where $\mathbb{R}^2$ is given the product foliation (or in other words, generic cascades $C \to \mathbb{R}^2 \twoheadrightarrow \mathbb{R}$). These are 2D Morse functions. The result of this analysis is well known (see Figure 2.3 for illustrations of the singularities), but it serves as another example for using the refined Thom–Boardman classification to derive normal forms for singularities.

This is not going to be too much different from Table 2.8 and Table 2.9, and the main difference is that the projection of generic 1-manifolds to $\mathbb{R}^2/\mathbb{R}$ must be a Morse function as well, so we will see local minima and maxima as well as the 2D version of the 1D Cerf singularity. The results are listed in Table 2.10 and Table 2.9. Once we later have all the singularities for curves on surfaces, we can see that this is a complete set of singularities for the shadows of those curves.

Let C be a 1-manifold. A germ $f \in C_p^\infty(C, \mathbb{R}^2)_q$ has type $S[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}]$ if the dropped rank of df_p is r_1 and if the dropped rank of df_p when projected to $\mathbb{R}^2/\mathbb{R}$ is r_2. These must satisfy $0 \le r_1 \le r_2 \le 1$, and the following are all the types along with their codimensions and simplified normal bundles:

Type	$K[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}]$	$L[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}]$	$\mathrm{Hom}(K[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}], L[\begin{smallmatrix}r_1\\r_2\end{smallmatrix}])$	Codim.
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]$	$0 \hookrightarrow 0$	0	0	0
$S[\begin{smallmatrix}0\\1\end{smallmatrix}]$	$0 \hookrightarrow T_pC$	$T_q\mathbb{R}^2/T_q\mathbb{R} \twoheadrightarrow T_q\mathbb{R}^2/T_q\mathbb{R}$	$\mathrm{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R})$	1
$S[\begin{smallmatrix}1\\1\end{smallmatrix}]$	$T_pC \hookrightarrow T_pC$	$T_q\mathbb{R}^2 \twoheadrightarrow T_q\mathbb{R}^2/T_q\mathbb{R}$	$\mathrm{Hom}(T_pC, T_q\mathbb{R}^2)$	2

Table 2.10: Classification of plane curve 2D Morse singularities.

Type	Codim.	Representative germ	See:
$S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	0	$f^*y_1 = 0$ $f^*y_2 = x$	Lemma 2.3.11
$S\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$	1	$f^*y_1 = x$ $f^*y_2 = x^2$	Lemma 2.3.12
$S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] \vert \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$	$1\vert 1$	$f_p^*y_1 = 0 \quad f_{p'}^*y_1 = x'$ $f_p^*y_2 = x \quad f_{p'}^*y_2 = x'$	Lemma 2.3.13

Table 2.11: Classification of plane curve 2D Cerf singularities.

Type	Codim.	Representative germ	See:	Suspends
$S\begin{bmatrix}0\\0\\0\end{bmatrix}$	0	$f^*y_1 = 0$ $f^*y_2 = x$	Lemma 2.3.14	$S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$
$S\begin{bmatrix}0\\1\\0\end{bmatrix}[0]$	1	$f^*y_1 = x$ $f^*y_2 = x^2$	Lemma 2.3.15	$S\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$
$S\begin{bmatrix}0\\1\\0\end{bmatrix}[1]$	2	$f^*y_1 = x$ $f^*y_2 = xt + x^3$	Lemma 2.3.16	
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right][0]$	2	$f^*y_1 = xt + x^3$ $f^*y_2 = x^2$	Lemma 2.3.17	
$S\begin{bmatrix}0\\0\\0\end{bmatrix} \vert \begin{bmatrix}0\\0\\0\end{bmatrix};[0\ 0]$	$1\vert 1$	$f_p^*y_1 = 0 \quad f_{p'}^*y_1 = x'$ $f_p^*y_2 = x \quad f_{p'}^*y_2 = x'$	Lemma 2.3.18	$S[0] \vert [0]$
$S\begin{bmatrix}0\\0\\0\end{bmatrix} \vert \begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0][0]$	$2\vert 2$	$f_p^*y_1 = 0 \quad f_{p'}^*y_1 = (x')^2 + t + O(x',t)^3$ $f_p^*y_2 = x \quad f_{p'}^*y_2 = x'$	Lemma 2.3.19	
$S\begin{bmatrix}0\\0\\0\end{bmatrix} \vert \begin{bmatrix}0\\1\\0\end{bmatrix}[0];[0]$	$2\vert 2$	$f_p^*y_1 = t + O(x,t)^2 \quad f_{p'}^*y_1 = x'$ $f_p^*y_2 = x \quad f_{p'}^*y_2 = (x')^2$	Lemma 2.3.20	
$S\begin{bmatrix}0\\0\\0\end{bmatrix} \vert \begin{bmatrix}0\\0\\0\end{bmatrix} \vert \begin{bmatrix}0\\0\\0\end{bmatrix}$ [a]	$2\vert 2\vert 2$	$f_p^*y_1 = x \quad f_{p'}^*y_1 = -x' \quad f_{p''}^*y_1 = t$ $f_p^*y_2 = x \quad f_{p'}^*y_2 = x' \quad f_{p''}^*y_2 = x''$	Lemma 2.3.21	

[a] Representative germ is to first order.

Figure 2.3: Illustrations of the 2D Morse singularities from Table 2.10.

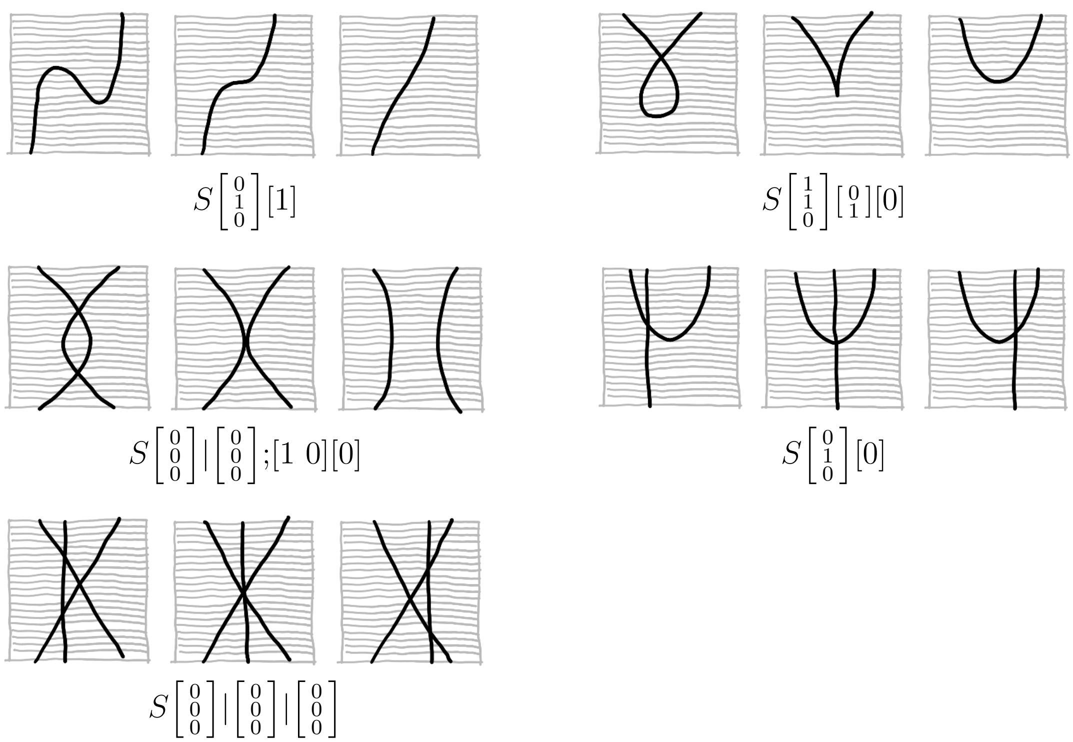


Figure 2.4: Illustrations of the 2D Cerf singularities from Table 2.11 except for suspensions. Each set of three diagrams is showing the $t = -1$, $t = 0$, and $t = 1$ frames of a movie. Interchange moves are in Figure 2.6.

Only the first two types occur generically. The first has no subtypes, and neither does the second due to codimension. Like usual, let $p = 0 \in \mathbb{R}$ and $q = 0 \in \mathbb{R}^2$, and let x the coordinate for $\mathbb{R}$ centered at p and (y_1, y_2) be coordinates for $\mathbb{R}^2$ centered at q.

Lemma 2.3.11. *If $f \in C_p^\infty(\mathbb{R}, \mathbb{R}^2)_q$ is a generic $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularity, then up to equivalence it is given by*

$$
\begin{aligned}
f^* y_1 &= 0 \\
f^* y_2 &= x
\end{aligned}
$$

Proof. For this type, $df_p = [\begin{smallmatrix}a\\b\end{smallmatrix}]$ for $a, b \in \mathbb{R}$ and $b \neq 0$. Using the coordinate change $\overline{x} = f_2(x)$ (which is valid by the inverse function theorem), we may assume $f^* y_2 = x$. The following is a foliation-respecting coordinate change:

$$
\begin{aligned}
\overline{y}_1 &= y_1 - f_1(y_2) \\
\overline{y}_2 &= y_2
\end{aligned}
$$

and $f^* \overline{y}_1 = f^* y_1 - f^* f_1(y_2) = f_1(x) - f_1(x) = 0$. This gives the desired form. $\square$

Lemma 2.3.12. *If $\in C_p^\infty(\mathbb{R}, \mathbb{R}^2)_q$ is a generic $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ singularity, then up to equivalence it is given by*

$$
\begin{aligned}
f^* y_1 &= x \\
f^* y_2 &= x^2
\end{aligned}
$$

Proof. For this type, after scaling y_1 we may assume that $df_p = [\begin{smallmatrix}1\\0\end{smallmatrix}]$. The second intrinsic derivative is

$$
\begin{aligned}
d^2 f_p : T_p\mathbb{R} &\to \operatorname{Hom}(T_p\mathbb{R}, T_q\mathbb{R}^2 / T_q\mathbb{R}) \\
v &\mapsto \left[\partial_2 f_2(p)\right] v,
\end{aligned}
$$

which for codimension reasons means $\partial_2 f_2(p) \neq 0$. Thus, $f_2(x) = h(x)x^2$ for some $h \in C_p^\infty(\mathbb{R})$ such that $h(x) \neq 0$. With the coordinate change $\overline{x} = \sqrt{|h(x)|}x$, we may assume that $f_2(x) = \pm x^2$. By the inverse function theorem, $\overline{y}_1 = y_1 \circ f_1$ and $\overline{y}_2 = y_2$ is a valid coordinate change, and $f^* \overline{y}_1 = y_1(f_1(x)) = x$. This gives the desired form. $\square$

The analysis of multijet singularities from Section 2.3.1 carries over to imply that only $S[\begin{smallmatrix}0\\*\end{smallmatrix}]||[\begin{smallmatrix}0\\*\end{smallmatrix}]$ singularities occur, where the $\{*\}$ signifies "no rank condition." Since $S[\begin{smallmatrix}0\\*\end{smallmatrix}]||[\begin{smallmatrix}0\\*\end{smallmatrix}]$ already has codimension 2, by further consideration of codimensions only $S[\begin{smallmatrix}0\\0\end{smallmatrix}]||[\begin{smallmatrix}0\\0\end{smallmatrix}]$ generically occurs.

Lemma 2.3.13. *If $f \in C_{\{p,p'\}}^\infty(C, \mathbb{R}^2)$ is a generic $S[\begin{smallmatrix}0\\0\end{smallmatrix}]||[\begin{smallmatrix}0\\0\end{smallmatrix}]$ multisingularity, then up to equivalence it is given by*

$$
\begin{aligned}
f_p^* y_1 &= 0 & \qquad f_{p'}^* y_1 &= x' \\
f_p^* y_2 &= x & \qquad f_{p'}^* y_2 &= x'
\end{aligned}
$$

Proof. Both germs have a nonzero differential in the y_2 direction, so we can use the inverse function theorem to reparameterize x and x' such that $f_p^* y_2 = x$ and $f_{p'}^* y_2 = x'$. Then we can cancel out $f_p^* y_1$ like in Lemma 2.3.11. Next, let $h \in \mathbb{C}_0^\infty(\mathbb{R}, \mathbb{R})_0$ be the germ such that $f_{p'}^* y_1 = h(x')$. Note that we can extend $x \mapsto x/h(x)$ to be a smooth function at 0 since $\partial_1 h(0) \neq 0$. We reparameterize with $\overline{y}_1 = y_1 y_2 / h(y_2)$ and $\overline{y}_2 = y_2$, which gives

$$f_p^* \overline{y}_1 = 0x/h(x) = 0 \qquad\qquad f_{p'}^* \overline{y}_1 = h(x')x'/h(x') = x'$$

This is gives the desired form. □

We now have to consider multijet singularities for germs that map to the same point after they are composed with the projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$. The analysis here is simple due to codimensions: the $S[\begin{smallmatrix}0\\1\end{smallmatrix}]$ and $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularities cannot occur with the same $\mathbb{R}^2/\mathbb{R}$ value. This is in accordance with the fact that $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ must be a Morse function, which is a weaker statement since we also have a condition on $S[\begin{smallmatrix}0\\0\end{smallmatrix}]|[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularities.

The results of the analysis of singularities of maps $C \to \mathbb{R}^2$ are in Table 2.10 and are illustrated in Figure 2.3.

2D Cerf singularities for curves

Now we consider Cerf singularities, which will be a refinement of the Cerf singularities we found in Section 2.3.1.

We are considering germs in $C_p^\infty(C \times \mathbb{R}, \mathbb{R}^2)$ where $C \times \mathbb{R}$ has the product foliation as does $\mathbb{R}^2$, up to Cerf equivalence. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ be the foliation-preserving projection onto leaf space. The type $S\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix} \subseteq J^1(C \times \mathbb{R}, \mathbb{R}^2)$ denotes those jets of germs f such that $\operatorname{drk} df_p|_{T_pC} = r_1$ and $\operatorname{drk} d(\pi \circ f)_p|_{T_pC} = r_2$. The table of simplified normal bundles and codimensions from the beginning of this section still applies, which we reproduce here to with minor changes to specialize the singularity symbols:

Type	$K\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$	$L\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$	$\operatorname{Hom}(K\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}, L\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix})$	Codim.
$S\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$0 \hookrightarrow 0$	0	0	0
$S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$0 \hookrightarrow T_pC$	$T_q\mathbb{R}^2/T_q\mathbb{R} \twoheadrightarrow T_q\mathbb{R}^2/T_q\mathbb{R}$	$\operatorname{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R})$	1
$S\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$T_pC \hookrightarrow T_pC$	$T_q\mathbb{R}^2 \twoheadrightarrow T_q\mathbb{R}^2/T_q\mathbb{R}$	$\operatorname{Hom}(T_pC, T_q\mathbb{R}^2)$	2

This time, all three are generically possible since the domain is two-dimensional. The first has no subtypes.

The $S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ type has subtypes $S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}[r]$ where $r = \operatorname{drk} d^2 f_p|_{T_pC}$. None of these have subtypes themselves, and $S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}[0]$ and $S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}[1]$ are the generic possibilities, respectively having codimensions 1 and 2.

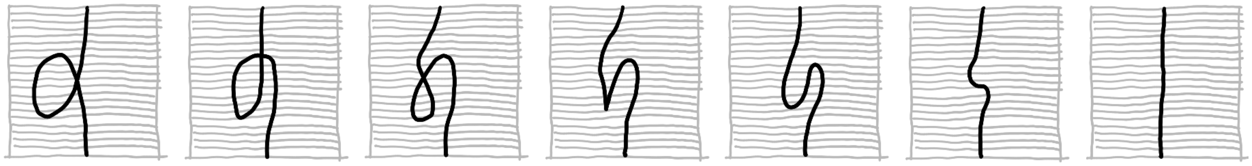


Figure 2.5: The $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ singularity type has codimension 3, so it does not generically occur since the domain of a Cerf function is two-dimensional. The singularity is unstable, and with a slight perturbation it decomposes into a collection of other Cerf singularities.

The $S\begin{bmatrix}1\\1\\0\end{bmatrix}$ type is a refinement of the Whitney umbrella singularity $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ from the previous section. Like before, we are justified in using $\operatorname{im} d^2 f_p|_{T_pC}$ as an additional subspace. The subtypes are $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}r\\1\end{smallmatrix}\right]$ where r is the dropped rank when the germ is considered as a $S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ singularity, — that is to say, it is the dropped rank of the projection of $d^2 f_p$ to a map $T_p(\mathbb{R}\times\mathbb{R}) \to \operatorname{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R})$:

$$\begin{tikzcd}
T_p(\mathbb{R}\times\mathbb{R}) \arrow[r, "d^2 f_p"] \arrow[d, dash] & \operatorname{Hom}(T_pC, T_q\mathbb{R}^2) \arrow[d, two heads] \\
T_p(\mathbb{R}\times\mathbb{R}) \arrow[r, "d^2 f_p"] & \operatorname{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R})
\end{tikzcd}$$

The 1 in the symbol is there to remind us that we have an additional subspace, and it denotes the fact that when projected to $\operatorname{im} d^2 f_p|_{T_pC}$ the dropped rank is 1. Only $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ generically occurs since $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ has codimension 3. Then $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right][0]$ is the only subtype of $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ that generically occurs. (See Figure 2.5 for a way to perturb $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.)

Lemma 2.3.14. *If $f \in C_p^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)_q$ is a generic $S\begin{bmatrix}0\\0\\0\end{bmatrix}$ singularity, then up to Cerf equivalence it is given by*

$$f^*y_1 = 0$$
$$f^*y_2 = x$$

Proof. The argument from Lemma 2.1.8 works to let us assume $f^*y_2 = x$. Then the idea from Lemma 2.3.11 works to clear out f^*y_1. □

Lemma 2.3.15. *If $f \in C_p^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)_q$ is a generic $S\begin{bmatrix}0\\1\\0\end{bmatrix}[0]$ singularity, then up to Cerf equivalence it is given by*

$$f^*y_1 = x$$
$$f^*y_2 = x^2$$

Proof. After Cerf equivalence, we may assume that $S\begin{bmatrix}0\\1\\0\end{bmatrix}[0](f)$ is locally $0 \times \mathbb{R}$ and furthermore that the differential vanishes on this submanifold. Then we can do a parameterized version of Lemma 2.3.12. □

Lemma 2.3.16. *If $f \in C^\infty_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)_q$ is a generic $S\begin{bmatrix}0\\1\\0\end{bmatrix}[1]$ singularity, then up to Cerf equivalence it is given by*

$$\begin{aligned} f^*y_1 &= x \\ f^*y_2 &= xt + x^3 \end{aligned}$$

Proof. After a reparameterization, we may assume that $df_p = \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$. The second intrinsic derivative is a map

$$\begin{aligned} d^2f_p : T_p(\mathbb{R} \times \mathbb{R}) &\to \operatorname{Hom}(T_p\mathbb{R}, T_q\mathbb{R}^2/T_q\mathbb{R}) \\ v &\mapsto \begin{bmatrix}\partial_{20}f_2(p) & \partial_{11}f_2(p)\end{bmatrix} v, \end{aligned}$$

where we identify the codomain with $\operatorname{Hom}(\mathbb{R}, \mathbb{R})$. Since $\operatorname{drk} d^2f_p|_{T_p\mathbb{R}} = 1$, then we have that $\partial_{20}f_2(p) = 0$ and $\partial_{11}f_2(p) \neq 0$. After a reparameterization we may assume that $\partial_{11}f_2(p) = 1$, and also $\partial_{02}f_2(p) = 0$. The third intrinsic derivative is

$$\begin{aligned} d^3f_p : T_p\mathbb{R} &\to \operatorname{Hom}(T_p\mathbb{R}, \operatorname{Hom}(T_p\mathbb{R}, T_q\mathbb{R}^2/T_q\mathbb{R})) \\ v &\mapsto \begin{bmatrix}\partial_{30}f_2(p)\end{bmatrix} v, \end{aligned}$$

and hence $\partial_{30}f_2(p) \neq 0$ since d^3f_p is generically surjective. With another reparameterization, we may additionally assume that $\partial_{30}f_2(p) = 6$. Thus, we have that

$$\begin{aligned} f^*y_1 &= x + r_1(x,t) \\ f^*y_2 &= xt + x^3 + r_2(x,t) \end{aligned}$$

with $r_1 \in \mathfrak{m}_p^2$ and $r_2 \in \mathfrak{m}_p^3$ with $\partial_{30}r_2(p) = 0$. Note that the lowest-order term of $f^*(y_1^i y_3^j)$ is $x^i y^j$, so by doing reparameterizations with $\overline{y}_1 = y_1 + c y_1^i y_3^j$ for suitable c and with $i + j \geq 2$ we may assume that $r_1 \in \mathfrak{m}_p^k$ for any $k \geq 2$ we wish, and this does not affect r_2. Write $r_2(x,t) = \sum_{i+j=3} s_{ij}(x,t)x^i t^j$ using the Hadamard lemma for some functions s_{ij}, with $s_{30} \in \mathfrak{m}_p$. We may assume $s_{03} \in \mathfrak{m}_p$ as well after a Cerf reparameterization. Then,

$$f^*y_2 = xt(1 + s_{21}(x,t)x + s_{12}(x,t)t) + x^3 + s_{30}(x,t)x^3 + s_{03}(x,t)t^3$$

We can substitute $\overline{x} = x(1 + s_{21}(x,t)x + s_{12}(x,t)t)$ and $\overline{t} = t$, which introduces terms of order ≥ 4 into f^*y_2 and terms of order ≥ 2 into f^*y_1. We are still able to assume that $r_1 \in \mathfrak{m}_p^k$ for any $k \geq 2$ we wish, and now we may additionally assume that $r_2 \in \mathfrak{m}_p^4$.

We can eliminate the t^4 term from r_2, and then we can eliminate the $x^i t^j$ terms $1 \leq i, j \leq 3$ and $i + j = 4$ using the same idea as above, but this time it introduces terms of order ≥ 5 into f^*y_2. We still possibly have a x^4 term in r_2 after this. Supposing $f^*y_2 = xt + x^3 + cx^4 + r_1(x,t)$

with $r_1 \in \mathfrak{m}_p^5$, then we can write $f^*y_2 = xt + x^3(1+cx) + r_1(x,t)$ and then substitute $\overline{x} = x(1+cx)^{1/3}$, which introduces $x^{2+i}t$ terms to f^*y_2. Substituting $x = \overline{x} + c_2\overline{x}^2 + c_3\overline{x}^3$ for suitable c_2 and c_3 cancels out these terms through the fourth order. Hence, we may assume $f^*y_2 = xt + x^3 + r_1(x,t)$ with $r_1 \in \mathfrak{m}_p^5$. Everything here adapts to being able to clear out terms to allow us to suppose that $r_1 \in \mathfrak{m}_p^k$ to any k we wish.

Thus, we have a sequence of equivalent germs that converges to the germ $(x,t) \mapsto (x, xt + x^3, t)$ in the Whitney C^∞ topology. Hence, it suffices to show that this germ is locally stable. We will use Lemma 1.4.14. We calculate the ideal $I = (t) + (xt + x^3, t)^2 = (x^6, t)$, and let $A = f_* C_p^\infty(T(\mathbb{R}\times\mathbb{R})) + IC_p^\infty(f^*T\mathbb{R}^3)$. It suffices to show that

$$C_p^\infty(f^*T\mathbb{R}^3) = A + f^*C_0^\infty(\mathbb{R}^3)\tfrac{\partial}{\partial y_1} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R})\tfrac{\partial}{\partial y_2} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R}^2)\tfrac{\partial}{\partial y_3}.$$

We first calculate that

$$\begin{aligned} f_*\tfrac{\partial}{\partial x_1} &= \tfrac{\partial}{\partial y_1} + (t+3x^2)\tfrac{\partial}{\partial y_2} \equiv_A \tfrac{\partial}{\partial y_1} + 3x^2\tfrac{\partial}{\partial y_2} \\ f_*\tfrac{\partial}{\partial x_2} &= x\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_3}. \end{aligned}$$

Thus, for all $k \in \mathbb{N}$ we have the reduction rules

$$\begin{aligned} x^k\tfrac{\partial}{\partial y_1} &\equiv_A -3x^{k+2}\tfrac{\partial}{\partial y_2} \\ x^k\tfrac{\partial}{\partial y_3} &\equiv_A -x^{k+1}\tfrac{\partial}{\partial y_2}, \end{aligned}$$

and hence we just need to check that we have $\frac{\partial}{\partial y_2}, \dots, x^5\frac{\partial}{\partial y_2}$ in the right-hand side of what we need to show. We have $x^k\frac{\partial}{\partial y_1}$ in the right-hand side for all $k \in \mathbb{N}$, thus we have $x^2\frac{\partial}{\partial y_2}, \dots, x^5\frac{\partial}{\partial y_2}$. Since we have $\frac{\partial}{\partial y_3}$ in the right-hand side we have $x\frac{\partial}{\partial y_2}$. We also have $\frac{\partial}{\partial y_2}$. We have therefore established the equality. □

Lemma 2.3.17. *If $f \in C_p^\infty(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)_q$ is a generic $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right][0]$ singularity, then up to Cerf equivalence it is given by*

$$\begin{aligned} f^*y_1 &= xt + x^3 \\ f^*y_2 &= x^2 \end{aligned}$$

Proof. The analysis and reparameterizations from Lemma 2.3.6 are all valid with a foliated codomain as well. Hence, given any such f we can create a sequence of germs that converge to the desired germ in the Whitney C^∞ topology. What remains is to use Lemma 1.4.14 to show that the desired germ is locally stable.

This time the ideal we have is $I = (t) + (x^2)^2 = (x^4, t)$, which is exactly the same as before. Letting $A = f_* C_p^\infty(T(\mathbb{R}\times\mathbb{R})) + IC_p^\infty(f^*T\mathbb{R}^3)$, it suffices to show that

$$C_p^\infty(f^*T\mathbb{R}^3) = A + f^*C_0^\infty(\mathbb{R}^3)\tfrac{\partial}{\partial y_1} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R})\tfrac{\partial}{\partial y_2} + f^*C_0^\infty(\mathbb{R}^3/\mathbb{R}^2)\tfrac{\partial}{\partial y_3}.$$

One can check that the rest of the proof of Lemma 2.3.6 applies. □

2D multijet Cerf singularities for curves

Now that we have found these singularities, we continue on to classify the multijet singularities. The codimension analysis for multijet singularities from Section 2.3.1 still applies, and these are the only possibilities due to codimensions:

Type	$S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix}$	$S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\1\\0\end{bmatrix}[0]$	$S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix}$
Codim.	3	4	6
Ambient dim.	4	4	6

Suppose $f \in C^\infty_{\{(p,t_0),(p',t_0)\}}(C \times \mathbb{R}, \mathbb{R}^2)_q$ with $p \neq p'$. Like before, to handle the $\beta^{-1}(\Delta^2_{\mathbb{R}^2})$ and $(\pi^{\times 2} \circ \alpha)^{-1}(\Delta^2_{\mathbb{R}})$ transversality conditions the relevant function is $F(x, x', t) = f(x, t) - f(x', t)$ whose differential is

$$dF_{(p,t_0),(p',t_0)} : T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R} \to T_q\mathbb{R}^2$$
$$v \mapsto \begin{bmatrix} \partial_{10}f_1(p) & -\partial_{10}f_1(p') & \partial_{01}f_1(p) - \partial_{01}f_1(p') \\ \partial_{10}f_2(p) & -\partial_{10}f_2(p') & \partial_{01}f_2(p) - \partial_{01}f_2(p') \end{bmatrix} v.$$

A $S\begin{bmatrix}0\\r_1\\0\end{bmatrix}\vert\begin{bmatrix}0\\r_2\\0\end{bmatrix}$ type is that, for the projection to $T_q\mathbb{R}^2/T_q\mathbb{R}$, the restriction to T_pC has dropped rank r_1 and the restriction to $T_{p'}C$ has dropped rank r_2.

The analysis of $S[\begin{smallmatrix}0\\0\end{smallmatrix}]\vert[\begin{smallmatrix}0\\0\end{smallmatrix}]$ carries over to $S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix}$, though we have to take care that we only do foliation-respecting changes of coordinates. We may assume that the differential has a matrix of the form

$$\begin{bmatrix} 0 & -a & b \\ 1 & -1 & 0 \end{bmatrix}$$

for some constants a and b. We define subtypes $S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[r_1\ r_2]$ where r_1 is the dropped rank when restricted to $T_pC \oplus T_{p'}C$ and r_2 is the dropped rank on all of $T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R}$. If $r_2 > 0$ then the codimension would be too high, so generically $r_2 = 0$. The type $S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[0\ 0]$ has no subtypes, and we next look at $S\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0]$. We get a second intrinsic derivative

$$d^2F_{(p,p',t_0)} : T_pC \oplus T_{p'}C \oplus T_{t_0}\mathbb{R} \to \mathrm{Hom}(K\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0], L\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0])$$

where $K\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0]$ is spanned by $(1, 1, 0)$ and $L\begin{bmatrix}0\\0\\0\end{bmatrix}\vert\begin{bmatrix}0\\0\\0\end{bmatrix};[1\ 0]$ is $T_q\mathbb{R}^2/T_q(0 \times \mathbb{R})$, so the total codimension for this singularity type is 4, and thus still generically possible. As a matrix we can write d^2F as

$$v \mapsto \begin{bmatrix} \partial_{20}f_1(p, t_0) & -\partial_{20}f_1(p', t_0) & \partial_{11}f_1(p, t_0) - \partial_{11}f_1(p', t_0) \end{bmatrix} v$$

We can restrict to $K\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\0\\0\end{bmatrix}$;[1 0] to get further subtypes, and for codimension reasons we generically need the restriction to be nonzero, hence $\partial_{20}f_1(p,t_0) \neq \partial_{20}f_1(p',t_0)$. Thus, we obtain the two types $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\0\\0\end{bmatrix}$;[0 0] and $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\0\\0\end{bmatrix}$;[1 0][0] from $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Next we move to type $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\1\\0\end{bmatrix}$. There are a few singularity types involved here: the type $S\begin{bmatrix}0\\*\\0\end{bmatrix}|\begin{bmatrix}0\\*\\0\end{bmatrix}$ where we disregard the foliation of the domain, the type $S\begin{bmatrix}0\\1\\0\end{bmatrix}$ for the point p', and the type $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\1\\0\end{bmatrix}$ itself. In a neighborhood of p', both $S\begin{bmatrix}0\\*\\0\end{bmatrix}|\begin{bmatrix}0\\*\\0\end{bmatrix}(f)$ and $S\begin{bmatrix}0\\1\\0\end{bmatrix}(f)$ are one-dimensional submanifolds intersecting at a point $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\1\\0\end{bmatrix}(f)$. The purpose of the next subtypes are to (1) ensure the second singularity is $S\begin{bmatrix}0\\1\\0\end{bmatrix}$[0] and (2) ensure that these two submanifolds intersect transversely. We may assume that the differential has a matrix of the form

$$\begin{bmatrix} 0 & 1 & \partial_{01}f_1(p) - \partial_{01}f_1(p') \\ 1 & 0 & \partial_{01}f_2(p) - \partial_{01}f_2(p') \end{bmatrix}$$

The kernel of this is $(\partial_{01}f_2(p) - \partial_{01}f_2(p'), \partial_{01}f_1(p) - \partial_{01}f_1(p'), -1)$, which indicates that the tangent space at p' of $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\*\\0\end{bmatrix}(f)$ is spanned by $(\partial_{01}f_1(p) - \partial_{01}f_1(p'), -1)$. As a $S\begin{bmatrix}0\\1\\0\end{bmatrix}$ point, we have a differential

$$\begin{aligned} df_{(p',t_0)} : T_{(p',t_0)}(C \times \mathbb{R}) &\to T_q\mathbb{R}^2 \\ v &\mapsto \begin{bmatrix} 1 & \partial_{01}f_1(p',t_0) \\ 0 & \partial_{01}f_2(p',t_0) \end{bmatrix} v \end{aligned}$$

with a second intrinsic derivative

$$\begin{aligned} d^2f_{(p',t_0)} : T_{(p',t_0)}(C \times \mathbb{R}) &\to \mathrm{Hom}(T_{(p',t_0)}C, T_q\mathbb{R}^2/T_q\mathbb{R}) \\ v &\mapsto \begin{bmatrix} \partial_{20}f_2(p',t_0) & \partial_{11}f_2(p',t_0) \end{bmatrix} v \end{aligned}$$

Hence $T_{(p',t_0)}S\begin{bmatrix}0\\1\\0\end{bmatrix}(f) = \ker d^2f_{(p',t_0)}$ is spanned by $(\partial_{11}f_2(p',t_0), -\partial_{20}f_2(p',t_0))$. By surjectivity, $\partial_{20}f_2(p',t_0) \neq 0$, for the two tangent spaces to be unequal, the condition we are looking for is $\partial_{20}f_2(p',t_0)(\partial_{01}f_1(p') - \partial_{01}f_1(p)) \neq \partial_{11}f_2(p',t_0)$. To implement this, we have a rank condition for $d^2f_{(p',t_0)}$ restricted to the subspace spanned by $(\partial_{01}f_1(p) - \partial_{01}f_1(p'), -1)$. By codimension considerations, the generic possibility is for the dropped rank to be 0. We denote this subtype by $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\1\\0\end{bmatrix}$[0];[0].

Lemma 2.3.18. *If $f \in C^\infty_{\{(p,0),(p',0)\}}(C \times \mathbb{R}, \mathbb{R}^2)_q$ defines a generic multigerm that is an $S\begin{bmatrix}0\\0\\0\end{bmatrix}|\begin{bmatrix}0\\0\\0\end{bmatrix}$;[0 0] multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$f^*_{(p,0)}y_1 = 0 \qquad f^*_{(p',0)}y_1 = x'$$
$$f^*_{(p,0)}y_2 = x \qquad f^*_{(p',0)}y_2 = x'$$

Proof. We may reparameterize such that the differential has the matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

We can reparameterize such that $S\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]|\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right];[0\ 0](f)$ is contained in $\{p,q\}\times\mathbb{R}$ near $(p,0)$ and $(q,0)$. Then we can reparameterize such that $f^*_{(p,0)}y_1 = 0$ and $f^*_{(p,0)}y_2 = x$, and then we can do a similar sort of reparameterization as in Lemma 2.3.13. ☐

Lemma 2.3.19. *If* $f \in C^\infty_{\{(p,0),(p',0)\}}(C\times\mathbb{R},\mathbb{R}^2)_q$ *defines a generic multigerm that is an* $S\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]|\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right];[1\ 0][0]$ *multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$\begin{aligned} f^*_{(p,0)}y_1 &= 0 & f^*_{(p',0)}y_1 &= (x')^2 + t + O(x',t)^3 \\ f^*_{(p,0)}y_2 &= x & f^*_{(p',0)}y_2 &= x' \end{aligned}$$

Proof. The argument from Lemma 2.3.9 applies. ☐

Lemma 2.3.20. *If* $f \in C^\infty_{\{(p,0),(p',0)\}}(C\times\mathbb{R},\mathbb{R}^2)_q$ *defines a generic multigerm that is an* $S\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]|\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right][0];[0]$ *multisingularity, then, up to Cerf equivalence, the multigerm is given by*

$$\begin{aligned} f^*_{(p,0)}y_1 &= t + O(x,t)^2 & f^*_{(p',0)}y_1 &= (x')^2 \\ f^*_{(p,0)}y_2 &= x & f^*_{(p',0)}y_2 &= x' \end{aligned}$$

Proof. There is a $S\left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right][0]$ singularity at $(p',0)$, so we can first apply the all the reparameterizations from Lemma 2.3.15 to let us assume that

$$\begin{aligned} f^*_{(p',0)}y_1 &= x' \\ f^*_{(p',0)}y_2 &= (x')^2. \end{aligned}$$

We may also assume that $f^*_{(p,0)}y_2 = x$. In consideration of $\partial_{20}f_2(p',t_0)(\partial_{01}f_1(p') - \partial_{01}f_1(p)) \neq \partial_{11}f_2(p',t_0)$, we have the condition $\partial_{01}f_1(p) \neq 0$, and we can assume it is 1. We can reparameterize the codomain to cancel any x term from $f^*_{(p,0)}y_1$, leading to

$$\begin{aligned} f^*_{(p,0)}y_1 &= t + h(x,t) & f^*_{(p',0)}y_1 &= x' + c(x')^2 \\ f^*_{(p,0)}y_2 &= x & f^*_{(p',0)}y_2 &= (x')^2 \end{aligned}$$

for some $h(x,t) \in \mathfrak{m}_p^2$ and $c \in \mathbb{R}$. There is an invertible function germ $g : \mathbb{R} \to \mathbb{R}$ such that $g(x' + c(x')^2) = x'$, and by substituting with $\overline{y}_1) = g(y_1)$ we have $f^*_{(p',0)}\overline{y}_1 = x'$. This does not change the form of $f^*_{(p,0)}y_1$. ☐

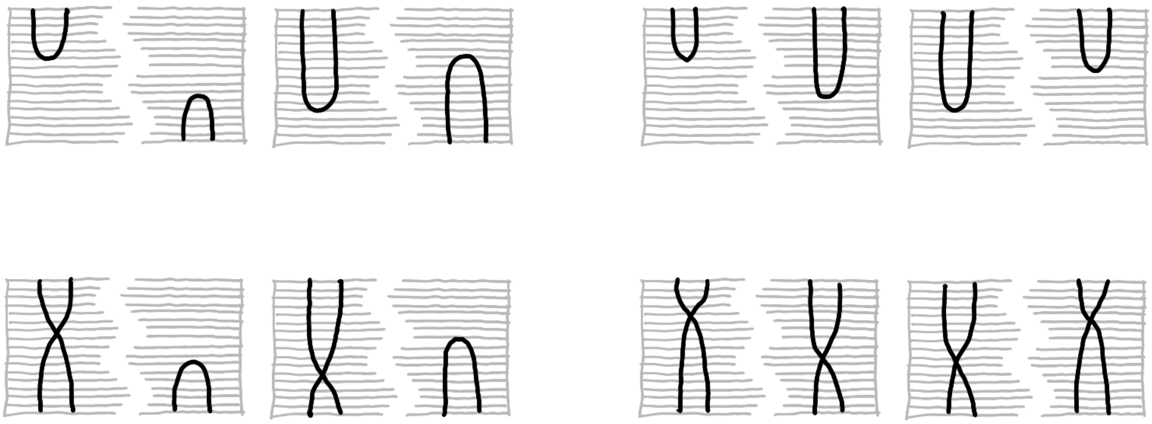


Figure 2.6: The four types of interchange moves, which are multijet Cerf singularities for the composition with the $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ projection. Each diagram shows the $t = -1$ and $t = 1$ frames of a movie, with the break indicating arbitrary horizotal separation in space.

Lemma 2.3.21. *If $f \in C^\infty(C \times \mathbb{R}, \mathbb{R}^2)$ is such that for three distinct points $p, q, r \in \mathbb{R}$ that $f(p, 0) = f(q, 0) = f(r, 0)$ and the germs of f at these three points define a generic multigerm of type $S\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} | \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} | \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then, up to Cerf equivalence, the multigerm is, to first order, given by*

$$\begin{aligned} f_p^* y_1 &= x & f_q^* y_1 &= -x' & f_r^* y_1 &= t \\ f_p^* y_2 &= x & f_q^* y_2 &= x' & f_r^* y_2 &= x'' \end{aligned}$$

Proof. The considerations are similar as before, but we need to be sure each germ is an $S\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ singularity, which means $\partial_{10} f_2$ is nonzero at each point. ☐

We have found all the multijet singularities for germs that map to the same point, but we have not yet considered multijet singularities for germs that map to the same point when projected to be maps $C \times \mathbb{R} \to \mathbb{R}^2/\mathbb{R}$ or $C \times \mathbb{R} \to \mathbb{R}^2/\mathbb{R}^2$.

We start with $C \times \mathbb{R} \to \mathbb{R}^2/\mathbb{R}$. Observe that none of the singularities listed in Figure 2.4 when projected can occur at the same time since they all have codimension-3. The next observation is that the suspensions of the $S[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]$ and $S[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]|[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$ singularities listed in Figure 2.3 when projected cannot occur at the same time and map to the same point as projections of the Figure 2.4 singularities, also due to codimension. What remains is considering pairs of projections of suspensions of the $S[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]$ and $S[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]|[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$ singularities. Each of these defines a 1-dimensional singularity manifold in $C \times \mathbb{R}$, and the next multijet singularity types are for these manifolds mapping to $\mathbb{R}^2\mathbb{R}$ transversely — these end up being *interchange moves*. We do not do the analysis here, but the result is that the images of the projections of these (multi-)singularity types can coincide at single points in time, and it is similar to the analysis of interchanges of Morse critical points (the type $S[{}^{1}_{0}][0]|[{}^{1}_{0}][0]$ multisingularities). We illustrate the possibilities in Figure 2.6 and we do not attempt to construct a naming convention for the types.

Lastly, we consider $C \times \mathbb{R} \to \mathbb{R}^2/\mathbb{R}$ for all the types considered to this point, and the only thing this does is allow us to say that, generically, the interchange moves do not occur at the same point in time. This completes the classification of the multijet singularities. Except for the interchange moves, the results are summarized in Table 2.11 and illustrated in Figure 2.4.

2.3.3 Graphs

We briefly mention how to modify the theory of singularities to handle 2D Morse and Cerf theory for diagrams of graphs in the plane. A *graph* Γ is a finite 1-dimensional CW complex with $V(\Gamma)$ its 0-cells (the *vertices*) and $E(\Gamma)$ its 1-cells (the *edges*). Graphs aren't manifolds. However, each 1-cell of a graph can be given a smooth manifold structure by regarding them as being diffeomorphic to the standard unit interval $[0,1] \subset \mathbb{R}$ (and, if we wish, we may use the smooth manifold germ $\mathcal{N}([0,1])$ so that it comes with an extention to some neighborhood).

We can regard Γ as a colimit over its cells in the usual way, which we describe here. Let E and V be a finite sets and let $s, t : E \to V$ be functions. For each $e \in E$ we have a map $f_e\{0,1\} \to V$ sending 0 to $s(e)$ and 1 to $t(e)$. Then Γ is the pushout in the following diagram:

$$\begin{tikzcd}
\bigsqcup_{e\in E}\{0,1\} \arrow[r, hook] \arrow[d, "\bigsqcup_{e\in E} f_e"] & \bigsqcup_{e\in E}[0,1] \arrow[d] \\
V \arrow[r] & \Gamma
\end{tikzcd}$$

Given this, we can give Γ something akin to a smooth structure, where $C^\infty(\Gamma)$ is a defined as a limit using the following pullback diagram from applying the C^∞ functor to the above diagram:

$$\begin{tikzcd}
\bigsqcup_{e\in E} C^\infty(\{0,1\}) & \bigsqcup_{e\in E} C^\infty([0,1]) \arrow[l] \\
C^\infty(V) \arrow[u] & C^\infty(\Gamma) \arrow[l] \arrow[u]
\end{tikzcd}$$

This is all to say that $C^\infty(\Gamma)$ consists of functions $\Gamma \to \mathbb{R}$ that, for all edges, its restriction to that edge is smooth. We regard $C^\infty(\Gamma)$ as a subring of $C^\infty(\bigsqcup_{e\in E}[0,1])$ when Γ has no isolated vertices — if there are isolated vertices, then each isolated vertex contributes an additional $\mathbb{R}$ direct summand. Similarly, for M a manifold we can define $C^\infty(\Gamma, M)$ by applying the $C^\infty(-, M)$ functor instead.

For M a manifold, we define $J^k(\Gamma, M)$ to be $J^k(\bigsqcup_{e\in E}[0,1], M)$. There is an additional projection $\alpha' : J^k(\Gamma, M) \to \Gamma$ using the map $\bigsqcup_{e\in E}[0,1] \to \Gamma$. For $f \in C^\infty(\Gamma)$, its *jet extension* is a map $j^k f : \bigsqcup_{e\in E}[0,1] \to J^k(\Gamma, M)$. This handles the fact that at a vertex there is one jet per incident edge.

We can adapt the Thom Transversality Theorem to the case of $J^k(\Gamma, M)$ and its jet extensions. Given a submanifold $W \subseteq J^k(\Gamma, M)$, we can break it up into one piece W_e per edge e for the part lying over the e copy of $[0,1]$. Then we can see that $T_{W_e} = \{f \in C^\infty(\Gamma, M) \mid j^k f \pitchfork W_e\}$ is a residual subset of $C^\infty(\Gamma, M)$ by modifying the proof of Theorem 1.2.9 to take care at the endpoints of e, where we argue that perturbations of $f|_e$ can be extended into perturbations of f in an arbitrary C^∞ neighborhood of f. Then, $T_W = \bigcap_{e\in E} T_{W_e}$ is a residual subset. Hence, by the following lemma, the set of functions in $C^\infty(\Gamma, M)$ whose jet extensions are transverse to any given countable collection of jet submanifolds is dense.

Lemma 2.3.22. *Suppose Γ is a finite graph. Then $C^\infty(\Gamma)$ is a Baire space.*

Proof. We may suppose that Γ has no isolated vertices since we can add isolated vertices to a graph without changing whether $C^\infty(\Gamma)$ is a Baire space. By Lemma 1.2.5, we need only show that $C^\infty(\Gamma)$ is closed with respect to the weak topology. Given an arbitrary $f \in C^\infty(\bigsqcup_{e\in E} C^\infty([0,1])) \setminus C^\infty(\Gamma)$, then there is a vertex v and two edges e and e' incident to it with conflicting values at their corresponding endpoints x and y. There exist disjoint open neighborhoods U_x of $f(x)$ and U_y of $f(y)$, and $\{g \in C^\infty(\bigsqcup_{e\in E}[0,1]) \mid g(x) \in U_x \text{ and } g(y) \in U_y\}$ is an open neighborhood of f in the weak topology. □

There is also a multijet transversality theorem, but in the definition of the multijet bundle we have to make sure to that our definition of fat diagonal is with respect to α', i.e., the jets have to be above distinct points of Γ.

Let us consider functions $f \in C^\infty(\Gamma, \mathbb{R}^2)$ where $\mathbb{R}^2$ has the product foliation. There is still a notion of germs for graphs and we can study and classify the singularities of f. The new thing is germs of diffeomorphisms at vertices — vertices must map to vertices since Γ is not smooth at them (so degree-2 vertices are indeed distinguished from interior points of edges) and also the degree of a vertex is a diffeomorphism invariant.

The analysis of singularities for non-vertex points has already been done in Section 2.3.2. What remains is the understanding the behavior of vertices and how vertices interact with other singularity types.

Let $v \in V(\Gamma)$ be a vertex. For each edge e incident to v, letting $x \in e$ be the corresponding endpoint, we can define $S_v[r]$ at x for whether $\operatorname{drk} d(\Gamma \to \mathbb{R}^2/\mathbb{R})_x = r$. If $r > 0$, then the codimension of the singularity type is 2, so it does not generically occur. Thus, we can assume that, in the image of f, edges always meet vertices transverse to the foliation. We can see that the classification of singularity types at v is then indexed by pairs (n, m) with $n+m$ the degree of v, where n is the number of edges incident from below and m the number of edges incident from above — this is as expected.

Furthermore, we can define multijet singularity types for whether vertices coincide with each other or with interiors of edges, or whether vertices have the same value in $\mathbb{R}^2/\mathbb{R}$ as other vertices or codimension-1 singularities. The upshot is that, due to their high codimension, none of these coincidences occur.

We could pursue the 2D Cerf theory of maps from graphs to the plane, which involves defining smooth functions on 2D CW complexes, but we will not pursue it here.

2.4 Classification of 1-dimensional singularities

Over the next few sections, we classify singularities of maps $\mathbb{R}^n \to \mathbb{R}^n$ for $1 \leq n \leq 3$, with $\mathbb{R}^n$ given the standard foliation. For the $n = 3$ case, we will only consider Cerf singularities since our main application is presenting 2D cobordism categories. This work is a recapitulation of the analysis in [SP09] but with special emphasis obtaining exact normal forms — i.e., on classifying the open orbits of $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)_0$ under the $\mathrm{Diff}(\mathbb{R}^n) \times \mathrm{Diff}(\mathbb{R} \subseteq \cdots \subseteq \mathbb{R}^n)$ action. Section 2.7 picks up with the classifications of maps of manifolds with a codimension-1 submanifold.

The $n = 1$ case is the simplest case: we consider smooth map germs $f : \mathbb{R} \to \mathbb{R}$ up to the action of $\mathrm{Diff}_p(\mathbb{R}) \times \mathrm{Diff}_q(\mathbb{R})$, with the germ centers $p = 0 \in \mathbb{R}$ for the domain and $q = f(p) = 0 \in \mathbb{R}$ for the codomain. This is a special case of Section 2.1, and we have it here for completeness. The results are summarized in Table 2.12.

Table 2.12: Classification of germs of 1-dimensional singularities.

Type	Codim.	Representative germ	See:
$S[0]$	0	$f^*y_1 = x_1$	Lemma 2.4.1
$S[1]$	1	$f^*y_1 = x_1^2$	Lemma 2.4.2

Table 2.13: Classification of germs of 2-dimensional singularities.

Type	Codim.	Representative germ	See:	Suspends
$S[\begin{smallmatrix}0\\0\end{smallmatrix}]$	0	$f^*y_1 = x_1$ $f^*y_2 = x_2$	Lemma 2.5.1	$S[0]$
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$	1	$f^*y_1 = x_1^2$ $f^*y_2 = x_2$	Lemma 2.5.2	$S[1]$
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$	2	$f^*y_1 = x_1x_2 + x_1^3$ $f^*y_2 = x_2$	Lemma 2.5.3	
$S[\begin{smallmatrix}1\\1\end{smallmatrix}][0]$	2	$f^*y_1 = x_1$ $f^*y_2 = \pm x_1^2 + x_2^2$	Lemma 2.5.4	

Table 2.14: First-order types of 2-dimensional singularities. We use a compact notation for subspaces of $T_p\mathbb{R}^2$ or quotients of $T_q\mathbb{R}^2$. For example, $\mathbb{R}_2$ is both $T_p(0\times\mathbb{R})$ and $T_q\mathbb{R}^2/T_q(\mathbb{R}\times 0)$.

Type	Repres. df_p	$K[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}]$	$L[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}]$	$\mathrm{Hom}(K[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}], L[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}])$	Codim.	Generic
$S[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$	$[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$	$0 \hookrightarrow \mathbb{R}_1$	$0 \twoheadrightarrow 0$	0	0	✓
$S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$	$[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}]$	$\mathbb{R}_1 \hookrightarrow \mathbb{R}_1$	$\mathbb{R}_1 \twoheadrightarrow 0$	$\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1)$	1	✓
$S[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}]$	$[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$	$\mathbb{R}_2 \hookrightarrow \mathbb{R}_{12}$	$\mathbb{R}_2 \twoheadrightarrow \mathbb{R}_2$	$\mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2)$	2	✓
$S[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}]$	$[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}]$	$\mathbb{R}_{12} \hookrightarrow \mathbb{R}_{12}$	$\mathbb{R}_{12} \twoheadrightarrow \mathbb{R}_{12}$	$\mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_{12})$	4	

Let $S[r] \subseteq J^1(\mathbb{R},\mathbb{R})$ denote those jets with dropped-rank r, and clearly only $r = 0$ and $r = 1$ occur.

Lemma 2.4.1. *If f is a $S[0]$ singularity, then f is equivalent to $f(x) = x$.*

Proof. Since $df_p \neq 0$, df_p is an isomorphism, so the inverse function theorem applies. □

Lemma 2.4.2. *If f is a $S[1]$ singularity, then f is equivalent to $f(x) = x^2$.*

Proof. Since $df_p = \big[0\big]$, the second intrinsic derivative at p is

$$\begin{aligned} d^2 f_p : \mathbb{R} &\to \mathrm{Hom}(\mathbb{R},\mathbb{R}) \\ v &\mapsto \big(w \mapsto w\, \big[\partial_2 f(p)\big]\, v\big) \end{aligned}$$

where we identify $T_p\mathbb{R} = T_q\mathbb{R} = \mathbb{R}$. For this to be surjective, $\partial_2 f(p) \neq 0$. Hence by Lemma 1.1.69, we have $f(x) = h(x)x^2$ for some smooth $h : \mathbb{R} \to \mathbb{R}$ with $h(p) \neq 0$. With $a = \frac{h(p)}{|h(p)|} = \pm 1$, the following is a valid coordinate change:

$$\overline{y} = ay \qquad\qquad \overline{x} = \sqrt{|h(x)|}x$$

and we see $f^*y = \overline{x}^2$. □

2.5 Classification of 2-dimensional singularities

We now consider smooth map germs $f : \mathbb{R}^2 \to (\mathbb{R} \subseteq \mathbb{R}^2)$ up to the action of $\mathrm{Diff}_p(\mathbb{R}^2) \times \mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2)$, with the germ centers $p = 0 \in \mathbb{R}^2$ for the domain and $q = f(p) = 0 \in \mathbb{R}^2$ for the codomain. The results are summarized in Table 2.13 and illustrated in Figure 2.7. This classification, in terms of stable cascades $\mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}$, appears in [Arn76, Theorem 5.1]. The classification, up to a weaker form of equivalence, also appears in [SP09].

The first-order classification of singularities is given by symbols $S[\begin{smallmatrix} a \\ b \end{smallmatrix}]$, where a is the dropped rank of df_p itself and b is the dropped rank of $\pi \circ df_P$, where $\pi : T\mathbb{R}^2 \to T\mathbb{R}^2/T\mathbb{R}$. These are summarized in Table 2.14. The representative df_p is obtained by applying some (foliation-preserving) diffeomorphisms that are linear transformations to the domain and

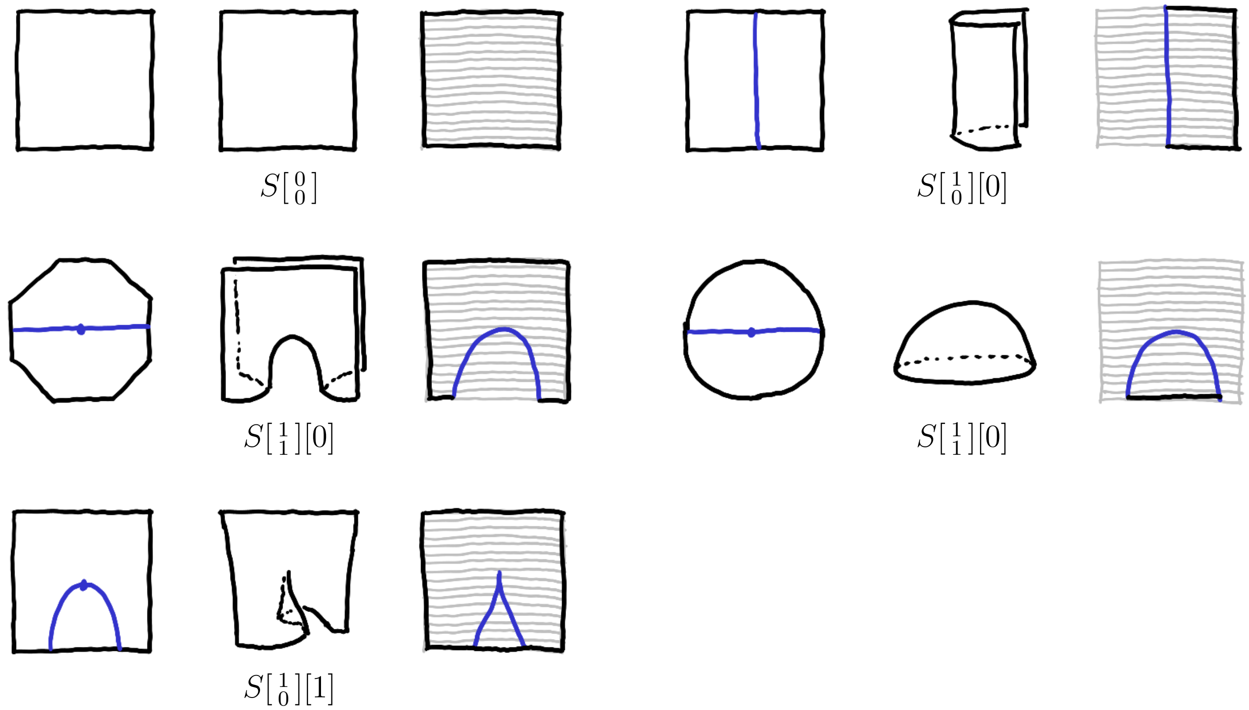


Figure 2.7: Illustrations of the 2D singularities from Table 2.13. Each set of three diagrams is showing (1) the domain, with singularity submanifolds in blue, (2) the domain "in flight," (3) the image in the foliated codomain. The $S[\begin{smallmatrix}1\\1\end{smallmatrix}][0]$ singularity has two variants from the choice of sign.

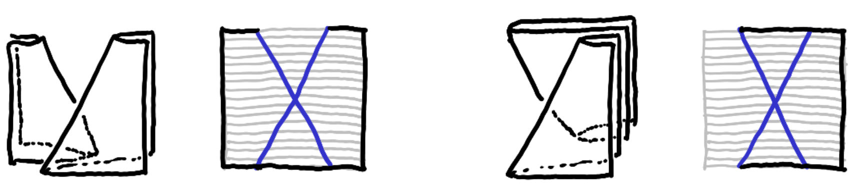


Figure 2.8: Illustrations of the 2D multigerm singularities, both variants of $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$.

codomain. The calculation of the normal bundle for the type uses a compact notation for kernels and cokernels — we give the correspondence in the following table rather than try to explain the rule:

	$T_p\mathbb{R}^2$	$T_q\mathbb{R}^2$
$\mathbb{R}_1$	$T_p(\mathbb{R}\times 0)$	$T_q\mathbb{R}^2/T_q(0\times\mathbb{R})$
$\mathbb{R}_2$	$T_p(0\times\mathbb{R})$	$T_q\mathbb{R}^2/T_q(\mathbb{R}\times 0)$
$\mathbb{R}_{12}$	$T_p\mathbb{R}^2$	$T_q\mathbb{R}^2$

Lemma 2.5.1. *If $f \in C^\infty_p(\mathbb{R}^2,\mathbb{R}^2)_q$ is a generic $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$ singularity, then it is equivalent to $f(x) = x$.*

Proof. Using a reparameterization that makes df_p the identity matrix like in Table 2.14, then the inverse function theorem gives an element $\varphi : \mathrm{Diff}(\mathbb{R}^2)$ such that $f\circ\varphi = \mathrm{id}$. ☐

2.5.1 Type $S[1;0]$

Let us consider the $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ type. Using a reparameterization that puts df_p into the form in Table 2.14, then by the implicit function theorem there is a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that $f_2(x_1, \psi(x_1, x_2)) = x_2$. The change of variables $x_1 = \overline{x}_1$ and $x_2 = \psi(\overline{x}_1, \overline{x}_2)$ gives

$$f^* y_2 = f_2(x_1, x_2) = f_2(\overline{x}_1, \psi(\overline{x}_1, \overline{x}_2)) = \overline{x}_2$$

and hence we may assume that $f_2(x) = x_2$.

The df_p matrix has $\ker df_p = \mathbb{R}_1$ and $\operatorname{coker} df_p = \mathbb{R}_1$, using the convention from the table. The second intrinsic derivative is then

$$\begin{aligned} d^2 f_p : \mathbb{R}^2 &\to \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1) \\ v &\mapsto \begin{bmatrix}\partial_{20} f_1(p) & \partial_{11} f_1(p)\end{bmatrix} v, \end{aligned}$$

where we identify $\mathrm{Hom}(\mathbb{R}_1,\mathbb{R}_1)$ with $\mathbb{R}$ for the purpose of this formula. We define the type $S[\begin{smallmatrix}1\\0\end{smallmatrix}][r]$ for $\operatorname{drk} d^2 f_p|_{\mathbb{R}_1} = r$. These are summarized in Table 2.15.

Lemma 2.5.2. *If $f \in C^\infty_p(\mathbb{R}^2,\mathbb{R}^2)_q$ is a $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ singularity, then there exist coordinates such that*

$$\begin{aligned} f^* y_1 &= x_1^2 \\ f^* y_2 &= x_2 \end{aligned}$$

Table 2.15: Second-order types for $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$.

Type	Representative $d^2 f_p$	Addl. Codim.	Codim.	Generic
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$	$\begin{bmatrix}2 & 0\end{bmatrix}$	0	1	✓
$S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$	$\begin{bmatrix}0 & 1\end{bmatrix}$	1	2	✓

Proof. By Lemma 1.1.69 there are functions $h_{ij} : \mathbb{R}^2 \to \mathbb{R}$ such that $f^*y_1 = h_{20}(x)x_1^2 + h_{11}(x)x_1x_2 + h_{02}(x)x_2^2$, and $h_{20}(p) \neq 0$ since $\partial_{20}f_1(p) \neq 0$. After a Morse Lemma style reparameterization, we may assume that $h_{20}(x) = \pm 1$ and $h_{11}(x) = 0$, and by possibly negating y_1 we fix $h_{20}(x) = 1$. From this, we calculate the local ring of the singularity:

$$C_p^\infty(\mathbb{R}^2)/C_p^\infty(\mathbb{R}^2)f^*(y_1, y_2) = C_p^\infty(\mathbb{R}^2)/(x_1^2 + h_{02}(x)x_2^2, x_2) \cong \mathbb{R}[x_1]/(x_1^2).$$

By the Malgrange Preparation Theorem (Theorem 1.1.73), 1 and x generate $C_p^\infty(\mathbb{R}^2)$ as a $C_q^\infty(\mathbb{R}^2)$-module. Hence there exist $\alpha, \beta \in C_q^\infty(\mathbb{R}^2)$ such that $h_{02}(x) = f^*\alpha + xf^*\beta$, and thus

$$x_1^2 + h_{02}(x) = x_1^2 + (f^*\alpha + xf^*\beta)x_2^2 = (x_1 + \tfrac{1}{2}x_2^2 f^*\beta)^2 + x_2^2 f^*\alpha - \tfrac{1}{4}x_2^4 f^*\beta^2.$$

Applying the change of variables

$$\begin{aligned} \overline{x}_1 &= x_1 + \tfrac{1}{2}x_2^2 f^*\beta & \qquad \overline{x}_2 &= x_2 \\ \overline{y}_1 &= y_1 - y_2^2\alpha + \tfrac{1}{4}y_2^4\beta^2 & \qquad \overline{y}_2 &= y_2 \end{aligned}$$

then $f^*\overline{y}_1 = \overline{x}_1^2$ and $f^*\overline{y}_2 = \overline{x}_2$, as desired. □

Lemma 2.5.3. *If $f \in C_p^\infty(\mathbb{R}^2, \mathbb{R}^2)_q$ is a $S[\begin{smallmatrix}1\\0\end{smallmatrix}][1]$ singularity, then there exist coordinates such that*

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_1^3 \\ f^*y_2 &= x_2 \end{aligned}$$

Proof. Since $\partial_{20}f_1(p) = 0$ and d^2f_p is surjective, we have $\partial_{11}f_1(p) \neq 0$. Rescaling y_1 we may assume $\partial_{11}f_1(p) = 1$, giving the "representative d^2f_p" in the table. The kernel of d^2f_p gives $T_pS[\begin{smallmatrix}1\\0\end{smallmatrix}](f) = \mathbb{R}_1$, and the third intrinsic derivative is a map $d^3f_p : \mathbb{R}_1 \to \mathrm{Hom}(\mathbb{R}_1, \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1))$. To simplify notation, identify $\mathrm{Hom}(\mathbb{R}_1, \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1))$ with $\mathbb{R}$:

$$\begin{aligned} d^3f_p : \mathbb{R}_1 &\to \mathbb{R} \\ v &\mapsto \partial_{30}f_1(p)v \end{aligned}$$

Surjectivity implies $\partial_{30} \neq 0$. This germ is precisely the $m = 1$ case of Lemma 2.1.10, hence there exists a Cerf change of coordinates to put f into the desired form (and moreover f is 3-determined). □

2.5.2 Type $S[1;1]$

Using the df_p from Table 2.14 for $S[\begin{smallmatrix}1\\1\end{smallmatrix}]$, then, using the convention from the table, the normal bundle for the second intrinsic derivative is isomorphic to the two-dimensional space $\mathrm{Hom}(\mathbb{R}_2 \hookrightarrow \mathbb{R}^2, \mathbb{R}_2 \twoheadrightarrow \mathbb{R}_2)$, where recall $\mathbb{R}_2$ denotes both $0 \times \mathbb{R}$ and $\mathbb{R}^2/(\mathbb{R} \times 0)$. This is isomorphic to $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}_2)$, hence the second intrinsic derivative is given by

$$\begin{aligned} d^2f_p : \mathbb{R}^2 &\to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}_2) \\ v &\mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{20}f_2(p) & \partial_{11}f_2(p) \\ \partial_{11}f_2(p) & \partial_{02}f_2(p) \end{bmatrix} v \right) \end{aligned}$$

Define subtypes $S[{}^1_1][r]$ by the dropped rank of the restriction to $d^2 f_P : \mathbb{R}_2 \to \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_2)$. We do not consider restrictions such as $\mathbb{R}_2 \to \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}_2)$ since they all have full rank due to $d^2 f_p$ being generically surjective. Note that even though $d^2 f_p$ is affected by symmetry of partial derivatives, since we are projecting to $\mathbb{R}_2 \to \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_2)$ the usual normal bundle calculations still apply for the third intrinsic derivative.

For $S[{}^1_1][1]$, then $\partial_{20} f_2(p) = 0$, so the third intrinsic derivative with respect to this subtype is

$$\begin{aligned} d^3 f_3 : \mathbb{R}_2 &\to \mathbb{R} \\ v &\mapsto \partial_{03} f_2(p) v \end{aligned}$$

This subtype has codimension equal to 3, so it does not generically occur.

Lemma 2.5.4. *If $f \in C^\infty_p(\mathbb{R}^2, \mathbb{R}^2)_q$ is a $S[{}^1_1][0]$ singularity, then there exist coordinates such that*

$$\begin{aligned} f^* y_1 &= x_1 \\ f^* y_2 &= a x_1^2 + x_2^2 \end{aligned}$$

where $a \in \{-1, 1\}$.

Proof. We have that $\partial_{02} f_2(p) \neq 0$ and that the matrix for $d^2 f_p$ is non-singular. Thus, we may apply a reparameterization in the style of the Morse Lemma using the variable order x_2, x_1 to let us assume that $f^* y_2 = \pm x_1^2 \pm x_2^2$. Negating y_2 if needed, then there exists some $a \in \{-1, 1\}$ such that $f^* y_2 = a x_1^2 + x_2^2$. The effect of these reparameterizations is that $f^* y_1 = x_1 + h(x)$ for some $h \in \mathfrak{m}_p^2$.

Consider for $\gtrsim 0$ the substitution $x_1 = c\overline{x}_1$, $x_2 = c\overline{x}_2$, $\overline{y}_1 = c y_1$, and $\overline{y}_2 = c^2 y_2$. Then

$$\begin{aligned} f^* \overline{y}_1 &= \overline{x}_1 + c^{-1} h(c\overline{x}_1, c\overline{x}_2) \\ f^* \overline{y}_2 &= a\overline{x}_1^2 + \overline{x}_2^2. \end{aligned}$$

Since $h \in \mathfrak{m}_p^2$, as $c \to 0$ then $c^{-1} h(c\overline{x}_1, c\overline{x}_2)$ converges to 0 in the C^∞ topology. Hence it suffices to show that the limiting case is locally stable, for which we will use We use Corollary 1.4.11.

Consider the germ with $f^* y_1 = x_1$ and $f^* y_2 = a x_1^2 + x_2^2$. We calculate the ideal $I = (f^* y_1^2, f^* y_2) = (x_1^2, a x_1^2 + x_2^2) = (x_1^2, x_2^2)$. Furthermore,

$$\begin{aligned} f_* \tfrac{\partial}{\partial x_1} &= \tfrac{\partial}{\partial y_1} + 2 a x_1 \tfrac{\partial}{\partial y_2} \\ f_* \tfrac{\partial}{\partial x_2} &= 2 x_2 \tfrac{\partial}{\partial y_2}. \end{aligned}$$

Let $A = f_* C^\infty_p(\mathbb{R}^2) + I C^\infty_p(f^* T\mathbb{R}^2)$. It suffices to show that

$$C^\infty_p(f^* T\mathbb{R}^2) = A + \mathbb{R}\tfrac{\partial}{\partial y_1} + \mathbb{R} x_1 \tfrac{\partial}{\partial y_1} + \mathbb{R}\tfrac{\partial}{\partial y_2}.$$

We see that every $x_1^i x_2^j \frac{\partial}{\partial y_k}$ is equivalent modulo $IC_p^\infty(f^*T\mathbb{R}^2)$ such a term with $0 \leq i, j \leq 1$. Using $f_* \frac{\partial}{\partial x_1}$, then modulo A we may assume that $k = 2$, and then $f_* \frac{\partial}{\partial x_1}$ gives us that if $j > 0$ the term is in A. By this reasoning, such terms that might not be in A are $\frac{\partial}{\partial y_2}$ and $x_1 \frac{\partial}{\partial y_2}$. The latter is equivalent modulo A to $-\frac{1}{2a} \frac{\partial}{\partial y_1}$, so we are done. □

2.5.3 Multijet singularities

We will mention the multigerm singularities that can occur, but we will not formally derive normal forms. Like usual, for codimension reasons generically the codimension-2 singularities cannot coincide with any other singularity type other than $S[\begin{smallmatrix}0\\0\end{smallmatrix}]$. The interesting type is $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$, which is a multijet singularity where two $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ types map to the same point (this is illustrated in Figure 2.8). The $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ type gives a curve of singularities, and the images of these curves can meet — generically, they must meet transversely.

2.6 Classification of 3-dimensional Cerf singularities

Our goal is to classify the 3-dimensional singularities that will help us determine the relations for an extended 2D TQFT. For this end, we do not need to consider all germs $f : \mathbb{R}^3 \to (\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, but rather those from maps $\Sigma \times I \mapsto \Sigma \times I$ that commute with the projection to I. Hence, we will instead consider smooth map germs $f : \mathbb{R}^3 \to (\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, up to the action of $\mathrm{Diff}_p(\mathbb{R}^3) \times \mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, with the germ centers p and q like before, and where $f_3(x) = x_3$. We call these *Cerf* singularities, which we hope does not cause confusion in context — these 3-dimensional Cerf singularities are the Cerf singularities for the 2-dimensional singularities classified in Section 2.5.

We analyze such germs as germs $f' \in C_p^\infty(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2)$ up to Cerf equivalence, where $\mathbb{R}^2 \times \mathbb{R}$ is given the product foliation $\mathbb{R}^2 \subseteq \mathbb{R}^3$ and $\mathbb{R}^2$ is given the standard $\mathbb{R} \subseteq \mathbb{R}^2$ foliation. Recall that Cerf equivalence means that we consider such germs up to equivalence as $\mathbb{R}^3 \to \mathbb{R}^3$ maps with foliated codomain. The results are summarized in Table 2.16, with germs represented in this f' form, and illustrated in Figure 2.9.

The first-order classification of maps $(\mathbb{R}^2 \subseteq \mathbb{R}^3) \to (\mathbb{R} \subseteq \mathbb{R}^2)$ up to Cerf equivalence is given in Table 2.17. The singularity types are denoted by $S\begin{bmatrix} r_1 \\ r_2 \\ 0 \end{bmatrix}$, where $r_1 = \operatorname{drk} df_p$ and $r_2 = \operatorname{drk} d(\pi \circ f)_p$, where $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ is the foliation-respecting projection. Like usual, the additional 0 in the symbol is a reminder that these maps are implicitly extended with $f_3(x) = x_3$ and a $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$ foliated codomain.

Lemma 2.6.1. *If $f \in C_p^\infty(\mathbb{R}^3, \mathbb{R}^2)_q$ is a $S\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}[0]$ singularity, then there exist coordinates such that*

$$f^* y_1 = x_1 \qquad\qquad f^* y_2 = x_2$$

Proof. This is essentially the same as Lemma 2.5.1. □

2.6.1 Type $S[1;0;0]$

After a reparameterization by linear maps in $\mathrm{Diff}_p(\mathbb{R}^2 \subseteq \mathbb{R}^3)$ and $\mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, the differential of a $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$ singularity can be put into the form as in Table 2.17:

$$df_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

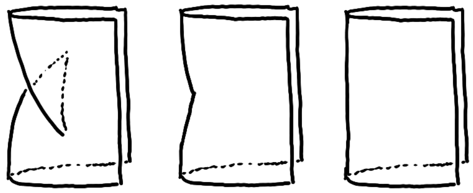
Using the implicit function theorem, we may reparameterize further so that $f_2(x) = x_2$. The normal bundle for this type is isomorphic to $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1)$ where, continuing with the convention from before, $\mathbb{R}_1$ denotes both $\mathbb{R} \times 0 \times 0$ and $\mathbb{R}^2/(0 \times \mathbb{R})$. Hence, identifying $\mathbb{R}$ with this normal bundle, the second intrinsic derivative is

$$\begin{aligned} d^2 f_p &: \mathbb{R}^3 \to \mathbb{R} \\ v &\mapsto \begin{bmatrix} \partial_{200} f_1(p) & \partial_{110} f_1(p) & \partial_{101} f_1(p) \end{bmatrix} v \end{aligned}$$

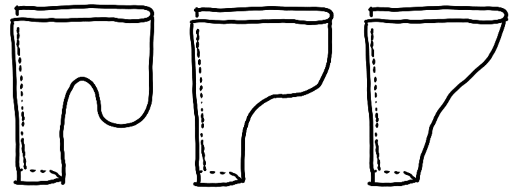
Table 2.16: Classification of germs of 3-dimensional Cerf singularities.

Type	Codim.	Representative germ	See:	Suspends[a]
$S\left[\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right]$	0	$f^*y_1 = x_1$ $f^*y_2 = x_2$	Lemma 2.6.1	$S[0], S\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\end{smallmatrix}\right]$	1	$f^*y_1 = x_1^2$ $f^*y_2 = x_2$	Lemma 2.6.2	$S[1], S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][0]$
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\end{smallmatrix}\right][0]$	2	$f^*y_1 = x_1x_2 + x_1^3$ $f^*y_2 = x_2$	Lemma 2.6.3	$S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right][1]$
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\end{smallmatrix}\right][1]$	3	$f^*y_1 = x_1x_2 + x_1^2x_3 + x_1^4 + O(x)^5$ $f^*y_2 = (1 + O(x)^2)x_2$	Lemma 2.6.4	
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	3	$f^*y_1 = x_1x_3 + x_1x_2^2 + ax_1^3 + O(x)^4$ $f^*y_2 = x_2$	Lemma 2.6.5	
$S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	2	$f^*y_1 = x_1$ $f^*y_2 = ax_1^2 + x_2^2 + O(x)^3$	Lemma 2.6.6	$S\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right][0]$
$S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$	3	$f^*y_1 = (1 + O(x))x_1$ $f^*y_2 = x_1x_3 + x_2^2 + x_1^3 + O(x)^4$	Lemma 2.6.7	
$S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$	3	$f^*y_1 = (1 + O(x))x_1$ $f^*y_2 = x_1x_2 + x_2^2x_3 + x_2^3 + O(x)^4$	Lemma 2.6.8	

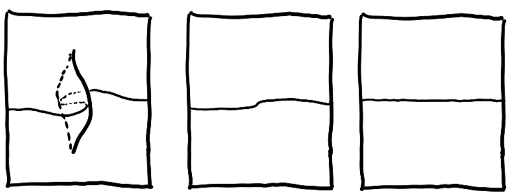
[a] See Tables 2.12 and 2.13.



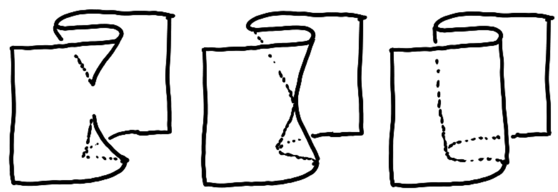
$S\begin{bmatrix}1\\0\\0\end{bmatrix}[1\ 0][1]$



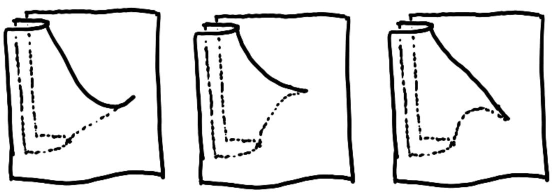
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$



$S\begin{bmatrix}1\\0\\0\end{bmatrix}[1\ 1]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$



$S\begin{bmatrix}1\\0\\0\end{bmatrix}[1\ 1]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$



$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$

Figure 2.9: Illustrations of the 3D Cerf singularities from Table 2.16. Each set of three diagrams is showing the $t = -1$, $t = 0$, and $t = 1$ movie frames. Suspensions of 2D singularities are not shown.

Table 2.17: First-order types of 3-dimensional Cerf singularities.

Type	Repres. df_p	Codim.	Generic
$S\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\left[\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right]$	0	✓
$S\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\left[\begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix}\right]$	1	✓
$S\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right]$	2	✓
$S\begin{bmatrix}2\\1\\0\end{bmatrix}$	$\left[\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right]$	4	

We define subtypes $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}r & s\end{smallmatrix}]$ where r is the dropped rank when $d^2 f_p$ is restricted to $\mathbb{R}_1$, and s is the same when restricted to $\mathbb{R}_{12}$. The possibilities are enumerated in Table 2.18. For the "representative $d^2 f_p$" column, we perform the usual kinds of changes-of-variables to put the matrix into the given form.

Lemma 2.6.2. *If $f \in C_p^\infty(\mathbb{R}^3, \mathbb{R}^2)_q$ is a $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}0 & 0\end{smallmatrix}]$ singularity, then there exist coordinates such that*

$$\begin{aligned} f^* y_1 &= x_1^2 \\ f^* y_2 &= x_2 \end{aligned}$$

Proof. The proof is essentially the same as Lemma 2.5.2, but we make use of $f_3 = x_3$ to help clear out more partial derivatives of f_1. $\square$

Type $S[1; 0; 0][1\ \ 0]$

The second intrinsic derivative for type $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][1\ \ 0]$ can be put in the form represented in Table 2.18:

$$\begin{aligned} d^2 f_p : \mathbb{R}^3 &\to \mathbb{R} \\ v &\mapsto \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} v \end{aligned}$$

The kernel of this map is $T_p S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$, and the normal bundle for $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 0\end{smallmatrix}]$ is isomorphic to $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}) \approx \mathbb{R}$, where as usual $\mathbb{R}_1 = \mathbb{R} \times 0 \times 0$. The third intrinsic derivative is given by

$$\begin{aligned} d^3 f_p : \mathbb{R} \times 0 \times \mathbb{R} &\to \mathbb{R} \\ v &\mapsto \begin{bmatrix} \partial_{300} f_1(p) & \partial_{201} f_1(p) \end{bmatrix} v \end{aligned}$$

We define subtypes $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 0\end{smallmatrix}][r]$ based on the dropped rank of $d^3 f_p$ when restricted to $\mathbb{R}_1$. These are summarized in Table 2.19 and will be analyzed in the following lemmas.

Table 2.18: Second-order types for $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$.

Type	Representative $d^2 f_p$	Addl. Codim.	Codim.	Generic
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}0 & 0\end{smallmatrix}]$	$[\begin{smallmatrix}2 & 0 & 0\end{smallmatrix}]$	0	1	✓
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 0\end{smallmatrix}]$	$[\begin{smallmatrix}0 & 1 & 0\end{smallmatrix}]$	1	2	✓
$S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 1\end{smallmatrix}]$	$[\begin{smallmatrix}0 & 0 & 1\end{smallmatrix}]$	2	3	✓

Lemma 2.6.3. *If $f \in C_p^\infty(\mathbb{R}^3, \mathbb{R}^2)_q$ is a $S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\;0\,][0]$ singularity, then there exist coordinates such that*

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_1^3 \\ f^*y_2 &= x_2 \end{aligned}$$

Proof. This is similar to Lemma 2.5.3 (and by extension the $m = 1$ case of Lemma 2.1.10), but we can make use of $f_3(x) = x_3$ to clear out more terms, and so there exists coordinates such that

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_1^3 + r(x) \\ f^*y_2 &= x_2 \end{aligned}$$

for some $r \in \mathfrak{m}_p^4$. The argument in Lemma 2.1.10 carries over for why the closure of the orbit of this singularity type contains the germ with $r = 0$, so it suffices to show that the germ with $r = 0$ is locally stable.

We use Lemma 1.4.14 with the germ as a map $\mathbb{R}^3 \to \mathbb{R}^3$. We have

$$\begin{aligned} f_*\tfrac{\partial}{\partial x_1} &= (x_2 + 3x_1^2)\tfrac{\partial}{\partial y_1} \\ f_*\tfrac{\partial}{\partial x_2} &= x_1\tfrac{\partial}{\partial y_1} + \tfrac{\partial}{\partial y_2} \\ f_*\tfrac{\partial}{\partial x_3} &= \tfrac{\partial}{\partial y_3} \end{aligned}$$

and we calculate the ideal $I = C_p^\infty(\mathbb{R}^3)f^*(\mathfrak{m}_q(\mathbb{R}^3/\mathbb{R}^2) + \mathfrak{m}_q(\mathbb{R}^3/\mathbb{R})^2) = (x_2^2, x_3)$. Letting $A = f_*C_p^\infty(T\mathbb{R}^3) + IC_p^\infty(f^*T\mathbb{R}^3)$, then it suffices to show that

$$C_p^\infty(f^*T\mathbb{R}^3) = A + f^*C^\infty(\mathbb{R}^3)\tfrac{\partial}{\partial y_1} + f^*C^\infty(\mathbb{R}^3/\mathbb{R})\tfrac{\partial}{\partial y_2} + f^*C^\infty(\mathbb{R}^3/\mathbb{R}^2)\tfrac{\partial}{\partial y_3}.$$

We see that $f_*C_p^\infty(T\mathbb{R}^3)$ gives $\frac{\partial}{\partial y_2} \equiv_A -x_1\frac{\partial}{\partial y_1}$ and $\frac{\partial}{\partial y_3} \equiv_A 0$. We also have $x_2^2\frac{\partial}{\partial y_1} \equiv_A 9x_1^4\frac{\partial}{\partial y_1}$, hence A contains $(x_1^4, x_2^2, x_3)\frac{\partial}{\partial y_1}$.

Using $f^*C^\infty(\mathbb{R}^3)\frac{\partial}{\partial y_1}$, we can get $\frac{\partial}{\partial y_1}$, $x_2\frac{\partial}{\partial y_1}$, and $x_1^2\frac{\partial}{\partial y_1}$. Using $f^*C^\infty(\mathbb{R}^3/\mathbb{R})\frac{\partial}{\partial y_2}$, we can get $x_1\frac{\partial}{\partial y_1}$ and $x_1^3\frac{\partial}{\partial y_1}$. This completes the proof. □

Table 2.19: Third-order types for $S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\;0\,]$.

Type	Representative d^3f_p	Addl. Codim.	Codim.	Generic
$S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\;0\,][0]$	$[\,6\;0\,]$	0	2	✓
$S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\;0\,][1]$	$[\,0\;2\,]$	1	3	✓

Lemma 2.6.4. *If $f \in C_p^\infty(\mathbb{R}^3, \mathbb{R}^2)_q$ is a $S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\ 0\,][1]$ singularity, then there exist coordinates such that*

$$f^* y_1 = x_1x_2 + x_1^2x_3 + x_1^4 + O(x)^5$$
$$f^* y_2 = (1 + g(x))x_2$$

where $g \in \mathfrak{m}_p^2$.

Proof. In this case, $\partial_{300}f_1(p) = 0$, so $\partial_{201}f_1(p) \neq 0$ by surjectivity of $d^3 f_p$. Hence by rescaling y_1 we obtain the $d^3 f_p$ listed in Table 2.19.

Consider now the fourth intrinsic derivative. We have $\ker d^3 f_p = T_p S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\ 0\,]$, and the normal bundle for $S\begin{bmatrix}1\\0\\0\end{bmatrix}[\,1\ 0\,][1]$ is isomorphic to $\mathbb{R}$, hence $d^4 f_p$ is given by

$$d^4 f_p : \mathbb{R}_1 \to \mathbb{R}$$
$$v \mapsto \partial_{400}f_1(p)v$$

By surjectivity, $\partial_{400}f_1(p) \neq 0$.

After a reparameterization of the form $\overline{y}_1 = y_1 + ay_2^2 + by_1y_2 + cy_2^2$, we may assume all partial derivatives through the second vanish for $f_1(x) - x_1x_2$. By Lemma 1.1.69, there are functions h_{ijk} such that

$$f_1(x) = x_1x_2 + \sum_{i+j+k=3} h_{ijk}(x)x_1^i x_2^j x_3^k$$

with $h_{300}(p) = 0$ and $h_{201}(p) = 1$. Reparameterizing with $x_1 = \overline{x_1} - \frac{1}{2}h_{102}(p)\overline{x_3}$ and then reparameterizing y_1 to cancel out the new x_2x_3 quadratic term, we may assume $h_{102}(p) = 0$. Additionally, we may assume $h_{003}(p) = 0$ by a reparameterization of the form $\overline{y}_1 = y_1 + ay_3^3$. Thus, by reparameterizing

$$\overline{x}_1 = x_1 + \sum_{\substack{i+j+k=3\\ j\geq 1}} h_{ijk}(x)x_1^i x_2^{j-1} x_3^k$$
$$\overline{x}_2 = x_2$$
$$\overline{x}_3 = x_3$$

we may assume

$$f_1(x) = x_1x_2 + h_{300}(x)x_1^3 + h_{201}(x)x_1^2x_3 + h_{102}(x)x_1x_3^2 + h_{003}(x)x_3^3 + O(x)^5$$

with $h_{201}(p) = 1$ and $h_{300}(p) = h_{102}(p) = h_{003}(p) = 0$. In particular, all partial derivatives through the third vanish for $f_1(x) - x_1x_2 - x_1^2x_3$.

A substitution of the form $x_1 = \overline{x}_1$, $x_2 = \overline{x}_2 + a\overline{x}_3^2$, $x_3 = \overline{x}_3$ along with $\overline{y}_1 = y_1$, $\overline{y}_2 = y_2 - ay_3^2$, $\overline{y}_3 = y_3$ lets us assume $\partial_{001}h_{102}(p) = 0$. Additionally, by rescaling all the variables, we may assume $\partial_{100}h_{300}(p) = 1$.

by Lemma 1.1.69 again, there are functions h_{ijk} with $h_{400}(p) = 1$ such that

$$\begin{aligned} f_1(x) &= x_1x_2 + x_1^2x_3 + h_{400}(x)x_1^4 + h_{310}(x)x_1^3x_2 + h_{301}(x)x_1^3x_3 \\ &\quad + h_{211}(x)x_1^2x_2x_3 + h_{202}(x)x_1^2x_3^2 + h_{112}(x)x_1x_2x_3^2 + h_{103}(x)x_1x_3^3 \\ &\quad + h_{013}(x)x_2x_3^3 + h_{004}(x)x_3^4 + O(x)^5 \\ &= (1 + h_{310}(x)x_1^2 + h_{211}(x)x_1x_3 + h_{112}(x)x_3^2)x_1x_2 \\ &\quad + (1 + h_{301}(x)x_1 + h_{202}(x)x_3)x_1^2x_3 \\ &\quad + h_{103}(x)x_1x_3^2 + h_{400}(x)x_1^4 + h_{013}(x)x_2x_3^3 + h_{004}(x)x_3^4 + O(x)^5. \end{aligned}$$

By a reparameterization of the y variables, we can assume $h_{013}(p) = 0$ and $h_{004}(p) = 0$. By virtue of Lemma 1.1.69, h_{103}, h_{013} and h_{004} are in $\mathfrak{m}_p$ and so their corresponding terms are in $\mathfrak{m}_p^5$. Similarly, $h_{400}(x)x_1^4 \in x_1^4 + \mathfrak{m}_p^5$. Together, these let us simplify the expression:

$$\begin{aligned} f_1(x) &= (1 + h_{310}(x)x_1^2 + h_{211}(x)x_1x_3 + h_{112}(x)x_3^2)x_1x_2 \\ &\quad + (1 + h_{301}(x)x_1 + h_{202}(x)x_3)x_1^2x_3 \\ &\quad + x_1^4 + O(x)^5. \end{aligned}$$

The substitution $\overline{x}_1 = \sqrt{|1 + h_{301}(x)x_1 + h_{202}(x)x_3|}x_1$ lets us write

$$f_1(x) = g(x)x_1x_2 + ax_1^2x_3 + x_1^4 + O(x)^5$$

with $g : \mathbb{R}^3 \to \mathbb{R}$ such that $g(p) = 1$ and $g - 1 \in \mathfrak{m}_p^2$ and with $a \in \{-1, 1\}$. Lastly, by $\overline{x}_2 = g(x)x_2$ and potentially negating y_3 we get the result. $\square$

Type $S[1; 0; 0][1\ 1]$

In this case, as represented in Table 2.18, the second intrinsic derivative is

$$\begin{aligned} d^2f_p &: \mathbb{R}^3 \to \mathbb{R} \\ &v \mapsto \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} v \end{aligned}$$

The kernel of this map is $T_pS\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$, and the normal bundle for $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 1\end{smallmatrix}]$ is isomorphic to $\mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R})$, where $\mathbb{R}_{12} = \mathbb{R}^2 \times 0$. The third intrinsic derivative is given by

$$\begin{aligned} d^3f_p &: \mathbb{R}_{12} \to \mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}) \\ &v \mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{300}f_1(p) & \partial_{210}f_1(p) \\ \partial_{210}f_1(p) & \partial_{120}f_1(p) \end{bmatrix} v \right) \end{aligned}$$

We define the subtypes $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right][\begin{smallmatrix}1 & 1\end{smallmatrix}][\begin{smallmatrix}a & b\\ b & c\end{smallmatrix}]$ by the dropped ranks for the restrictions $d^3f_p : V \to \mathrm{Hom}(W, \mathbb{R})$ with $V, W \in \{\mathbb{R}_1, R_{12}\}$. By reason of codimension, $a = b = c = 0$ is the only possibility.

Lemma 2.6.5. *If f is a $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}0 & 0\\0 & 0\end{smallmatrix}\right]$ singularity, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= x_1 x_3 + x_1 x_2^2 + a x_1^3 + O(x)^4 \\ f^* y_2 &= x_2 \end{aligned}$$

where $a \in \{\pm 1\}$.

Proof. After a reparameterization, the Hessian for f_1 is

$$Hf_1(p) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

After another reparameterization, the matrix for $d^3 f_p$ is

$$\begin{bmatrix} \partial_{300} f_1(p) & \partial_{210} f_1(p) \\ \partial_{210} f_1(p) & \partial_{120} f_1(p) \end{bmatrix} = \begin{bmatrix} 6a & 0 \\ 0 & 2b \end{bmatrix}$$

with $a, b \in \{-1, 1\}$. After yet more tedious reparameterizations, we can get

$$f_1(x) = x_1 x_3 + a x_1^3 + b x_1 x_2^2 + O(x)^4.$$

Changing signs of variables yields the result. ☐

2.6.2 Type $S[1;1;0]$

After a reparameterization by a linear map in $\mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, the differential of a $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$ singularity can be put into the form as in Table 2.17:

$$df_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the implicit function theorem, we may reparameterize further so that $f_1(x) = x_1$. The normal bundle for this type is isomorphic to $\mathrm{Hom}(\mathbb{R}_2 \hookrightarrow \mathbb{R}_{12}, \mathbb{R}_2 \twoheadrightarrow \mathbb{R}_2)$ where $\mathbb{R}_2$ denotes both $0 \times \mathbb{R} \times 0$ and $\mathbb{R}^2/(\mathbb{R} \times 0)$. This is isomorphic to the two-dimensional space $\mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2)$. Hence, the second intrinsic derivative is

$$\begin{aligned} d^2 f_p &: \mathbb{R}^3 \to \mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2) \\ v &\mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{200} f_2(p) & \partial_{110} f_2(p) & \partial_{101} f_2(p) \\ \partial_{110} f_2(p) & \partial_{020} f_2(p) & \partial_{011} f_2(p) \end{bmatrix} v \right) \end{aligned}$$

We define subtypes $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}r & s\\s & t\end{smallmatrix}\right]$ where r, s, and t are dropped ranks according to the following scheme:

$$\begin{array}{r c} & \begin{array}{cc} \mathbb{R}_2 & \mathbb{R}_{12} \end{array} \\ \begin{array}{r} \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_2) \\ \mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2) \end{array} & \begin{bmatrix} * & * \\ * & * \end{bmatrix} \end{array} \qquad \text{for coranks of } \begin{bmatrix} \partial_{020} h(p) & \partial_{110} h(p) \\ \partial_{110} h(p) & \partial_{200} h(p) \end{bmatrix}$$

For example, s is the dropped rank for both $d^2 f_p : \mathbb{R}^2 \to \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_2)$ and $d^2 f_p : \mathbb{R}_2 \to \mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2)$. The cases where the second intrinsic derivative are surjective are listed in Table 2.20, and as usual we reparameterize to get a "representative $d^2 f_p$" — both a and b are -1 or 1.

Lemma 2.6.6. *If f is a $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$ singularity, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= x_1 \\ f^* y_2 &= a x_1^2 + x_2^2 + O(x)^3 \end{aligned}$$

where $a \in \{-1, 1\}$.

Proof. This is essentially the same as Lemma 2.5.4, where we can use a reparameterization with $\overline{y}_2 = y_2 + c y_3^2$ for some c to remove a x_3^2 term. □

Lemma 2.6.7. *If f is a $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ singularity, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= (1 + g(x)) x_1 \\ f^* y_2 &= x_1 x_3 + x_2^2 + x_1^3 + O(x)^4 \end{aligned}$$

where $g \in \mathfrak{m}_p$.

Proof. By usual sorts of parameter changes, the $d^2 f_p$ can be put into the form as show in Table 2.20. We can furthermore put the Hessian of f_2 at p into the form

$$H f_2(p) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2a & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with $a \in \{-1, 1\}$. Using Morse-lemma-like reparameterizations and Lemma 1.1.69, there are functions $h_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f_2(x) = h_{200}(x) x_1^2 + a x_2^2 + h_{002}(x) x_3^2 + h_{101}(x) x_1 x_3$$

Table 2.20: Second-order types for $S\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Type	Representative $d^2 f_p$	Addl. Codim.	Codim.	Generic
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2a&0&0\\0&2b&0\end{smallmatrix}\right]$	0	2	✓
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0&b\\0&2a&0\end{smallmatrix}\right]$	1	3	✓
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&a&0\\a&0&0\end{smallmatrix}\right]$	1	3	✓
$S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}2a&0&0\\0&0&b\end{smallmatrix}\right]$	2	4	

where $h_{002}(p) = 0$ and $h_{101}(p) = 1$.

The kernel of $d^2 f_p$ is $T_P S\begin{bmatrix}1\\1\\0\end{bmatrix}$, which is $\mathbb{R}$, and the normal bundle for the subtype at p is isomorphic to $\mathrm{Hom}(\ker d^2 f_p|_{\mathbb{R}^2}, \operatorname{coker} d^2 f_p|_{\mathbb{R}^2})$. We have $\ker d^2 f_p|_{\mathbb{R}^2} = \mathbb{R}$ and $\operatorname{coker} d^2 f_p|_{\mathbb{R}^2} \approx \mathrm{Hom}(\mathbb{R}, \mathbb{R}_2)$, so it is 1-dimensional. Hence, we can write the third intrinsic derivative as

$$\begin{aligned} d^3 f_p : \mathbb{R} &\to \mathbb{R} \\ v &\mapsto \partial_{300} f_2(p) v \end{aligned}$$

and thus $\partial_{300} f_2(p) \neq 0$, and by rescaling variables we may assume $\partial_{300} f_2(p) = 6$. By Lemma 1.1.69, there are functions $h_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{aligned} f_2(x) = x_1 x_3 + a x_2^2 &+ h_{300}(x) x_1^3 + h_{210}(x) x_1^2 x_2 + h_{201}(x) x_1^2 x_3 \\ &+ h_{102}(x) x_1 x_3^2 + h_{012}(x) x_2 x_3^2 + h_{003}(x) x_3^3 \\ &+ h_{111}(x) x_1 x_2 x_3 \end{aligned}$$

and $h_{300}(p) = 1$. Using another Morse-lemma-like reparameterization, we may assume $h_{210} = h_{021} = h_{111} = 0$, and we may also assume $h_{003}(p) = 0$ and $h_{300}(p) = 0$.

$$\begin{aligned} f_2(x) &= x_1 x_3 + a x_2^2 + x_1^3 + h_{201}(x) x_1^2 x_3 + h_{102}(x) x_1 x_3^2 + O(x)^4 \\ &= (1 + h_{201}(x) x_1 + h_{102}(x) x_3) x_1 x_3 + a x_2^2 + x_1^3 + O(x)^4 \end{aligned}$$

Reparameterizing $\overline{x}_1 = (1 + h_{201}(x) x_1 + h_{102}(x) x_3) x_1$ yields the result. □

Lemma 2.6.8. *If f is a $S\begin{bmatrix}1\\1\\0\end{bmatrix}\left[\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right]$ singularity, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= (1 + g(x)) x_1 \\ f^* y_2 &= x_1 x_2 + x_2^2 x_3 + x_2^3 + O(x)^4 \end{aligned}$$

where $g \in \mathfrak{m}_p$.

Proof. By usual sorts of parameter changes, the $d^2 f_p$ can be put into the form as show in Table 2.20. With $a \in \{-1, 1\}$, we can furthermore put the Hessian of f_2 at p into the form

$$H f_2(p) = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identifying the normal bundle at p with $\mathbb{R}$, the third intrinsic derivative is

$$\begin{aligned} d^3 f_p : \mathbb{R}_3 &\to \mathbb{R} \\ v &\mapsto \partial_{021} f_2(p) v \end{aligned}$$

and thus $\partial_{021} f_2(p) \neq 0$. Restricting $d^2 f_p$ to $\mathbb{R}_2$ in the codomain, we get a map

$$\begin{aligned} d^2 f_p &: \mathbb{R}^3 \to \operatorname{Hom}(\mathbb{R}_2, \mathbb{R}_2) \\ &v \mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{110} f_2(p) & \partial_{020} f_2(p) & \partial_{011} f_2(p) \end{bmatrix} v\right) \end{aligned}$$

and we have already established that this matrix is $\left[\begin{smallmatrix} a & 0 & 0 \end{smallmatrix}\right]$. The third intrinsic derivative of f with respect to this is

$$\begin{aligned} d^3 f_p &: \mathbb{R}_{23} \to \mathbb{R} \\ &v \mapsto \begin{bmatrix} \partial_{030} f_2(p) & \partial_{021} f_2(p) \end{bmatrix} v \end{aligned}$$

There are subtypes based on the dropped rank when restricted to R_2. Since $S\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a codimension-3 singularity, by reason of codimension, we may assume $\partial_{030} f_2(p) \neq 0$.

By Morse-lemma-like reparameterizations and Lemma 1.1.69, there are functions $h_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f_2(x) = a x_1 x_2 + h_{020}(x) x_2^2 + h_{011}(x) x_2 x_3 + h_{002}(x) x_3^2$$

and $h_{020}(p) = h_{011}(p) = h_{002}(p) = 0$. Further application of Lemma 1.1.69 gives

$$\begin{aligned} f_2(x) = a x_1 x_2 &+ h_{120}(x) x_1 x_2^2 + h_{030}(x) x_2^3 + h_{021}(x) x_2^2 x_3 \\ &+ h_{111}(x) x_1 x_2 x_3 + h_{012}(x) x_2 x_3^2 \\ &+ h_{102}(x) x_1 x_3^2 + h_{003}(x) x_3^3 \end{aligned}$$

with $h_{030}(p) = b$ and $h_{021}(p) = c$ where $b, c \neq 0$. Reparameterizing, we may assume $h_{003}(p) = 0$, hence

$$\begin{aligned} f_2(x) = (a &+ h_{120}(x) x_2 + h_{111}(x) x_3) x_1 x_2 + b x_2^3 + c x_2^2 x_3 \\ &+ h_{012}(x) x_2 x_3^2 + h_{102}(x) x_1 x_3^2 + O(x)^4 \end{aligned}$$

A substitution with $x_1 = \overline{x}_1 + k\overline{x}_3^2$ for some k (followed by a reparameterization of the y variables to clear out the resulting $\overline{x}_3^2$ in $f^* y_1$) lets us assume $h_{012}(p) = 0$. Similarly, one with $x_2 = \overline{x}_2 + k\overline{x}_3^2$ for some k lets us assume $h_{102}(p) = 0$. Hence, substituting $\overline{x}_1 = (a + h_{120}(x) x_2 + h_{111}(x) x_3) x_1$ gives

$$\begin{aligned} f_1(x) &= (a^{-1} + g(x)) x_1 \\ f_2(x) &= x_1 x_2 + b x_2^3 + c x_2^2 x_3 + O(x)^4 \end{aligned}$$

with $g \in \mathfrak{m}_p$. Scaling variables gives the result. ∎

2.6.3 Multijet singularities

There are some important multijet singularities that occur now at this higher dimension. The codimension-3 singularities of course do not generically map to the same point or occur at the same point in time. More interestingly, the $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]|[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$ singularities (see Figure 2.8 themselves have multijet singularities similar to those moves from Figure 2.4 that are like the Reidemeister II and Reidemeister III moves — the images of the singularity submanifolds may become momentarily tangent, and three may coincide for a single moment in time. Another interesting move is that the suspensions of the codimension-2 two-dimensional singularities can momentarily intersect suspensions of $S[\begin{smallmatrix}1\\0\end{smallmatrix}][0]$, which is when these singularity points move past a fold. There are also many interchange moves to change the temporal ordering of codimension-2 singularities. We do not enumerate them here.

2.7 Classification of $(0+1)$-dimensional singularities

We now extend the work from Sections 2.4 to 2.6 to the case of singularities of manifolds containing a codimension-1 submanifold, which we will pursue for the remainder of the chapter. It should be said that the way we add rank conditions for a submanifold is to take a submanifold in the jet bundle that lies over the given submanifold — a consequence to this is that the codimension of a first-order singularity is normal bundle dimension plus the

Table 2.21: Classification of germs of $(0+1)$-dimensional singularities.

Type	Codim.[a]	Representative germ	See:
$S[\,0\;0\,]$	1	$f^*y_1 = x_1$	Lemma 2.7.1

[a] The convention is that the codimension is with respect to the manifold, not the submanifold.

Table 2.22: Classification of germs of $(1+1)$-dimensional singularities.

Type	Codim.	Representative germ[b]	See:	Suspends
$S[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}]$	1	$f^*y_1 = x_2$ $f^*y_2 = x_1$	Lemma 2.8.1	$S[\,0\;0\,]$
$S[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}]$	2	$f^*y_1 = x_1$ $f^*y_2 = x_1^2 + x_2$	Lemma 2.8.2	
$S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\,0\;0\,]$	2	$f^*y_1 = x_1x_2 + x_2^2 + O(x)^3$ $f^*y_2 = x_1$	Lemma 2.8.3	

[b] The submanifold is parameterized by x_1.

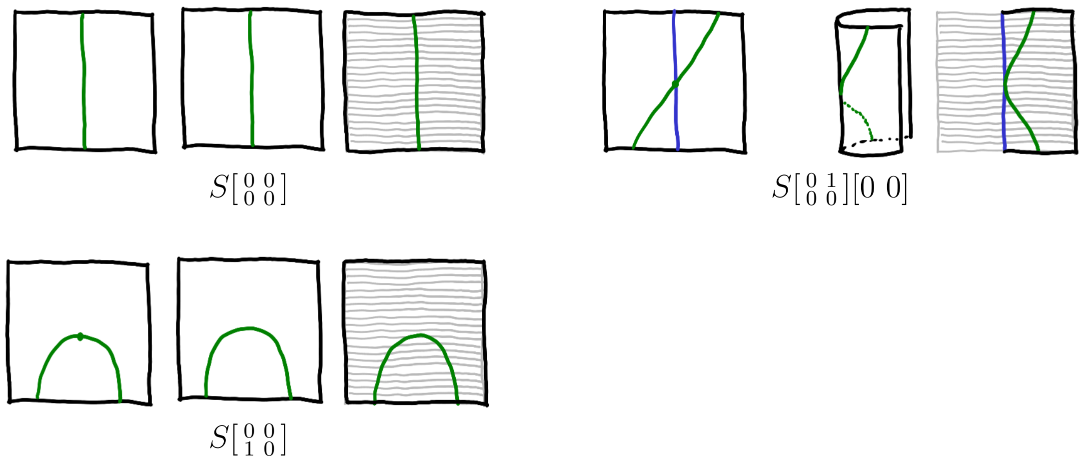


Figure 2.10: Illustrations of the $(1+1)$-dimensional singularities from Table 2.22. This is using the same convention as Figure 2.7, but the 1-dimensional submanifold is depicted in green.

codimension of the submanifold.

In this section, we consider 0-dimensional submanifolds in a 1-manifold. In particular, we consider generic germs $f : \mathbb{R} \to \mathbb{R}$ with a fixed 0-dimensional submanifold $\mathbb{R}^0 \subseteq \mathbb{R}$ in the domain. As usual, let $p = 0 \in \mathbb{R}$ be the center of the germ, where $p \in \mathbb{R}^0$, and let $q = f(p) = 0$. The classification is done up to $\mathrm{Diff}_p(\mathbb{R}) \times \mathrm{Diff}_q(\mathbb{R})$ where the $\mathrm{Diff}_p(\mathbb{R})$ component leaves the submanifold invariant (which is trivially satisfied in the 0-dimensional case).

The classification is easy. In general, we let $S[\begin{smallmatrix} a & b \end{smallmatrix}]$ be the submanifold of $J^1(\mathbb{R}, \mathbb{R})$ when pulled back along the inclusion $\mathbb{R}^0 \hookrightarrow \mathbb{R}$, where a denotes the dropped rank of the differential when restricted to $\mathbb{R}^0$ (only 0 is nontrivial) and b denotes the usual dropped rank; hence $S[\begin{smallmatrix} 0 & b \end{smallmatrix}] \subseteq S[b]$. In particular, $S[\begin{smallmatrix} 0 & 0 \end{smallmatrix}] = \alpha^{-1}(\mathbb{R}^0) \cap S[0]$ and $S[\begin{smallmatrix} 0 & 1 \end{smallmatrix}] = \alpha^{-1}(\mathbb{R}^0) \cap S[1]$. These are submanifolds of codimensions 1 and 2, and the latter does not generically occur.

Lemma 2.7.1. *If f is a $S[\begin{smallmatrix} 0 & 0 \end{smallmatrix}]$ singularity, then it is equivalent to a germ with*

$$f^* y_1 = x_1$$

Proof. The proof in Lemma 2.4.1 goes through; every element of $\mathrm{Diff}_p(\mathbb{R})$ fixes $\mathbb{R}^0$. $\square$

2.8 Classification of $(1+1)$-dimensional singularities

In this section, we consider germs $f : (\mathbb{R} \subseteq \mathbb{R}^2) \to (\mathbb{R} \subseteq \mathbb{R}^2)$ where $\mathbb{R}$ in the domain is considered to be a submanifold (and not a leaf of the product foliation). Let $p = 0 \in \mathbb{R} \subseteq \mathbb{R}^2$ in the domain and $q = f(p) = 0$. The classification is done up to $\mathrm{Diff}_p(\mathbb{R}^2; \mathbb{R}) \times \mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2)$

where $\mathrm{Diff}_p(\mathbb{R}^2;\mathbb{R})$ indicates those elements of $\mathrm{Diff}_p(\mathbb{R})$ that leave the submanifold $\mathbb{R}$ invariant. While $\mathrm{Diff}_p(\mathbb{R} \subseteq \mathbb{R}^2) \subseteq \mathrm{Diff}_p(\mathbb{R}^2;\mathbb{R})$, this is a strict inclusion since diffeomorphisms do not need to preserve any foliation of the submanifold's tubular neighborhood. The classification of germs is summarized in Table 2.22 and illustrated in Figure 2.10.

To define the first-order singularity types, let $C = \mathbb{R}$ and $\Sigma = \mathbb{R}^2$ in the domain to help distinguish them from the codomain. The differential df_p is an element of $\mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2)$, and we consider the four images of df_p in the following diagram of hom sets:

$$\begin{tikzcd}
\mathrm{Hom}(T_pC, T_q\mathbb{R}^2) \arrow[d, two heads] & \mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2) \arrow[l, two heads] \arrow[d, two heads] \\
\mathrm{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R}) & \mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2/T_q\mathbb{R}) \arrow[l, two heads]
\end{tikzcd}$$

The type $S[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}] \subseteq S[\begin{smallmatrix} c \\ d \end{smallmatrix}]$ indicates that the images of df_p lie in the following submanifolds:

$$\begin{matrix}
\mathrm{Hom}^a(T_pC, T_q\mathbb{R}^2) & \mathrm{Hom}^c(T_p\Sigma, T_q\mathbb{R}^2) \\
\\
\mathrm{Hom}^b(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R}) & \mathrm{Hom}^d(T_p\Sigma, T_q\mathbb{R}^2/T_q\mathbb{R})
\end{matrix}$$

We think of the first column as coming from fiber bundles over C rather than over Σ. The subset of those elements of $\mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2)$ whose images lie in the above submanifolds is a submanifold as well, and they assemble to form a submanifold of $J^1(\Sigma, \mathbb{R}^2)$ over C. For jet transversality, we automatically have transversality in the horizontal direction, and what remains is transversality in the vertical direction. Let K_a, K_b, K_c, K_d and L_a, L_b, L_c, L_d respectively be the kernels and cokernels of the images of df_p, where the indices suggest which images they correspond to. The second intrinsic derivative in this situation is a map

$$d^2 f_p : T_pC \to \mathrm{Hom}\left(\begin{tikzcd} K_a \arrow[r, hook] \arrow[d, hook] & K_c \arrow[d, hook] \\ K_b \arrow[r, hook] & K_d \end{tikzcd}, \begin{tikzcd} L_a \arrow[r, two heads] \arrow[d, two heads] & L_c \arrow[d, two heads] \\ L_b \arrow[r, two heads] & L_d \end{tikzcd} \right)$$

Restricting a $S[\begin{smallmatrix} c \\ d \end{smallmatrix}]$ submanifold to C yields a submanifold of one higher codimension, hence only $S[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$ and $S[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$ (see Table 2.14) generically occur over the submanifold. For example, if $a = 1$ (and thus $b = 1$), then the normal bundle would contain $\mathrm{Hom}(T_pC, \mathbb{R}^2)$, so it would be at least a codimension-3 phenomenon and generically not occur.

The generic first-order singular types are listed in Table 2.23. The codimensions are with respect to $\Sigma = \mathbb{R}^2$, and the "representative df_p" column is from applying linear coordinate changes that leave $C = \mathbb{R}$ invariant.

Lemma 2.8.1. *If f is a singularity of type $S[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_2 \\ f^*y_2 &= x_1 \end{aligned}$$

Proof. By the implicit function theorem, there is a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that, for all x_1 and x_2 in a neighborhood of 0, $f_2(\psi(x_1, x_2), x_2) = x_1$. The change of variables

$$\begin{aligned} x_1 &= \psi(\overline{x}_1, \overline{x}_2) \\ x_2 &= \overline{x}_2 \end{aligned}$$

corresponds to an element of $\mathrm{Diff}_p(\mathbb{R}^2; \mathbb{R})$, and hence we may assume $f_2(x) = x_1$. The map $x \mapsto (f_1(x_2, x_1), x_2)$ is an element of $\mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2)$, and using its inverse to reparameterize the codomain lets us assume $f(x_1, 0) = (0, x_1)$ for all x_1 in the domain. Hence, f itself is a valid coordinate change for the domain, the inverse of which gives the result. $\square$

2.8.1 Type $S[0\ 0; 1\ 0]$

Following Table 2.23, we reparameterize such that the differential is

$$\begin{aligned} df_p &: \mathbb{R}^2 \to \mathbb{R}^2 \\ v &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v \end{aligned}$$

The normal bundle for the second intrinsic derivative over C is then

$$\mathrm{Hom}\left(\begin{array}{c} \begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & 0 \arrow[d, hook] \\ \mathbb{R}_1 \arrow[r, hook] & \mathbb{R}_1 \end{tikzcd} \end{array}, \begin{array}{c} \begin{tikzcd} \mathbb{R}_2 \arrow[r, two heads] \arrow[d, two heads] & 0 \arrow[d, two heads] \\ \mathbb{R}_2 \arrow[r, two heads] & 0 \end{tikzcd} \end{array} \right) = \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2)$$

Hence the second intrinsic derivative is

$$\begin{aligned} d^2 f_p &: \mathbb{R}_1 \to \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2) \\ v &\mapsto \left(w \mapsto w^T \left[\partial_{20} f_2(p)\right] v\right) \end{aligned}$$

Surjectivity implies $\partial_{20} f_2(p) \neq 0$.

Table 2.23: First-order types of $(1+1)$-dimensional singularities.

Type	Representative df_p	Codim.[a]	Generic
$S\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$	1	✓
$S\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$	2	✓
$S\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]$	2	✓
$S\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$	3	

[a] We give the codimension with respect to the manifold, not the submanifold.

Lemma 2.8.2. *If f is a singularity of type $S[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1 \\ f^*y_2 &= x_1^2 + x_2 \end{aligned}$$

Proof. By Lemma 1.1.69, there are functions $h_{ij} : \mathbb{R}^2 \to \mathbb{R}$ such that $f_2(x) = x_2 + h_{20}(x)x_1^2 + h_{11}(x)x_1x_2 + h_{02}(x)x_2^2$ with $h_{20}(p) \neq 0$. After a reparameterization, we may assume $f_2(x) = x_2 + x_1^2 + h_{02}(x)x_2^2$. Furthermore, the reparameterization with $x_1 = \overline{x}_1$ and $x_2 = \overline{x}_2 + h_{02}(x)\overline{x}_2^2$ fixes $\mathbb{R}$, and with it we may further assume $f_2(x) = x_2 + x_1^2$. Note that the ideal (f_1, f_2) is (x_1, x_2), hence by the Malgrange Preparation Theorem there exists some smooth function α such that $x_1 = \alpha(f_1, f_2)$. The reparameterization with $\overline{y}_1 = \alpha(y_1, y_2)$ and $\overline{y}_2 = y_2$ gives the desired form. □

2.8.2 Type $S[0\ 1; 0\ 0]$

Using the parameterization like in Table 2.23, assume

$$\begin{aligned} df_p : \mathbb{R}^2 &\to \mathbb{R}^2 \\ v &\mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} v \end{aligned}$$

The component of the normal bundle corresponding to the submanifold in this case is trivial, so instead we start with this as a $S[\begin{smallmatrix}1\\0\end{smallmatrix}]$ singularity. The normal bundle at p in this case is $\mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1)$, hence the second intrinsic derivative is

$$\begin{aligned} d^2f_p : \mathbb{R}^2 &\to \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1) \\ v &\mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{11}f_1(p) & \partial_{02}f_1(p) \end{bmatrix} v\right) \end{aligned}$$

We have both $\mathbb{R}$ (from the submanifold's tangent bundle) and $\mathbb{R}_2$ available for restriction, hence we have a natural splitting for $\mathbb{R}^2$. Let $S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}a&b\end{smallmatrix}]$ denote the subtypes where a is the dropped rank from restricting to $\mathbb{R}$ and b is the dropped rank from restricting to $\mathbb{R}_2$. Since $S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}a&b\end{smallmatrix}] \subseteq \alpha^{-1}(\mathbb{R}) \cap S[\begin{smallmatrix}1\\0\end{smallmatrix}][b]$, by consulting Table 2.15 only $b = 0$ generically occurs, and $S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}a&0\end{smallmatrix}]$ has codimension at least 2 in $J^2(\mathbb{R}^2, \mathbb{R}^2)$.

If $a = 1$, then $\partial_{11}f_1(p) = 0$ implies the third intrinsic derivative would have a 1-dimensional codomain, giving a total codimension of 3. This does not generically occur, hence we may assume $a = b = 0$.

Lemma 2.8.3. *If f is a singularity of type $S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\end{smallmatrix}]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_2^2 + O(x)^3 \\ f^*y_2 &= x_1 \end{aligned}$$

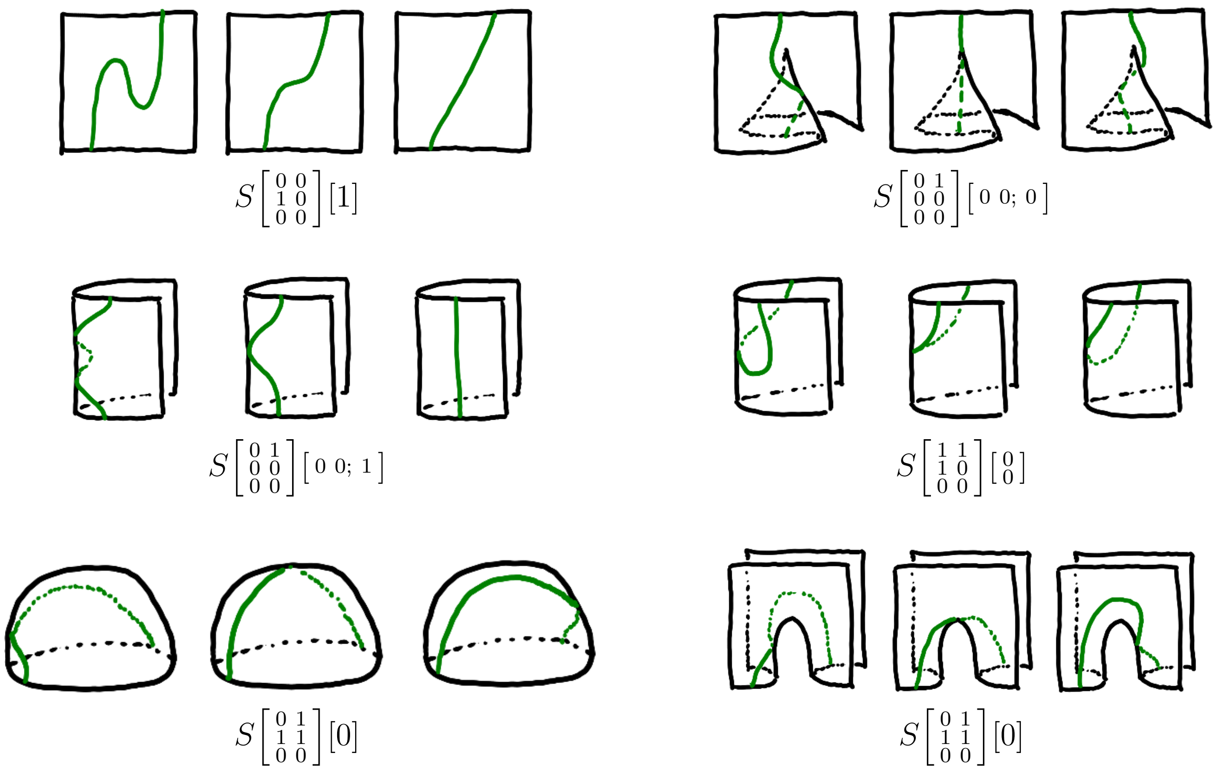


Figure 2.11: Illustrations of the $(2+1)$-dimensional singularities from Table 2.24. This is using the same convention as Figure 2.10, and each group of three diagrams depicts the $t=-1$, $t=0$, and $t=1$ movie frames.

Proof. By the implicit function theorem, we may assume $f_2(x) = x_1$. The above considerations, and further reparameterizations, let us assume the Hessian of f_1 is of the form

$$Hf_1(p) = \begin{bmatrix} 0 & 1 \\ 1 & 2a \end{bmatrix}$$

with $a \in \{-1, 1\}$. After negating variables as needed, this yields

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_2^2 + h(x_1, x_2) \\ f^*y_2 &= x_1 \end{aligned}$$

for some $h \in \mathfrak{m}_p^4$. □

2.9 Classification of $(2+1)$-dimensional Cerf singularities

In this section, we now consider germs of a two-dimensional submanifold in $\mathbb{R}^3$ with codomain $(\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$ for a three-dimensional decomposition. Like in Section 2.6, since we are wanting to derive relations between surfaces, what we are actually considering are curves C in a two-manifold Σ and then studying generic germs $(C \subseteq \Sigma) \times I \to (\mathbb{R} \subseteq \mathbb{R}) \times I$ that commute with the projection to I.

We again work with germs up to Cerf equivalence. Thinking of the curve as $\mathbb{R} \subseteq \mathbb{R}^2$, then we are considering germs $f : (\mathbb{R} \times 0 \times \mathbb{R} \subseteq \mathbb{R}^3) \to (\mathbb{R} \subseteq \mathbb{R}^2)$, implicitly extended with $f_3(x) = x_3$. These are classified up to the action of $\mathrm{Diff}_p(\mathbb{R}^2 \subseteq \mathbb{R}^3; \mathbb{R} \times 0 \times \mathbb{R}) \times \mathrm{Diff}_q(\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3)$, with p and q as usual. The results are summarized in Table 2.24 and Figure 2.11.

Valid reparameterizations need much more care: we need to ensure that we preserve the $\mathbb{R}^2 \subseteq \mathbb{R}^3$ product foliation in the domain while also sending $\mathbb{R} \times 0 \times \mathbb{R}$ to itself. In particular,

Table 2.24: Classification of germs of $(2+1)$-dimensional Cerf singularities.

Type	Codim.	Representative germ[a]	See:	Suspends[b]
$S\left[\begin{smallmatrix}0&0\\0&0\\0&0\end{smallmatrix}\right]$	1	$f^*y_1 = x_2$ $f^*y_2 = x_1$	Lemma 2.9.1	$S[\begin{smallmatrix}0&0\end{smallmatrix}], S[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}]$
$S\left[\begin{smallmatrix}0&0\\1&0\\0&0\end{smallmatrix}\right][0]$	2	$f^*y_1 = x_1$ $f^*y_2 = x_1^2 + x_2$	Lemma 2.9.2	$S[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}]$
$S\left[\begin{smallmatrix}0&0\\1&0\\0&0\end{smallmatrix}\right][1]$	3	$f^*y_1 = x_1$ $f^*y_2 = x_2 + x_1x_3 + x_1^3 + O(x)^4$	Lemma 2.9.3	
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}0&0;&0\end{smallmatrix}]$	2	$f^*y_1 = x_1x_2 + x_2^2 + O(x)^3$ $f^*y_2 = x_1$	Lemma 2.9.4	$S[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}][\begin{smallmatrix}0&0\end{smallmatrix}]$
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}1&0;&0\end{smallmatrix}]$	3	$f^*y_1 = x_1x_2 + x_2^2x_3 + x_2^3 + O(x)^4$ $f^*y_2 = x_1$	Lemma 2.9.5	
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}0&0;&1\end{smallmatrix}]$	3	$f^*y_1 = x_2x_3 + x_2^2 + x_1^2x_2 + O(x)^4$ $f^*y_2 = x_1$	Lemma 2.9.6	
$S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}0\\0\end{smallmatrix}]$	3	$f^*y_1 = x_1^2 + O(x)^4$ $f^*y_2 = x_2 + x_1x_3 + x_1^2 + x_1^3$	Lemma 2.9.7	
$S\left[\begin{smallmatrix}0&1\\1&1\\0&0\end{smallmatrix}\right][0]$	3	$f^*y_1 = x_1 + O(x)^3$ $f^*y_2 = x_1^2 + x_1x_2 + ax_2^2 + x_1x_3 + O(x)^3$	Lemma 2.9.8	

[a] The submanifold is parameterized by x_1, and the "time" parameter is x_3. Recall we assume $f^*y_3 = x_3$.
[b] See Tables 2.21 and 2.22.

if $\varphi \in \mathrm{Diff}_p(\mathbb{R}^2 \subseteq \mathbb{R}^3; \mathbb{R} \times 0 \times \mathbb{R})$, then there are functions $\psi_1, \varphi_2, \psi_3$ such that

$$\begin{aligned}\varphi_1(x_1, x_2, x_3) &= \psi_1(x_1, x_2, x_3)\\ \varphi_2(x_1, x_2, x_3) &= \psi_2(x_1, x_2, x_3)\\ \varphi_3(x_1, x_2, x_3) &= \psi_3(x_3)\end{aligned}$$

and with $\psi_2(x_1, 0, x_3) = 0$ for all x_1 and x_3 in the domain.

The first-order singularity types are defined like they were in Section 2.8. We define $S\left[\begin{smallmatrix} a & c \\ b & d \\ 0 & 0 \end{smallmatrix}\right]$ for the dropped ranks of f, where a, b, c, and d play a similar role as they did in $S\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$, where the first column is for restrictions to submanifold C, but now the second is for restrictions to the leaf Σ of the product foliation of $\Sigma \times I$. In particular, we consider the images of the restriction of df_p to $T_p\Sigma$ in

$$\begin{tikzcd} \mathrm{Hom}(T_pC, T_q\mathbb{R}^2) \arrow[d, two heads] & \mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2) \arrow[l, two heads] \arrow[d, two heads] \\ \mathrm{Hom}(T_pC, T_q\mathbb{R}^2/T_q\mathbb{R}) & \mathrm{Hom}(T_p\Sigma, T_q\mathbb{R}^2/T_q\mathbb{R}) \arrow[l, two heads] \end{tikzcd}$$

This is defined so that $S\left[\begin{smallmatrix} a & c \\ b & d \\ 0 & 0 \end{smallmatrix}\right] \subseteq S\left[\begin{smallmatrix} c \\ d \\ 0 \end{smallmatrix}\right]$. As we have been doing for Cerf singularities, the final row of 0's is to distinguish these from the $(1+1)$-dimensional singularities.

The first-order singularity types are summarized in Table 2.25. We start the enumeration by using the generic types from Table 2.17, none of which can be excluded by codimension when intersected with $\alpha^{-1}(C \times \mathbb{R})$. Codimensions in the table are one more than the dimensions of the normal bundles for the second intrinsic derivatives over $C \times \mathbb{R}$.

Lemma 2.9.1. *If f is a singularity of type $S\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_2\\ f^*y_2 &= x_1\end{aligned}$$

Proof. An argument similar to the one in Lemma 2.8.1 applies: we apply a coordinate change to the codomain so we may assume that $f(x_1, 0, x_3) = (0, x_1, x_3)$, and then use f itself as a coordinate change for the domain. □

2.9.1 Type $S[0\ 0; 1\ 0; 0\ 0]$

As in Table 2.25, for a $S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ singularity there is a linear change of coordinates such that the differential has the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The normal bundle in this case is $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2)$, hence the second intrinsic derivative is

$$\begin{aligned} d^2 f : \mathbb{R} \times 0 \times \mathbb{R} &\to \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2) \\ v &\mapsto \left(w \mapsto w^T \left[\partial_{200} f_2(p) \quad \partial_{101} f_2(p)\right] v\right) \end{aligned}$$

Define the subtypes $S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right][r]$ where r is the dropped rank when restricted to $\mathbb{R}_1$.

Lemma 2.9.2. *If f is a singularity of type $S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right][0]$, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= x_1 \\ f^* y_2 &= x_1^2 + x_2 \end{aligned}$$

Proof. For this subtype, there is a change of coordinates such that the matrix for $d^2 f_p$ is $\left[\begin{smallmatrix} 2 & 0 \end{smallmatrix}\right]$. The kernel of $d^2 f_p$ is $T_p S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and is equal to $0 \times 0 \times \mathbb{R}$, thus $S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right](f)$ is a 1-dimensional submanifold of $\mathbb{R} \times 0 \times \mathbb{R}$. Let $\gamma : \mathbb{R} \to \mathbb{R} \times 0 \times \mathbb{R}$ be a parameterization of this submanifold with $\gamma_3(t) = t$ for all t. Reparameterizing with $x_1 = \overline{x}_1 + \gamma_1(\overline{x}_3)$, $x_2 = \overline{x}_2$, and $x_3 = \overline{x}_3$, we may assume the singularity submanifold is $0 \times 0 \times \mathbb{R}$ exactly. Furthermore, reparameterizing with

$$\begin{aligned} \overline{y_1} &= y_1 - f_1(0, 0, y_3) \\ \overline{y_2} &= y_2 - f_2(0, 0, y_3) \\ \overline{y_3} &= y_3 \end{aligned}$$

we may assume that $f(0, 0, x_3) = (0, 0, x_3)$ for all x_3 in the domain. Rescaling x_1 using a function of x_3, we may assume $d^2 f_{(0,0,x_3)}$ has the matrix $\left[\begin{smallmatrix} 2 & 0 \end{smallmatrix}\right]$ for all x_3 in the domain.

Table 2.25: First-order types of $(2+1)$-dimensional singularities.

Type	Representative df_p	Codim.[a]	Generic
$S\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$	1	✓
$S\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$	2	✓
$S\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right]$	2	✓
$S\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right]$	3	✓
$S\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$	3	✓
$S\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$	4	

[a] We give the codimension with respect to $\Sigma \times I$.

Applying Lemma 1.1.69 to f_2 as a function of just x_1 and x_2, then since $f_2(0,0,x_3) = 0$, there are functions $h_{ij} : \mathbb{R}^3 \to \mathbb{R}$ such that $f_2(x) = x_2 + h_{20}(x)x_1^2 + h_{11}(x)x_1x_2 + h_{02}(x)x_2^2$. By a coordinate change, we may assume $f_2(x) = x_2 + x_1^2 + h_{02}(x)x_2^2$. The result is from adapting the rest of the argument from Lemma 2.8.2. □

Lemma 2.9.3. *If f is a singularity of type $S\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}[1]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1 \\ f^*y_2 &= x_2 + x_1x_3 + x_1^3 + O(x)^4 \end{aligned}$$

Proof. For this subtype, we may assume the matrix for d^2f_p is $[\,0\ 1\,]$. We now consider the third intrinsic derivative. Restricted to $\mathbb{R}_1$, the kernel is $\mathbb{R}_1$ and the cokernel is $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2) \approx \mathbb{R}$, so we get

$$\begin{aligned} d^3f_p : \mathbb{R}_1 &\to \mathbb{R} \\ v &\mapsto \partial_{300}f_2(p)v \end{aligned}$$

Hence, since this is surjective we may assume $\partial_{300}f_2(p) \neq 0$, and by a reparameterization that $\partial_{300}f_2(p) = 6$.

By Lemma 1.1.69 there are functions $h_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f_2(x) = x_2 + \sum_{i+j+k=2} h_{ijk}(x)x_1^i x_2^j x_3^k$$

such that $h_{200}(p) = 0$ and $h_{101}(p) = 1$. Reparameterizing with

$$\overline{x}_1 = h_{101}(x)x_1 \qquad \overline{x}_2 = \left(1 + \sum_{\substack{i+j+k=2 \\ j>0}} h_{ijk}(x)x_1^i x_2^{j-1} x_3^k\right) x_2 \qquad \overline{x}_3 = x_3$$

then we may assume

$$f_2(x) = x_2 + h_{200}(x)x_1^2 + x_1x_3 + h_{002}(x)x_3^2.$$

Like usual, we may assume $h_{002}(p) = 0$ by a coordinate change with $\overline{y}_2 = y_2 - h_{002}(p)y_3^2$. Thus, there are functions $h_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{aligned} f_2(x) = x_2 + x_1x_3 &+ h_{300}(x)x_1^3 + h_{210}(x)x_1^2x_2 + h_{201}(x)x_1^2x_3 \\ &+ h_{102}(x)x_1x_3^2 + h_{012}(x)x_2x_3^2 + h_{003}(x)x_3^3 \end{aligned}$$

Again, we may reparameterize to eliminate terms with a factor of x_2 or x_1x_3 beyond the first two, and we may assume $h_{003}(p) = 0$. We obtain

$$f_2(x) = x_2 + x_1x_3 + h_{300}(x)x_1^3 + h_{003}(x)x_3^3.$$

Since $h_{300}(p) = 1$, this is $f_2(x) = x_2 + x_1x_3 + x_1^3 + O(x)^4$. □

2.9.2 Type $S[0\ 1; 0\ 0; 0\ 0]$

As in Table 2.25, for a $S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]$ singularity there is a linear change of coordinates such that the differential has the matrix

$$\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$$

Since f is locally an immersion when restricted to the submanifold (like in Section 2.8.2), we may as well consider this first as a $S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$ singularity. The normal bundle is $\mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1)$, hence the second intrinsic derivative is

$$\begin{aligned} d^2 f_p : \mathbb{R}^3 &\to \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1) \\ v &\mapsto \left(w \mapsto w^T \left[\partial_{110} f_1(p) \quad \partial_{020} f_1(p) \quad \partial_{011} f_1(p)\right] v\right) \end{aligned}$$

Define subtypes $S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}r\ s;\ t\end{smallmatrix}\right]$ where $r = \operatorname{drk} d^2 f_p|_{\mathbb{R}_2}$, $s = \operatorname{drk} d^2 f_p|_{\mathbb{R}_{12}}$, and $t = \operatorname{drk} d^2 f_p|_{\mathbb{R}_1}$. This notation is designed so that $S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}r\ s;\ t\end{smallmatrix}\right] \subseteq S\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}r\ s\end{smallmatrix}\right]$. Since $\mathbb{R}_1 = T_pC$ and $\mathbb{R}_2$ split $\mathbb{R}_{12} = T_p\Sigma$, the s value is determined by r and t. The possibilities are listed in Table 2.26.

Using the implicit function theorem like before, we may reparameterize x_1 to let us assume $f_2(x) = x_1$.

Lemma 2.9.4. *If f is a singularity of type $S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\ 0;\ 0\end{smallmatrix}\right]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_2^2 + O(x)^3 \\ f^*y_2 &= x_1 \end{aligned}$$

Proof. There is a linear reparameterization such that, for some constants $a_{ijk} \in \mathbb{R}$,

$$f_1(x) = a_{200}x_1^2 + x_1x_2 + a_{101}x_1x_3 + x_2^2 + a_{011}x_2x_3 + a_{002}x_3^2 + O(x)^2.$$

Table 2.26: Second-order types for $S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]$.

Type	Representative $d^2 f_p$	Addl. Codim.	Codim.[a]	Generic
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\ 0;\ 0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1\ 2\ 0\end{smallmatrix}\right]$	0	2	✓
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\ 0;\ 0\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}1\ 0\ 0\end{smallmatrix}\right]$	1	3	✓
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}0\ 0;\ 1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\ 2\ *\end{smallmatrix}\right]$	1	3	✓
$S\left[\begin{smallmatrix}0&1\\0&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}1\ 1;\ 1\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}0\ 0\ 1\end{smallmatrix}\right]$	2	4	

[a] We give the codimension with respect to $\Sigma \times I$.

Reparameterizing with $\overline{y}_1 = y_1 - a_{200}y_2^2 - a_{101}y_2y_3 - a_{002}y_3^2$ lets us assume

$$f_1(x) = x_1x_2 + x_2^2 + a_{011}x_2x_3 + O(x)^2.$$

The coordinate change with $\overline{x}_1 = x_1 + a_{011}x_3$ and $\overline{y}_1 = y_1 - a_{011}y_3$ gives the result. □

Lemma 2.9.5. *If f is a singularity of type $S\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}[\,1\ 0;\ 0\,]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1x_2 + x_2^2x_3 + x_2^3 + O(x)^4 \\ f^*y_2 &= x_1 \end{aligned}$$

Proof. Restricted to $\mathbb{R}_2$, the kernel of d^2f_p is $\mathbb{R}_2$ and the cokernel is $\mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1)$, and restricted to $\mathbb{R}_1$ both of these are trivial. Hence, the normal bundle for calculating d^3f_p is $\mathrm{Hom}(\mathbb{R}_2, \mathrm{Hom}(\mathbb{R}_2, \mathbb{R}_1)) \cong \mathbb{R}$. We have

$$\begin{aligned} d^3f_p : \mathbb{R}_3 &\to \mathbb{R} \\ v &\mapsto \partial_{021}f_1(p)v \end{aligned}$$

hence $\partial_{021}f_1(p) \neq 0$.

We may also consider f as a $S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}[\,1\ 0\,]$ singularity. Consulting Table 2.19, by consideration of codimension it must be a $S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}[\,1\ 0\,][0]$ singularity. To make use of this, we calculate d^3f_p for this type. The normal bundle is the same, but now

$$\begin{aligned} d^3f_p : 0 \times \mathbb{R}^2 &\to \mathbb{R} \\ v &\mapsto \begin{bmatrix} \partial_{030}f_1(p) & \partial_{021}f_1(p) \end{bmatrix} v \end{aligned}$$

The subtype $S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}[\,1\ 0\,][0]$ corresponds to the dropped rank of the restriction of this to $\mathbb{R}_2$, thus $\partial_{030}f_1(p) \neq 0$. By scaling the coordinate variables, we may assume $\partial_{021}f_1(p) = 4$ and $\partial_{030}f_1(p) = 6$.

Using reparameterizations similar to Lemma 2.9.4, then by Lemma 1.1.69 there are constants $a_{ijk} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f_1(p) = x_1x_2 + x_2^2x_3 + x_2^3 + \sum_{i+j+k=3} a_{ijk}x_1^ix_2^jx_3^k + O(x)^4$$

where $a_{021} = 0$ and $a_{030} = 0$. A reparameterization with $\overline{y}_1 = y_1 - \sum_{i+k=3} a_{i0k}y_2^ix_3^k$ lets us assume that

$$f_1(p) = x_1x_2 + x_2^2x_3 + x_2^3 + \sum_{\substack{i+j+k=3 \\ j>0}} a_{ijk}x_1^ix_2^jx_3^k + O(x)^4.$$

Using the change of coordinates

$$\overline{x}_1 = x_1 \qquad \overline{x}_2 = x_2\left(1 + \sum_{\substack{i+j+k=3\\ i>0, j>0}} a_{ijk} x_1^{i-1} x_2^{j-1} x_3^k\right) \qquad \overline{x}_3 = x_3$$

we may furthermore assume that

$$f_1(p) = x_1x_2 + x_2^2x_3 + x_2^3 + a_{012}x_2x_3^2 + O(x)^4.$$

Lastly, the coordinate change

$$\begin{aligned} x_1 &= \overline{x}_1 - a_{012}\overline{x}_3^2 & x_2 &= \overline{x}_2 & x_3 &= \overline{x}_3 \\ \overline{y}_1 &= y_1 & \overline{y}_2 &= y_2 + a_{012}y_3^2 & \overline{y}_3 &= y_3 \end{aligned}$$

gives the desired result. □

Lemma 2.9.6. *If f is a singularity of type $S\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 0; & 1 \end{smallmatrix}\right]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_2x_3 + x_2^2 + x_1^2x_2 + O(x)^4 \\ f^*y_2 &= x_1 \end{aligned}$$

Proof. In this type, the second intrinsic derivative gives $\partial_{110}f_1(p) = 0$ and $\partial_{020}f_1(p) \neq 0$. The second intrinsic derivative also needs to be surjective restricted to $C \times \mathbb{R}$, which gives $\partial_{011}f_1(p) \neq 0$. Passing to the third intrinsic derivative, we have a map

$$d^3f_p : T_pC \to \operatorname{Hom}(\ker d^2f_p|_{T_pC}, \operatorname{coker} d^2f_p|_{T_pC})$$

which computes $\partial_{210}f_1(p)$, and hence generically $\partial_{210}f_1(p) \neq 0$. Rescaling variables, $\partial_{020}f_1(p) = 2$, $\partial_{011}f_1(p) = 1$, and $\partial_{210}f_1(p) = 2$. After the usual reparameterizations, the can see that the 2-jet of f_1 is

$$f_1(p) = x_2x_3 + x_2^2 + O(x)^3.$$

There are constants $a_{ijk} \in \mathbb{R}$ with $a_{210} = 0$ such that, after a reparameterization like in Lemma 2.9.5,

$$f_1(p) = x_2x_3 + x_2^2 + x_1^2x_2 + \sum_{\substack{i+j+k=3\\ j>0}} a_{ijk}x_1^ix_2^jx_3^k + O(x)^4.$$

The change of coordinates

$$\overline{y}_1 = y_1 - a_{120}y_1y_2 - a_{021}y_1y_3 \qquad \overline{y}_2 = y_2 \qquad \overline{y}_3 = y_3$$

lets us furthermore assume that

$$f_1(p) = x_2x_3 + x_2^2 + x_1^2x_2 + a_{030}x_2^3 + a_{012}x_2x_3^2 + a_{111}x_1x_2x_3 + O(x)^4,$$

and with $x_1 = \overline{x}_1 - \frac{1}{2}a_{111}\overline{x}_3$ and $\overline{y}_2 = y_2 + \frac{1}{2}a_{111}y_3$, that

$$f_1(p) = x_2x_3 + x_2^2 + x_1^2x_2 + a_{030}x_2^3 + a_{012}x_2x_3^2 + O(x)^4.$$

Then, with $\overline{x}_2 = x_2(1 + a_{012}x_3)$, that

$$f_1(p) = x_2x_3 + x_2^2 + x_1^2x_2 + a_{030}x_2^3 + a_{021}x_2^2x_3 + O(x)^4.$$

Lastly, the change of coordinates

$$\begin{array}{lll} x_1 = \overline{x}_1 & x_2 = \overline{x}_2(1 - \frac{1}{2}a_{030}(x_2 - x_3) - a_{021}x_3) & x_3 = \overline{x}_3 \\ \overline{y}_1 = y_1 + (a_{021} - \frac{1}{2}a_{030})y_3 & \overline{y}_2 = y_2 & \overline{y}_3 = y_3 \end{array}$$

gives the desired form. □

2.9.3 Type $S[1\ 1; 1\ 0; 0\ 0]$

As in Table 2.25, for an $S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right]$ singularity there is a linear change of coordinates such that the differential has the matrix

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

The normal bundle at p for the second intrinsic derivative, is

$$\mathrm{Hom}\left(\begin{tikzcd} \mathbb{R}_1 \arrow[r, hook] \arrow[d, hook] & \mathbb{R}_1 \arrow[d, hook] \\ \mathbb{R}_1 \arrow[r, hook] & \mathbb{R}_1 \end{tikzcd}, \begin{tikzcd} \mathbb{R}_{12} \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R}_1 \arrow[d, two heads] \\ \mathbb{R}_2 \arrow[r, two heads] & 0 \end{tikzcd}\right) = \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}^2) \approx \mathbb{R}^2$$

so we write the second intrinsic derivative as

$$\begin{aligned} d^2f_p : \mathbb{R} \times 0 \times \mathbb{R} &\to \mathbb{R}^2 \\ v &\mapsto \begin{bmatrix}\partial_{200}f_1(p) & \partial_{101}f_1(p)\\ \partial_{200}f_2(p) & \partial_{101}f_2(p)\end{bmatrix} v \end{aligned}$$

The normal bundle at p canonically splits as $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}^2) \cong \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1) \oplus \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2)$, and we define subtypes $S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right]\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ according to the following scheme:

$$\begin{array}{r c} & \mathbb{R}_1 \\ \begin{array}{r}\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_1)\\ \mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2)\end{array} & \begin{bmatrix} * \\ * \end{bmatrix} \end{array}$$

Since $S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right]$ is already codimension-3, the only subtype that generically occurs is $S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}0\\0\end{smallmatrix}]$. Hence, $\partial_{200}f_1(p) \neq 0$ and $\partial_{200}f_2(p) \neq 0$. We may rescale variables so that $\partial_{200}f_1(p) = 2$ and $\partial_{200}f_2(p) = 2$. By the usual changes of coordinates, we may assume $\partial_{110}f_1(p) = 0$ and $\partial_{101}f_1(p) = 0$. Since d^2f_p is surjective, then $\partial_{101}f_2(p) \neq 0$, which after rescaling variables we may assume is 1.

Just like for Lemma 2.3.6, along $\mathbb{R} \times 0 \times \mathbb{R}$ we have another third intrinsic derivative from using the cokernel of $d^2f_p|_{T_p\mathbb{R}}$. Projecting to the cokernel for this restriction can be represented by $v \mapsto [\partial_{200}f_1(p) - \partial_{200}f_2(p)]v$, and then the third intrinsic derivative for this is a map

$$\begin{aligned} T_p(\mathbb{R} \times 0 \times 0) &\mapsto \mathbb{R} \\ v &\mapsto \left[\partial_{300}f_1(p) - \partial_{300}f_2(p)\right] v \end{aligned}$$

and so for codimension reasons we need $\partial_{300}f_1(p) \neq \partial_{300}f_2(p)$.

Lemma 2.9.7. *If f is a singularity of type $S\left[\begin{smallmatrix}1&1\\1&0\\0&0\end{smallmatrix}\right][\begin{smallmatrix}0\\0\end{smallmatrix}]$, then it is equivalent to a germ with*

$$\begin{aligned} f^*y_1 &= x_1^2 + O(x)^4 \\ f^*y_2 &= x_2 + x_1x_3 + x_1^2 + x_1^3 + O(x)^4 \end{aligned}$$

Proof. This is a matter of clearing out terms to the fourth order, given what we have so far deduced about the 3-jet. ☐

2.9.4 Type $S[0\ 1; 1\ 1; 0\ 0]$

As in Table 2.25, for an $S\left[\begin{smallmatrix}0&1\\1&1\\0&0\end{smallmatrix}\right]$ singularity there is a linear change of coordinates such that the differential has the matrix

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The normal bundle at p for the second intrinsic derivative, is

$$\operatorname{Hom}\left(\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & \mathbb{R}_2 \arrow[d, hook] \\ \mathbb{R}_1 \arrow[r, hook] & \mathbb{R}_{12} \end{tikzcd}, \begin{tikzcd} \mathbb{R}_2 \arrow[r, two heads] \arrow[d, two heads] & \mathbb{R}_2 \arrow[d, two heads] \\ \mathbb{R}_2 \arrow[r, two heads] & \mathbb{R}_2 \end{tikzcd}\right) = \operatorname{Hom}(\mathbb{R}_{12}, \mathbb{R}_2) \approx \mathbb{R}^2$$

so we write the second intrinsic derivative as

$$\begin{aligned} d^2(f|_{\mathbb{R}\times 0\times\mathbb{R}})_p : T_p(\mathbb{R} \times 0 \times \mathbb{R}) &\to \operatorname{Hom}(\mathbb{R}_{12}, \mathbb{R}_2) \\ v &\mapsto \begin{bmatrix}\partial_{200}f_2(p) & \partial_{101}f_2(p)\\ \partial_{110}f_2(p) & \partial_{011}f_2(p)\end{bmatrix} v \end{aligned}$$

We can restrict this to $\mathbb{R}_1$ and $\mathrm{Hom}(\mathbb{R}_1, \mathbb{R}_2)$, and so $\partial_{200} f_2(p) \neq 0$ due to codimensions. We can clear out $\partial_{101} f_2(p)$ using a Morse-like substitution — this may introduce some first-order t terms in f_1 but these can be eliminated by a Cerf reparameterization.

We also have a second intrinsic derivative for this as a $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$ singularity:

$$\begin{aligned} d^2 f_p : T_p\mathbb{R}^3 &\to \mathrm{Hom}(\mathbb{R}_{12}, \mathbb{R}_2) \\ v &\mapsto \left(w \mapsto w^T \begin{bmatrix} \partial_{200} f_2(p) & \partial_{110} f_2(p) & \partial_{101} f_2(p) \\ \partial_{110} f_2(p) & \partial_{020} f_2(p) & \partial_{011} f_2(p) \end{bmatrix} v \right) \end{aligned}$$

For codimension reasons, this must be a $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$, so $\partial_{020} f_2(p) \neq 0$ and the matrix

$$\begin{bmatrix} \partial_{200} f_2(p) & \partial_{110} f_2(p) \\ \partial_{110} f_2(p) & \partial_{020} f_2(p) \end{bmatrix}$$

is non-singular. If it were the case that $\partial_{110} f_2(p) = 0$ or $\partial_{011} f_2(p) = 0$, then the tangent space to $S\left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right](f)$, which is $\ker d^2 f_p$, would contain $\mathbb{R}_1$ or $\mathbb{R}_3$, and so we could take a third intrinsic derivative for the $\mathbb{R} \times 0 \times \mathbb{R}$ submanifold, which would exceed our codimension limit, and so generically $\partial_{110} f_2(p) \neq 0$ and $\partial_{012} f_2(p) \neq 0$.

For sake of having a symbol, we will write $S\left[\begin{smallmatrix}0&1\\1&1\\0&0\end{smallmatrix}\right][0]$ for this type.

Lemma 2.9.8. *If f is a singularity of type $S\left[\begin{smallmatrix}0&1\\1&1\\0&0\end{smallmatrix}\right][0]$, then it is equivalent to a germ with*

$$\begin{aligned} f^* y_1 &= x_1 + O(x)^3 \\ f^* y_2 &= x_1^2 + x_1 x_2 + a x_2^2 + x_1 x_3 + O(x)^3 \end{aligned}$$

for some $a \in \{\pm 1\}$.

Proof. This is a matter of clearing out terms to the third order, given what we have so far deduced about the 2-jet. □

2.9.5 Multijet singularities

The multijet singularities of surfaces with embedded curves have no surprises. We still have all the multijet singularities for surfaces without embedded curves, of course, but now we have moves for having codimension-2 singularities pass across images of the curves as well as Reidemeister II and III moves for images of curves and between images of curves and the fold curves.

Chapter 3

Presentations of cobordism categories

The goal of this chapter is to provide presentations of cobordism categories using the classifications of singularities we have carried out in Chapter 2. We first explain how to obtain generators and relations for the cobordism 2-category of surfaces with embedded curves as a monoidal 2-category. We then explain how to adapt the chambering graphs and foams from [SP09] to give a presentation as a *symmetric* monoidal 2-category. In principal, one could study 3D Morse functions of surfaces to obtain a presentation as a symmetric monoidal 2-category in a natural way, but a tradeoff is that we would need to work out the Cerf theory of the multijet singularities — maps $\mathbb{R}^2 \to \mathbb{R}^3$ can generically have curves of transverse double points, and these have associated Roseman-like moves, but with the added complication that these double point curves are generic with respect to the $\mathbb{R}^3$ foliation.

The focus on this chapter is merely to give a presentation of these categories. While we will cover some theory to make special use of our strict adherence to classifing singularities with respect to the relevant diffeomorphism groups, we do not include the theory of 2-categories and their presentations, and instead we direct the reader to the account in [SP09] for symmetric monoidal 2-categories. We also mention [Dor18] for the general principle of decompositions using n-cubes with a "globularity condition" — globular sets by themselves have a deficiency where, for example, horizontal composition of 2-cells is indistinguishable from whiskering the 2-cells and then vertically composing. By working with such globular cubes, we can be sure that, if needed for coherency arguments, explicit reassociation occurs (such reassociation is the role of the interchange moves).

3.1 Globular sets

Part of the data of a category is a directed graph, and any generalization of categories to higher categories needs a generalization of directed graphs to combinatorial structures with cells of higher dimensions. Simplicial sets are one option, where an n-cell is an n-simplex. Globular sets are another, where an n-cell is an n-globe, which is best thought of as being the combinatorial model of a closed n-ball given a CW structure with two k-cells for all

$0 \leq k < n$ and a single n-cell, but where for each pair of k-cells one is the "source" and the other is the "target." This structure naturally lends itself to composition operations. We draw the following material from [Lei03].

Definition 3.1.1. The *globe category* $\mathbb{G}$ is the category whose objects are denoted by $[n]$ for all $n \in \mathbb{N}$ and whose morphisms are generated by $\sigma_n : [n] \to [n+1]$ and $\tau_n : [n] \to [n+1]$ for all $n \in \mathbb{N}$, subject to the relations, for all $n \in \mathbb{N}$,

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$$
$$\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$$

That is, both σ_n and τ_n equalize the pair of maps σ_{n+1} and τ_{n+1}. ◊

Definition 3.1.2. A *globular set* X is a functor $\mathbb{G}^{\mathrm{op}} \to \mathrm{Set}$. The *category of globular sets* gSet is the category $\mathrm{Fun}(\mathbb{G}^{\mathrm{op}}, \mathrm{Set})$.

We write X_n for the set $X([n])$ of n-cells of X, and we write $s = X(\sigma)$ and $t = X(\tau)$ for the *source* and *target* maps, respectively:

$$s : X_{n+1} \to X_n$$
$$t : X_{n+1} \to X_n.$$

We say X is an *n-globular set* if $X_i = \varnothing$ for all $i > n$. ◊

Example 3.1.3. The *standard n-globe* $\mathbb{G}^n$ is the globular set defined by

$$\mathbb{G}^n_0 = \{e^+_0, e^-_0\} \quad \ldots \quad \mathbb{G}^n_{n-1} = \{e^+_{n-1}, e^-_{n-1}\} \qquad \mathbb{G}^n_n = \{e_n\}$$

with $s, t : \mathbb{G}^n_{i+1} \to \mathbb{G}^n_i$ defined by $s(x) = e^+_i$ and $t(x) = e^-_i$ for all $1 \leq i < n$ and $x \in \mathbb{G}^n_{i+1}$. The cells in $\mathbb{G}^n$ are in correspondence with the cells in the closure an n-dimensional cell of the n-fold suspension of S^0 as a CW complex.

Another characterization of this globular set $\mathbb{G}^n$ is as the functor $\mathrm{Hom}^{\mathbb{G}}(-, [n])$. Morphisms in $\mathbb{G}$ are uniquely represented in form id, σ^k, or τ^k with $k > 1$, and so $\mathrm{Hom}^{\mathbb{G}}([i], [n])$ has two elements if $i < n$, one if $i = n$, and zero if $i > n$. This means that if X is a globular set, then by the Yoneda lemma

$$\mathrm{Hom}(\mathbb{G}^n, X) \approx \mathrm{Hom}(\mathrm{Hom}^{\mathbb{G}}(-, [n]), X) \approx X_n.$$

That is to say, n-cells of X are in one-to-one correspondence to maps from $\mathbb{G}^n$ to X. ◊

Example 3.1.4. Every globular set can be reconstructed by gluing together its contituent globes. This is due to the sheaf version of Lemma 1.1.15. If X is a globular set, then its category of elements $\mathrm{el}(X)$ has as objects pairs $([n], x)$ for $x \in X_n$, and as morphism sets

$$\mathrm{el}(X)(([n], x), ([m], y)) = \{f \in \mathbb{G}([m], [n]) \mid X(f)x = y\}.$$

Then

$$X \approx \mathrm{colim}^{\mathrm{gSet}}_{([n],x) \in \mathrm{el}(X)^{\mathrm{op}}} \mathbb{G}^n,$$

where $\mathbb{G}^n$ is the standard n-globe. One can also geometrically realize a globular set in this way by replacing $\mathbb{G}^n$ in the colimit by, for example, a CW complex with the same cell structure and taking the colimit in the category of CW complexes. ◊

3.2 Manifolds with faces and halated manifolds

In foliated $\mathbb{R}^2$, a subset $S \subseteq \mathbb{R}^2$ is a 2-*cube* if there is some $g \in \mathrm{Diff}(\mathbb{R} \subseteq \mathbb{R}^2)$ such that $S = g[0,1]^2$. Our basic strategy for decomposing a generic function $f : \Sigma \to [0,1]^2$ with $[0,1]^2$ foliated is to find some decomposition of $[0,1]^2$ into 2-cubes such that each pre-image is a manifold with corners that satisfy a "globularity condition," where the $\{0,1\} \times [0,1]$ sides could, in principal, be crushed to $\{0,1\} \times \{0\}$, which makes them well-suited for a 2-categorical (rather than a double categorical) decomposition. In this section we go into detail about the version of manifolds with corners we use.

Let us review how manifolds with faces are defined in the literature. Recall that an n-*manifold with corners* is a second-countable Hausdorff space M locally modeled on the principal orthant $\mathbb{R}^n_{\geq 0} \subseteq \mathbb{R}^n$ such that transition maps are the restrictions of elements of $\mathrm{diff}^\infty(\mathbb{R}^n)$ to $\mathbb{R}^n_{\geq 0}$ (that is, they are local homeomorphisms of $\mathbb{R}^n_{\geq 0}$ that extend to diffeomorphisms on an open neighborhood in $\mathbb{R}^n$). For each $x \in M$, we define $c(x) \in \mathbb{N}$ to be the number of coordinates in $\varphi(x)$ that are zero, where $\varphi : M \looparrowright \mathbb{R}^n_{\geq 0}$ is any chart with $x \in \operatorname{dom} \varphi$, and this number does not depend on φ. A *connected face* of M is the closure of a connected component of $\{x \in X \mid c(x) = 1\}$, and a *face* is a union of disjoint connected faces. The global structure of manifolds with corners is not generally well-behaved since a connected face might be self-incident. Introduced in [J$\ddot{6}$8], a *manifold with faces* M is a manifold with corners such that each $x \in M$ belongs to $c(x)$ different connected faces. Every face of a manifold with faces is itself a manifold with faces. Adding additional structure to this definition, a $\langle k \rangle$-*manifold* for $k \in \mathbb{N}$ is a manifold with faces M along with an ordered k-tuple $(M_0, \ldots, M_{k-1})$ of faces such that $M_0 \cup \cdots \cup M_{k-1}$ contains every face of M and for all $i \neq j$, $M_i \cap M_j$ is a (possibly nonempty) face of both M_i and M_j. These were noticed to be useful in [Lau00] for the cobordism theory of manifolds with corners, and they were put to use in [LP08] and [Mor07] for extended bordism categories.

In our variant, we wish to (1) build horizontal and vertical faces into the definition and (2) use manifold germs so that gluing along these faces do not require any arbitrary choices. We now continue with definitions.

Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ and $n \in \mathbb{N}$. Each subset $f \subseteq \{1, \ldots, n\}$ defines a subspace $H_f = \mathrm{Span}\{e_i \mid i \in f\}$, and the intersection $H_f \cap \mathbb{R}^n_{\geq 0}$ is the f-*face* of $\mathbb{R}^n_{\geq 0}$. Each $x \in \mathbb{R}^n_{\geq 0}$ is associated to the face $f_x = \{i \in \mathbb{N} \mid 1 \leq i \leq n \text{ and } x_i \neq 0\}$.

Consider the pseudogroup $\Gamma^n_{\geq 0}$ of $\mathbb{R}^n$ that is the sub-pseudogroup of $\mathrm{diff}^\infty(\mathbb{R}^n)$ consisting of those local diffeomorphisms $\varphi : \mathbb{R}^n \looparrowright \mathbb{R}^n$ such that, for all $x \in \operatorname{dom} \varphi$,

1. $x \in \mathbb{R}^n_{\geq 0}$ if and only if $\varphi(x) \in \mathbb{R}^n_{\geq 0}$, and
2. for all $f \subseteq \{1, \ldots, n\}$, if $x \in H_f$, then $\varphi(x) \in H_f$.

Suppose M is a manifold with $\Gamma^n_{\geq 0}$ structure. Let $\mathrm{core}(M) \subseteq M$ consist of those points $x \in M$ for which there exists a chart $\varphi : M \looparrowright \mathbb{R}^n$ with $x \in \operatorname{dom} \varphi$ such that $\varphi(x) \in \mathbb{R}^n_{\geq 0}$, a property that does not depend on the choice of φ. For $x \in \mathrm{core}(M)$, we let f_x be $f_{\varphi(x)}$, where φ is once again a chart with $x \in \operatorname{dom} \varphi$.

Such manifolds have, for each $f \subseteq \{1,\dots,n\}$, a closed set $M_f = \{x \in \mathrm{core}(M) \mid f_x = f\}$. The manifold with corners $\mathrm{core}(M)$ is a manifold with faces and furthermore an $\langle n\rangle$-manifold when using the tuple of faces $(M_{\{1,\dots,n\}-i\}})_{i=1}^n$.

Definition 3.2.1. Let $n \in \mathbb{N}$, and consider the category of manifolds with $\Gamma^n_{\geq 0}$ structures. For $f \subseteq \{0,\dots,n\}$, a *halated* (f,n)*-manifold*[1] is a smooth manifold germ $(M', \mathcal{N}(M'_f))$ for M' a manifold with $\Gamma^n_{\geq 0}$ structure. When the ambient dimension is unambiguous, we call this a halated f-manifold, or when $f = \{1,\dots,n\}$ we call this a halated n-manifold.

If M is a halated f-manifold represented by $(M', \mathcal{N}(M'_f))$, for each $f' \subseteq f$ let $M_{f'}$ be the halated f'-manifold $(M', \mathcal{N}(M'_{f'}))$, called the f'*-face* of M. There is a canonical inclusion $M_{f' \subseteq f} : M_{f'} \hookrightarrow M_f$ induced by the identity on M'.

A diffeomorphism of halated f-manifolds is an isomorphism of the smooth manifold germs that is representable by $\Gamma^n_{\geq 0}$ diffeomorphisms. Put another way, $(M, \mathcal{N}(M_f))$ is diffeomorphic to $(N, \mathcal{N}(M_g))$ if there exist open neighborhoods $U \supseteq M_f$ and $V \supseteq M_g$ and a $\Gamma^n_{\geq 0}$ diffeomorphism $f : U \to V$ such that $f(M_f) = M_g$. $\Diamond$

Remark 3.2.2. Suppose $f \subseteq \{1,\dots,n\}$ and M is a halated f-manifold. There is a well-defined $(n - |f|)$-dimensional normal bundle to M by taking the smooth manifold germ $(M', \mathcal{N}(M'_f))$ for M and computing the normal bundle to M'_f in M'. The normal bundle is trivial, trivialized by the tangent spaces of each $H_{\{i\}}$ for $i \notin f$ at M'_f. ■

Definition 3.2.3. For M a halated f-manifold represented by $(M', \mathcal{N}(M'_f))$, let $\mathrm{core}(M) = M'_f$, which is the forgetful functor from halated manifolds to C^∞ manifolds. $\Diamond$

This extra information in a halation is unique up to noncanonical diffeomorphism as the following lemma shows:

Lemma 3.2.4. *Suppose that $f \subset \{1,\dots,n\}$, that M and N are halated f-manifolds, and that $h : \mathrm{core}(M) \to \mathrm{core}(M)$ is a diffeomorphism such that $c(f(x)) = c(x)$ for all $x \in M$. Then M and N are diffeomorphic.*

Proof. Represent M as $(M', \mathcal{N}(M'_f))$ and N as $(N', \mathcal{N}(N'_f))$. Both $\mathrm{core}(M')$ and $\mathrm{core}(N')$ are $\langle n\rangle$-manifolds, and by [Lau00] there exists a collar neighborhood in the following sense. For each $s \subset t \subseteq \{1,\dots,n\}$ there exists an embedding

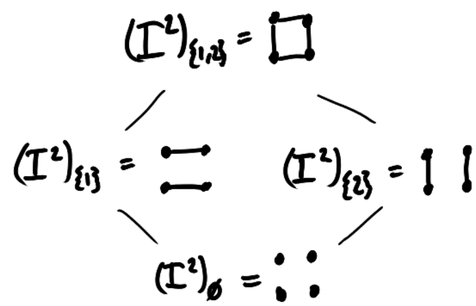
$$c_{s\subset t} : (\mathbb{R}^n_{\geq 0})_{s^c} \times M'_s \to (\mathbb{R}^n_{\geq 0})_{t^c} \times M'_t$$

with the property that $c_{s\subset t}|_{(\mathbb{R}^n_{\geq 0})_{t^c} \times M'_s}$ is the inclusion map $\mathrm{id} \times M'_{s\subseteq t}$, and furthermore the family of all these embeddings is functorial. The vector field integration argument that produces these can be extended to provide a bicollar neighborhood of $\mathrm{core}(M')$ in M'. By taking such a bicollar neighborhood for N' as well, we can extend the diffeomorphism h into some neighborhoods of M'_f and N'_f. □

[1]"Halated" as in "with a halo."

Definition 3.2.5. Suppose M is a halated (f, m)-manifold and N is a halated (g, n)-manifold, with M represented by $(M', \mathcal{N}(M'_f))$ and N by $(N', \mathcal{N}(N'_g))$. Let $f \sqcup g \subseteq \{1, \ldots, m+n\}$ denote the set $f \cup \{i+m \mid i \in g\}$. We define $M \times N$ to be $(M' \times N', \mathcal{N}(M'_f \times N'_g))$, which is a halated $(f \sqcup g, m+n)$-manifold. $\diamondsuit$

Example 3.2.6. We give the unit interval $I = [0, 1]$ the structure of a halated 1-manifold. The interval has the set $\{0, 1\}$ as its $\varnothing$-face. Then, for $n \in \mathbb{N}$, we have I^n as a halated $(\{1, \ldots, n\}, n)$-manifold. For example, I^2 is a halated 2-manifold with four faces (the top face being I^2 itself):



Each of these pictures should be regarded as being manifold germs in $\mathbb{R}^2$. $\diamondsuit$

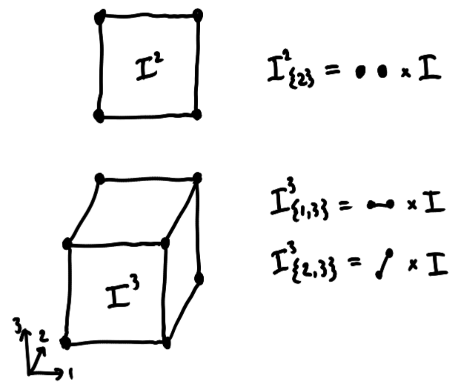
Definition 3.2.7. A *globular* halated $(\{1, \ldots, k\}, n)$-manifold is comprised of the following inductively defined data:

- A halated $(\{1, \ldots, k\}, n)$-manifold M and a family of halated $(f, n-1)$-manifolds $M^\circ(f)$ for all $f \subsetneq \{1, \ldots, k-1\}$.
- A decomposition of the face $M_{\{1,\ldots,k-1\}}$ into a disjoint union of globular halated $(\{1, \ldots, k-1\}, n)$-manifolds sM and tM, respectively called the *source* and *target* of M.
- For all $f' \subseteq f \subseteq \{1, \ldots, k-1\}$ an embedding $M^\circ(f') \hookrightarrow M^\circ(f)$ of halated manifolds such that the embeddings are functorial.
- For all $f \subsetneq \{1, \ldots, k-1\}$ a diffeomorphism $h_f : M^\circ(f) \times I \to M_f$, such that $h_f(M^\circ(f) \times [0, 1/2))_{\{1,\ldots,k-1\}}$ is in sM and $h_f(M^\circ(f) \times (1/2, 1])_{\{1,\ldots,k-1\}}$ is in tM, and furthermore that for all $f' \subseteq f \subseteq \{1, \ldots, k-1\}$ these maps fit into a commutative diagram

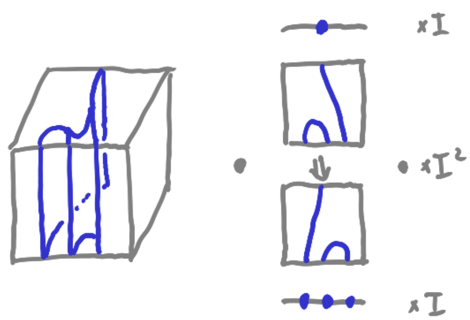
$$\begin{tikzcd} M^\circ(f') \times I \arrow[r, "h_{f'}"] \arrow[d, hook] & M_{f' \cup \{k\}} \arrow[d, hook] \\ M^\circ(f) \times I \arrow[r, "h_f"] & M_{f \cup \{k\}} \end{tikzcd}$$

We call each *collapse face* $M_{f\cup\{k\}}$ a *suspension* of $M^\circ(f)$ and each $M^\circ(f)$ a *collapse* of $M_{f\cup\{k\}}$.[2] We call the maps h_f the *collapse diffeomorphisms*. ♢

Example 3.2.8. The halated manifold I^n is globular, as illustrated in the following diagrams for I^2 and I^3.



A sort of picture to keep in mind is a foliation-respecting singularity type:



This, more precisely, is a globular halated manifold with an embedded codimension-1 submanifold. The submanifold itself is a globular halated manifold as well, whose faces are the intersections with the faces of I^3. ♢

Remark 3.2.9. Globular halated n-manifolds are like globes in that sources and targets satisfy $ssM = stM$ and $tsM = ttM$. The data also allows one to crush faces in the I direction of each collapse diffeomorphisms. Crushed globes have been used for globular set description of cobordisms, see for example pinched products in [MW12]. A benefit to not crushing cubes into globes is that whiskering does not require dealing with singular manifolds. ■

Remark 3.2.10. Our definition of globular halated n-manifolds is roughly equivalent to the definition of the haloed bordisms defined in [SP09]. The role of a series of co-oriented halations is played by halated (f, n)-manifolds, where the $\Gamma^n_{\geq 0}$ structure supplies all the co-orientations. ■

[2] N.B. This terminology is meant to evoke the sense of suspensions of singularities rather than suspensions of topological spaces.

3.3 Cobordism bicategory

We give some details about the cobordism bicategory for surfaces, but we point the reader toward [SP09] for more elaboration using a similar setup.

The collection of halated n-manifolds have n different directions for composition operations, one per $(n-1)$-dimensional face. In our definition of halated n-manifolds, we required that charts preserve the entire subspaces $H_f = \operatorname{Span}\{e_i \mid i \in f\}$, which gives us a well-defined reflection operation:

Definition 3.3.1. Suppose M is a halated (f, n)-manifold. For every subset $s \subseteq \{1, \dots, n\} \setminus f$, there is halated (f, n)-manifold sM defined by taking all charts for M and flipping all the coordinates contained in s. $\diamondsuit$

Suppose we have halated (f, n)-manifolds M and N, and furthermore suppose, for $f' \subset f$ with $|s| = |f| - 1$ that we have an (f', n) manifold A and embeddings $f_M : A \hookrightarrow M$ and $f_N : \{i\}A \hookrightarrow N$ for some $i \in f \setminus f'$ that respect the halation structure such that f_M and f_N map onto a connected component of the f'-faces. Then we claim we can form the pushout

$$\begin{tikzcd} A \arrow[r, hook, "f_M"] \arrow[d, hook, "\{i\}f_N"] & MM \arrow[d, hook] \\ N \arrow[r, hook] & M \cup_A N \end{tikzcd}$$

where $\{i\}f_N$ indicates the map from including A into $\{i\}N_f$ rather than $\{i\}A$ into N_f. Furthermore, this pushout has the property that $\operatorname{core}(M \cup_A N) = \operatorname{core}(M) \cup_{\operatorname{core}(A)} \operatorname{core}(N)$, which is a sense in which halations yield canonical gluings. It should be said, though, that we still need to choose a representative for the resulting halated manifold.

The reason we can form this pushout is that we can represent M as a manifold germ $(M', \mathcal{N}(M'_f))$ and N as $(N', \mathcal{N}(N'_f))$, then use the existence of a bicollar neighborhood for M' and N' to glue M' and N' along some open neighborhood of the image of A contained in this bicollar neighborhood. Writing $M' \cup_A N'$ for this glued manifold temporarily, then $M \cup_A N$ is $(M' \cup_A N', \mathcal{N}(M'_f \cup N'_f))$, where we identify M'_f and N'_f with their images in $M' \cup_A N'$. One can check that this construction yields a halated (f, n)-manifold.

For each $n \in \mathbb{N}$ and $k \leq n$, let $\operatorname{Bord}_{n,k}$ be the set of compact globular halated $(\{1, \dots, k\}, n)$-manifolds, where *compact* means the core of the underlying halated manifold is compact. Then $\operatorname{Bord}_n = \bigsqcup_{k=0}^n \operatorname{Bord}_{n,k}$ is something like a quasicategory, in that compositions exist but without any promise of associativity or other coherence. We mention these compositions here.

For $M, N \in \operatorname{Bord}_{n,k}$, we have a subset $\operatorname{Bord}_{n,k+1}(M, N)$ of all $F \in \operatorname{Bord}_{n,k+1}$ such that $sF = M$ and $tF = \{n\}N$. Then for $L, M, N \in \operatorname{Bord}_{n,k}$, there is a well-defined map $\operatorname{Bord}_{n,k+1}(M, N) \times \operatorname{Bord}_{n,k+1}(L, M) \to \operatorname{Bord}_{n,k+1}(L, N)$ defined on the underlying halated manifolds by $(F', F) \mapsto F' \cup_M F$. To extend this to a map of globular halated manifolds, we glue the collapse diffeomorphisms end-to-end using some fixed identification $I \cup_p I \equiv I$, where p is the one-point $(\varnothing, 1)$-halated manifold that embeds into 0 in the first I and 1 in the second — note that we can do this gluing since the diffeomorphisms are diffeomorphisms of halated manifolds.

Remark 3.3.2. Of course, this composition operation is not associative. This is due to the fact that gluing three copies of I two at a time is non-assocative. We could potentially remedy this by having globular halated n-manifolds use arbitrary closed intervals $[0, a]$ for $a > 0$ rather than fixing $a = 1$, since then there is an associative way to compose such intervals. ■

There is also, for each $n \in \mathbb{N}$, a monoidal composition $\mathrm{Bord}_{n,k}(M, N) \times \mathrm{Bord}_{n,k}(M', N') \to \mathrm{Bord}_{n,k}(M \sqcup M', N \sqcup N')$ by taking disjoint unions. It is worth pointing out that this operation is also not associative since disjoint unions require an encoding, for example $X \sqcup Y = \{(0, x) \mid x \in X\} \cup \{(1, x) \mid y \in Y\}$.

We have another operation, which is *whiskering*. For $F \in \mathrm{Bord}_{n,k+1}(M, N)$ and $L \in \mathrm{Bord}_{n,k}$ such that $ssF = \{k-1\}tL$, then we can use the collapse diffeomorphism for F to glue $L \times I$ along sF, yielding an element of $\mathrm{Bord}_{n,k+1}(M \cup_{tL} L, N \cup_{tL} L)$. Note that $N \cup_{tL} L$ uses the collapse isomorphism to do the gluing, since N and L might otherwise be incompatible.

The essence of the cobordism n-category Cob_n is that we take $\mathrm{Bord}_{n,k}$ to be the k-morphisms for all $0 \leq k \leq n$, modulo the relation that n-morphisms are equivalent if they are diffeomorphic, and with identity maps given by $M \times I \in \mathsf{Cob}_n(M, M)$. Then for each $M, N \in (\mathsf{Cob}_n)_k$, the set $\mathsf{Cob}_n(M, N)$ has the structure of an $(n-k)$-category since we get weak associativity of compositions.

This was a brief explanation of cobordism categories using globular halated manifolds, and since our goal is to give a presentation of some concrete cobordism categories we will stop here.

3.4 Gradient flows

Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function and let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R}$ be the standard projection. Like in 1D Morse theory, we can use gradient flows to decompose Σ into well-understood pieces, which we describe how to do in this section. See Figures 3.2 to 3.4 for illustrations of all the pieces that we found through our classification of singularities; this includes singularities of surfaces with embedded curves.

Generic 2D Morse functions have a stratification of the domain into locally closed singularity submanifolds. These singularity submanifolds have the property that f and $\pi \circ f$ are immersions when restricted to them. Furthermore, there are multijet singularity submanifolds associated to subsets of points that have the same image, and these have the same properties.

Hence, a singularity has codimension 2 if and only if $d(\pi \circ f)$ restricted to its singularity submanifold has rank 0.

Definition 3.4.1. Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function. A vector field $V \in C^\infty(T\Sigma)$ is *subordinate* to the singularities if for all singularity submanifolds $S \subseteq \Sigma$ and $x \in S$, we have $V_x \in T_x S$. $\Diamond$

Lemma 3.4.2. *Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function. There exists a Riemannian metric on Σ such that the gradient vector field $\nabla(\pi \circ f)$ is subordinate to the singularities and vanishes only at codimension-2 singularities.*

Proof. For each $p \in \Sigma$, the germ $[f]_p$ is equivalent to one of the singularities in our classification, and so there exists a chart neighborhood U of p such that $f|_U$ has coordinates given by one of the normal forms. One can see that in each case there exists a Riemannian metric such that the gradient vector field $\nabla(\pi \circ f|_U)$ is subordinate to the singularities in U. Then we can get a Riemannian metric from these using the standard partition of unity and paracompactness argument, and this Riemannian metric has the desired properties. $\square$

Lemma 3.4.3. *Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function. Let $t_0 < t_1$ be real numbers such that $(\pi \circ f)^{-1}([t_0, t_1])$ intersects no codimension-2 singular submanifolds. Then there exists a diffeomorphism $h : f^{-1}(t_0) \times I \to f(\pi \circ f)^{-1}([t_0, t_1])$ such that h restricts to diffeomorphisms $f^{-1}(t_0) \times \{0\} \to f(\pi \circ f)^{-1}(\{t_0\})$ and $f^{-1}(t_1) \times \{1\} \to f(\pi \circ f)^{-1}(\{t_1\})$. Furthermore, each singularity submanifold in $(\pi \circ f)^{-1}([t_0, t_1])$ is a union of the $h(x \times I)$ sets it intersects.*

Proof. Give Σ a Riemannian metric such that $\nabla(\pi \circ f)$ is subordinate to the singularities. Since there are no codimension-2 singular submanifolds in $(\pi \circ f)^{-1}([t_0, t_1])$, the gradient is non-vanishing in this region. We then take the flow of $\nabla(\pi \circ f)/|\nabla(\pi \circ f)|^2$, for time in the interval $[0, t_1 - t_0]$, which after identifying this interval with I yields the desired diffeomorphism since this is the height-parameterized flow like in Morse theory. The last claim of the lemma follows from the vector field being subordinate to the singularities. $\square$

Now we understand what occurs between codimension-2 singularities, and now we need a description of what occurs at codimension-2 singularities. To do this, we develop structures we use for the decomposition.

Definition 3.4.4. For $n \in \mathbb{N}$, suppose $\mathbb{R}^n$ is given the standard $\mathbb{R} \subseteq \cdots \subseteq \mathbb{R}^n$ flag foliation. An *n-cube* is a set of the form $g[0,1]^n$ for some $g \in \mathrm{Diff}(\mathbb{R} \subseteq \cdots \subseteq \mathbb{R}^n)$. We give n-cubes the structure of a globular halated n-manifold through its identification with $[0,1]^n$ as a subset of $\mathbb{R}^n$. $\Diamond$

Remark 3.4.5. In the following, we will be somewhat loose with terminology and speak of "isomorphism." What we mean is that we are constructing particular decompositions and carrying around a combinatorial description of it. $\blacksquare$

Theorem 3.4.6. *Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function. For every $p \in \mathbb{R}^2$ and open neighborhood $U \supseteq p$ there exists a 2-cube $C \subseteq U$ such that the smooth manifold germ $(\Sigma, f^{-1}(C))$ and inherits the structure of globular halated 2-manifold, which is to say the*

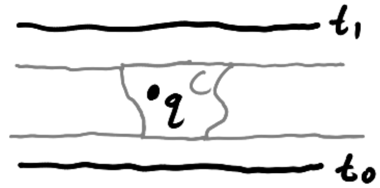


Figure 3.1: When there is a codimension-2 singularity, there is a cube containing the value q and a corresponding decomposition that gives the surface a description as a whiskering by a 1-manifold in Cob_2.

preimage of each f-face of C is an f-face of $f^{-1}(C)$ and the collapse diffeomorphisms for the globular structure are pulled back to $f^{-1}(C)$. Furthermore, this globular halated 2-manifold is isomorphic to one of the standard ones from Figure 3.2.

Proof. For each $q \in \mathbb{R}^2$, the preimage $f^{-1}(q)$ intersects one or more singularity submanifolds. Shrink U until $f^{-1}(U)$ is a disjoint union of neighborhoods, one per point in $f^{-1}(q)$. If any $p \in f^{-1}(p)$ is a codimension-2 singularity, then we can disregard it in the following analysis since $f^{-1}(C)$ for any 2-cube $C \subseteq U$ will lift to a diffeomorphic cube in a neighborhood of p. Let $S \subseteq f^{-1}(q)$ be the set of non-codimension-2 singularities. We have analyzed all the possibilities through our analysis of multigerms, and for each of theme one can find a 2-cube C in U with the desired property. See Figure 3.2 for illustrations of the globular structure of each singularity type and how it maps to a 2-cube in $\mathbb{R}^2$. □

Remark 3.4.7. Given such a globular halated 2-manifold in Σ, note that the faces that are collapsed are all in the codimension-0 singularity submanifold, which follows from the fact that the collapse diffeomorphims are pulled back through f, so they yield diffeomorphisms of those faces onto faces in $\mathbb{R}^2$, and importantly faces are halated, so this extends to a diffeomorphism of an open neighborhood of those faces. ■

Lemma 3.4.8. *Suppose $f : \Sigma \to \mathbb{R}^2$ is a generic 2D Morse function and that for real numbers $t_0 < t_1$ there exists exactly one codimension-2 (multi)singularity p in $(\pi \circ f)^{-1}([t_0, t_1])$. Let C be a cube in $\mathbb{R} \times (t_0, t_1)$ containing $f(p)$ of the type from Theorem 3.4.6, and let $s_0, s_1 \in (t_0, t_1)$ with $s_0 < s_1$ be the y_2 coordinates of the sides of C. Using C, decompose $\mathbb{R} \times [t_0, t_1]$ into five regions as depicted in Figure 3.1. Then (1) $f^{-1}(C)$ is isomorphic to the disjoint union of one of of the standard globular halated 2-manifolds associated to codimension-2 singularities and some number of identity morphisms (call this disjoint union A) (2) $f^{-1}(\mathbb{R} \times [s_0, s_1])$ is isomorphic to a left- and right- whiskering of A (call this B), and (3) $f^{-1}(\mathbb{R} \times [t_0, t_1])$ is isomorphic to B.*

Proof. Let C be as stated. When we construct a Riemannian metric such that $\nabla(\pi \circ f)$ is subordinate to the singularities, we may use the collapse faces of C as additional "singularities" for the gradient to be subordinate to (and in general, for other dimensions, we can ask for the gradient to lie in the I direction from the collapse diffeomorphisms). Have done so, the resulting flows follow the collapse faces, so we can use the flow from Lemma 3.4.3 for the two regions L and R of $\mathbb{R} \times [s_0, s_1]$ outside C. Hence, $f^{-1}(\mathbb{R} \times [s_0, s_1])$ is the whiskering of $f^{-1}(C)$ by $f^{-1}(L \cap (\mathbb{R} \times \{s_0\})$ and $f^{-1}(L \cap (\mathbb{R} \times \{s_0\})$. Furthermore, since codimension-2 singularities do not occur simultaneously, we can "peel off" identity morphisms from $f^{-1}(C)$. For the rest, note that in the $\mathbb{R} \times [t_0, s_0]$ and $\mathbb{R} \times [s_1, t_1]$ regions we can apply Lemma 3.4.3. □

***Warning* 3.4.9.** While we mention disjoint unions, we are not claiming to be decomposing these pieces using the monoidal structure of the cobordism 2-category. There is still an issue of separating the manifold into sheets, which the next section comments upon. ♢

This lemmas gives a way to decompose any surface into whiskerings of 1-manifolds composed with one of the standard 2-manifolds associated to codimension-2 singularities. What remains is to decompose 1-manifolds. For generic f, when $t \in \mathbb{R}$ is such that $\mathbb{R} \times \{0\}$ contains no codimension-2 singularities, then we can restrict to studying the restriction of f to $f^{-1}(\mathbb{R} \times \{0\}) \to \mathbb{R} \times \{0\}$. This is itself generic, so we can continue the decomposition. In this case, it is simply a Morse function of 1-manifolds, so we skip it here.

Hence, we have described a way to use gradient flows to decompose a surface Σ that has a 2D Morse function $f : \Sigma \to \mathbb{R}^2$ into pieces from Figure 3.2. And of course, since 2D Morse functions are dense, this means we can decompose every compact surface in such a way by perturbing the constant-zero function.

For relations, we apply a similar process to 3D Morse functions $\Sigma \times I \to \mathbb{R}^3$. This time we look for 3-cubes in $\mathbb{R}^3$ that contain a codimension-3 singularity and decompose $\Sigma \times I$ using gradient flows. We have illustrations of all the codimension-3 singularities and a globular halated 3-manifold structure that maps to a 3-cube in Figures 3.2 to 3.4.

We mention that decompositions for the case of surfaces with embedded curves is handled in essentially the same way since we can treat the curves as if they were singular submanifolds.

3.5 Symmetric monoidal structure

One aspect we did not deal with in the previous section is decomposing manifolds using the symmetric monoidal structure of the cobordism category. There is a fundamental problem that needs solving. For $f : \Sigma \to \mathbb{R}^2$ a 2D Morse function, let Σ° be the preimage of $\mathbb{R}^2$ minus any singular values. This open submanifold is a collection of connected components, the *sheets*, that f embeds in $\mathbb{R}^2$. To yield a monoidal decomposition, a naive approach would be to totally order the sheets, but the total ordering needs to be compatible with the singularities in the sense that singularities cannot occur between sheets at points where there is another sheet between them. This quickly leads to impossibilities.

What Schommer-Pries proposed is to cut apart the sheets into chambers with tools called chambering graphs (for Σ) and chambering foams (for $\Sigma \times I$). One way to conceptualize these is that we want a function $\sigma : \Sigma \to \mathbb{R}$ that is generic with respect to f, and then we use the collection of double points (the chambering graph) to decompose Σ. This is

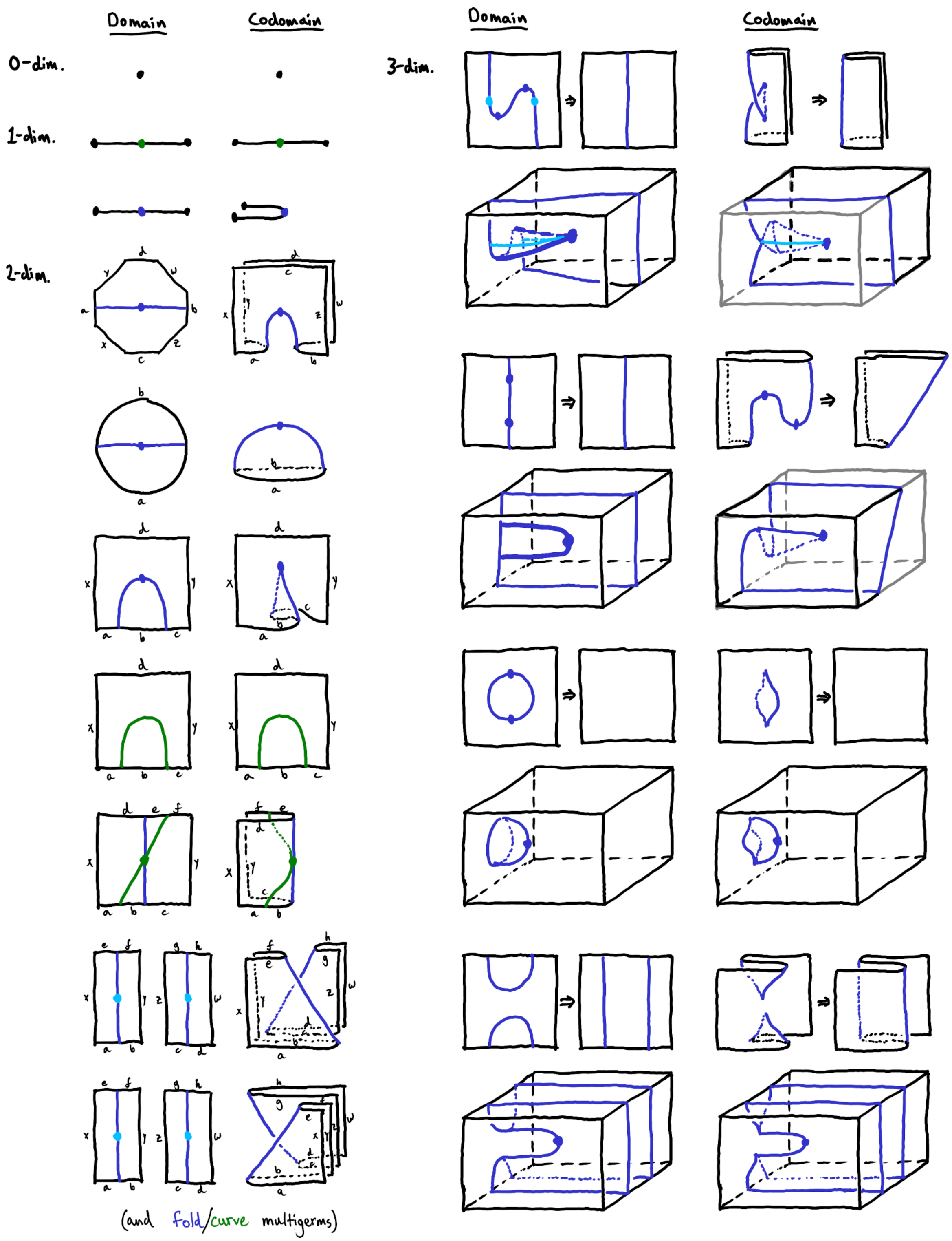


Figure 3.2: Illustrations of singularities with choices of neighborhoods such that the codomain is an n-cube and the domain is a manifold with faces. Suspensions are not shown, and not all multigerm singularities are present. Continued in Figure 3.3.

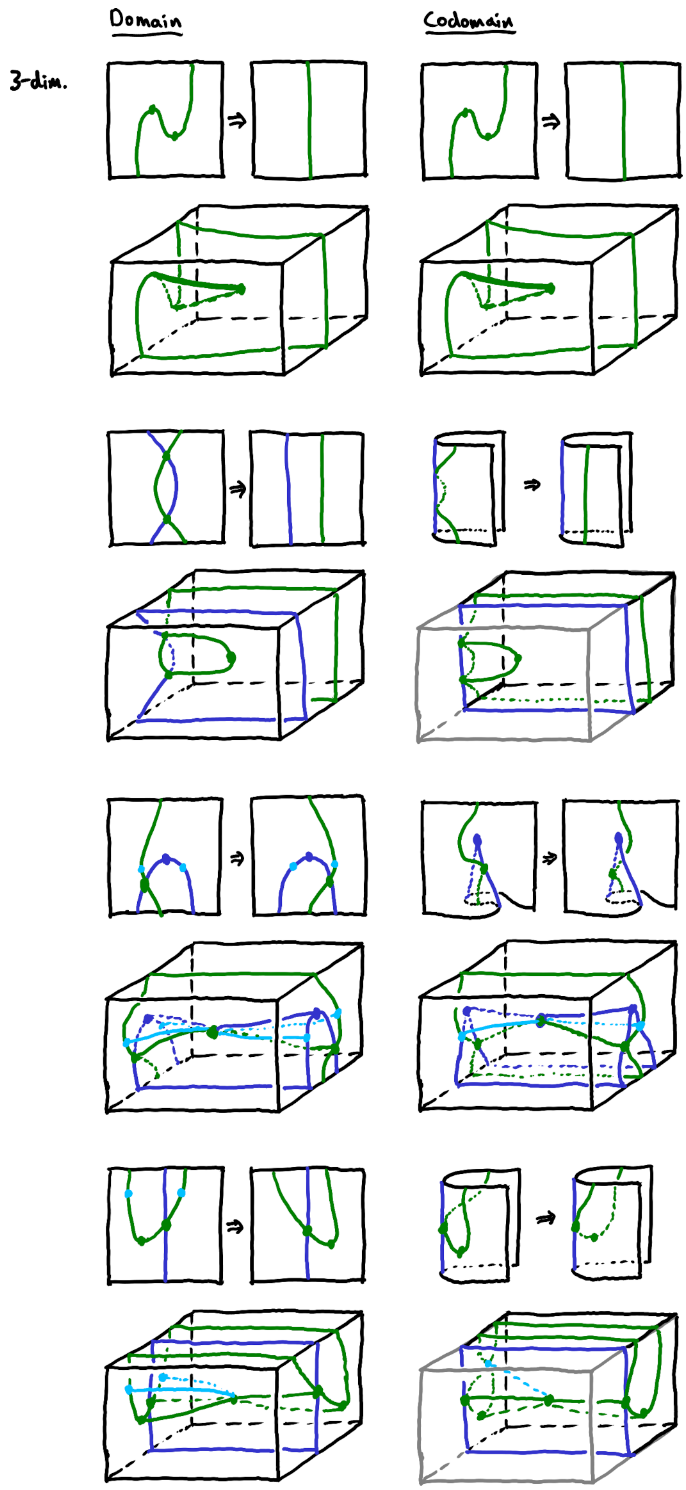


Figure 3.3: Continuation of Figure 3.2, and continued in Figure 3.4.

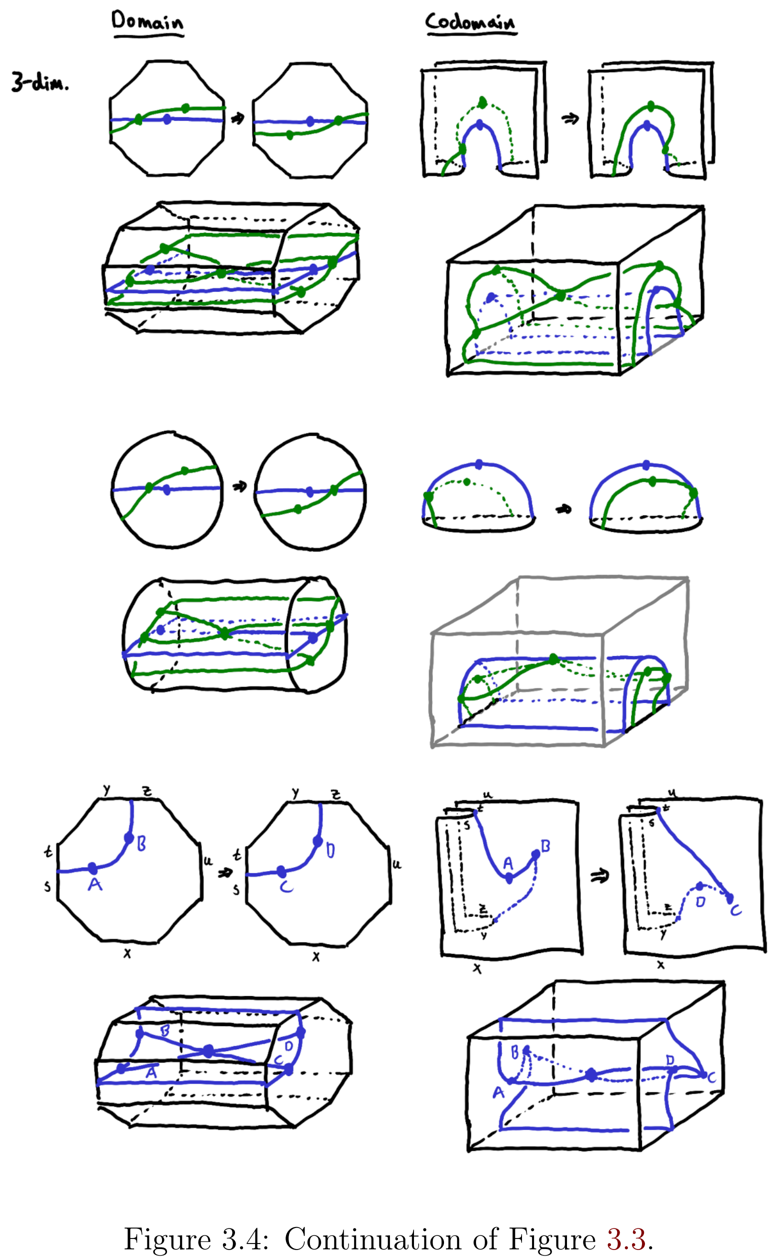


Figure 3.4: Continuation of Figure 3.3.

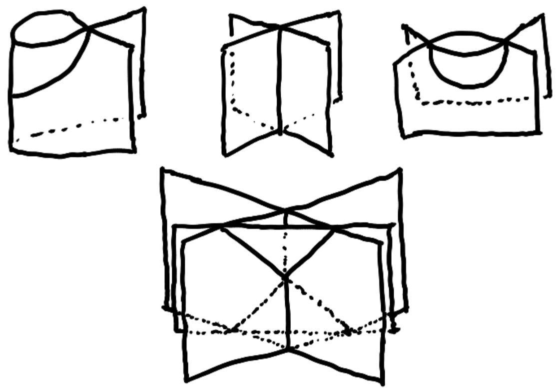


Figure 3.5: The 2-morphisms that are the *symmetry generators* for the 2D cobordism 2-category, which are all from the analysis of Cerf functions of curves in the plane from Section 2.3.2. The second and fourth are consequences of symmetric monoidal 2-categories.

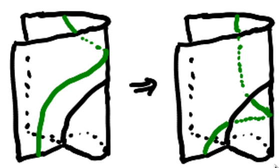


Figure 3.6: The 3-morphism that the double point relation for surfaces with embedded curves. We do not show the ones that are consequences of symmetric monoidal 2-categories.

equivalent to studying 3D Morse functions $f : \Sigma \to \mathbb{R}^3$, where the projection onto $\mathbb{R}^3/\mathbb{R}$ gives all the singularities we have found so far, and the additional (multi)singularities yield all the symmetry generators. The component f_1 is precisely the σ function.

We can deduce what the additional singularities and multigerm singularities must be, since for codimension reasons a 3D Morse function cannot have double points occuring at a codimension-3 singularity. This means that, away from such codimension-3 points the 3D Morse function is a Cerf function for curves in surfaces, which we already analyzed in Section 2.3.2.

Hence, by cutting up sheets along these loci of double points we have a way to totally order the remaining sheets and glue them back together using generators for the cobodism 2-category.

What remains is finding the relations between these types of singularities and the singularities we have found so far. We refer to [SP09, Figure 3.8] for now for the relations, and for embedded curves in surfaces one can deduce that there must be a move like Figure 3.6 and that all other relations on symmetry generators are consequences of symmetric monoidal

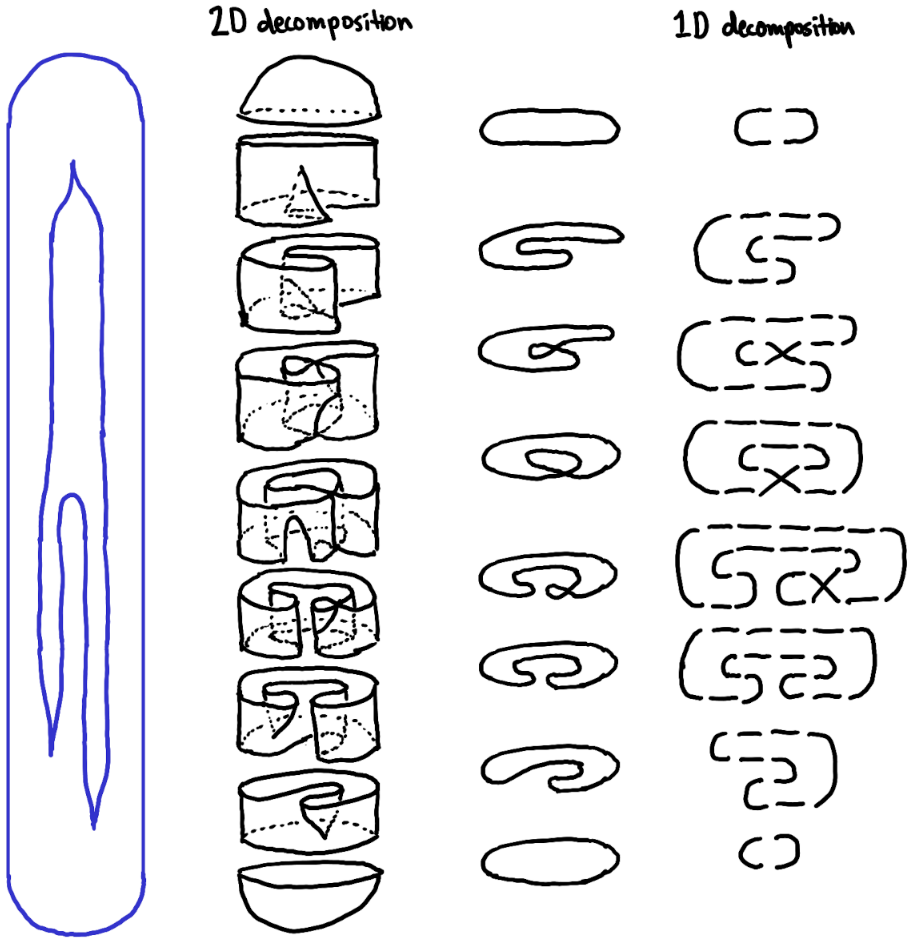
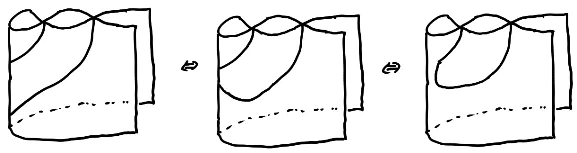


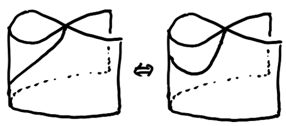
Figure 3.7: Decomposition of a 2D Morse function for $\mathbb{RP}^2$. Each vertical slice (a 2-morphism) has a single 2-morphism generator. The 1-morphisms between these slices are shown, and then these are sliced horizontally, with a single 1-morphism generator per slice. To the left, in blue, is a 2D Morse graphic showing the images of the singular points; many intermediate moves are for bookkeeping in the symmetric monoidal 2-category.

2-categories.

Remark 3.5.1. Although they are sufficent, from the perspective of Cerf theory of $\mathbb{R}^2 \to \mathbb{R}^3$ functions, the relations in the lower-left corner of [SP09, Figure 3.8] are not generic since with a slight perturbation the relation decomposes into a sequence of more obviously stable relations:



This uses a relation that is not present in the figure, the double-point flip:



Taking horizontal slices through time, the knot theoretic interpretation of this move is that doing a single Reidemeister I move is equivalent to do a Reidemeister II move and then a Reidemeister I move. ■

Example 3.5.2. In Figure 3.7 we decompose $\mathbb{R}\mathrm{P}^2$ as an element of the symmetric monoidal 2-category of 2D cobordisms. It is sliced into 2-morphisms, with one generating 2-morphism per piece, and then the intervening 1-morphisms are sliced, with one generating 1-morphism per piece. ◊

Chapter 4

Applications

4.1 Black-white extended 2D TQFTs

This author's original motivation for computing a presentation for the symmetric monoidal 2-category of surfaces with embedded curves was to study a particular extended 2D TQFT with defects, which we describe in this section. In this TQFT, oriented surfaces are decomposed into a black and a white region, separated by a common 1-dimensional submanifold. We extend this TQFT in Section 4.2 with an additional 0-dimensional defect type to encode surface graph invariants as extended TQFTs.

These are a sort of "surface algebra" analogue of the planar algebras described in [Jon], in that planar algebras come with a natural black/white checkerboard coloring, and so we extend this structure to diagrams on surfaces. Black-white 2D TQFTs, as surfaces constructed from patchworks of planar algebra diagrams, were studied in [KPS], and also related are the black-white "picture TFQTs" in [FNWW].

Consider the class of compact oriented surfaces with embedded 1-dimensional submanifolds such that the complement of the submanifold is given a locally constant function coloring each point either black or white. For surfaces with boundary, we require that the 1-manifold intersects the boundary transversely. We call these *black-white surfaces*, the points colored black or white the *black region* and the *white region*, respectively, and the 1-manifold the *strings*. There is a cobordism 2-category for these.

Definition 4.1.1. Let $\mathsf{BW2Cob}$ denote the symmetric monoidal 2-category for cobordisms of black-white surfaces, where a k-morphism is an globular halated $(\{1,\ldots,k\},2)$-manifold arising as a manifold germ of a black-white surface such that the defect meets each face transversely. $\Diamond$

Remark 4.1.2. In [SP09], Schommer-Pries showed how to carry out a characterization of extended oriented 2D cobordisms as an example of more sophisticated techniques that work for more sophisticated structures on manifolds. Simplifying this, we say an orientation of a surface Σ is a function μ that assigns to each $x \in \Sigma$ a generator $\mu_x \in H_2(\Sigma, \Sigma \setminus \{x\}) \approx \mathbb{Z}$

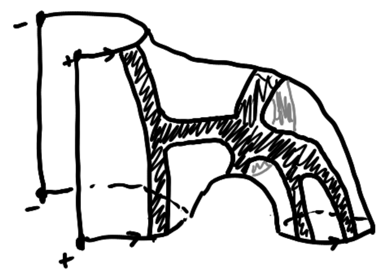


Figure 4.1: A 2-morphism in BW2Cob. Using the bottom-to-top convention, this is a morphism from the disjoint union of an interval and a circle to an interval, where each interval is separated into two white and one black regions and the circle is separated into two white and two black regions. The 2-morphism has four strings.

such that for all $x \in \Sigma$ there exists a neighborhood $U \subseteq \Sigma$ homeomorphic to a disk such that there exists a generator $\mu_U \in H_2(\Sigma, \Sigma \setminus U) \approx \mathbb{Z}$ such that for all y, the natural map $H_2(\Sigma, \Sigma \setminus U) \to H_2(\Sigma, \Sigma \setminus \{y\})$ maps μ_U to μ_x. We use this notion of orientation for BW2Cob, and there is no difficulty in gluing such functions together when doing the various compositions the category provides.

Additionally, the black-white coloring can be represented as the additional structure of a codimension-0 submanifold representing the black region. There is no difficulty in gluing such data. ■

Example 4.1.3. For a representative example, see Figure 4.1. ♢

Theorem 4.1.4. *The category* BW2Cob *is the symmetric monoidal* 2*-category presented by the morphisms and relations illustrated in Figure 4.2.*

Proof. The additional structure on the cobordisms (orientations and colorings) is simple enough that in our 2D Morse functions and Cerf functions, we can simply ignore it, classify singularity types, decompose, and then work out how the structure should be reattached to the pieces, which can be done without any choices. We take the classification of singularities for surfaces with embedded curves and then consider all possible 2-colorings and orientations. We illustrate the results in Figure 4.2, modulo obvious symmetries. □

Definition 4.1.5. Let $\mathcal{C}$ be a symmetric monoidal 2-category. A *topological quantum field theory* (or TQFT) for BW2Cob is a symmetric monoidal homomorphism BW2Cob $\to \mathcal{C}$.[1] ♢

In the next sections we review the algebraic theory to make sense of some example TQFTs, and we will continue in Section 4.1.2.

[1]Refer to [Sta13] for definitions, reproduced in [SP09, Appendix C].

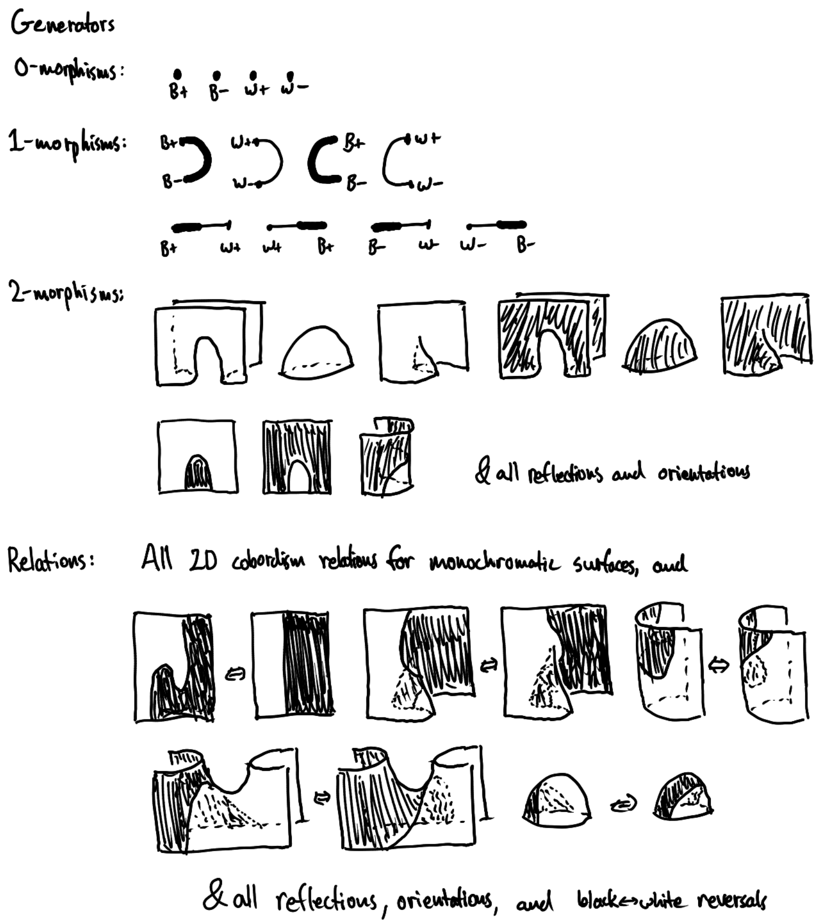


Figure 4.2: Generators of $\mathsf{BW2Cob}$. For 2-morphisms and 3-morphisms, we gives representatives up to vertical and horizontal reflection, orientation changes, and black-white reversals. We do not include symmetry generators or identities, which come from the definition of a symmetric monoidal 2-category.

4.1.1 Morita category

Of particular interest for the target category for a TQFT is the *Morita category* k-Alg$_2$ where, for k a commutative ring, objects are k-algebras, 1-morphisms are bimodules, and 2-morphisms are bimodule homomorphisms (intertwiners). Over the next few subsections we review properties of this category as well as Frobenius algebras, which play a role in the classification of k-Alg$_2$-valued extended 2D TQFTs.

Given k-algebras A, B, and C and bimodules ${}_AM_B$ and ${}_BN_C$, the composition of the bimodules is given by tensor products over the intermediate algebra B:

$$A \xleftarrow{M} B \xleftarrow{N} C \quad = \quad A \xleftarrow{M\otimes_B N} C.$$

One can think of k-Alg$_2$ as being a generalization of the category of algebras and their homomorphisms, since if $f : A \to B$ is an algebra homomorphism, then we have the bimodule ${}_AB_B$ with the A action given by $ab = f(a)b$ for $a \in A$ and $b \in B$.

Tensor products of k-algebras gives a monoidal structure to the category, where k is the monoidal unit. This is compatible with the morphisms and 2-morphisms in the following way. For ${}_AM_B$ and ${}_CN_D$ bimodules, the tensor product of $M \boxtimes N = M \otimes_k N$ is a $(A \otimes_k C, B \otimes_k D)$ bimodule:

$$A \otimes_k C \xleftarrow{M\boxtimes N} B \otimes_k D.$$

Furthermore, if $f : {}_AM_B \to {}_AM'_B$ and $g : {}_CN_D \to {}_CN'_D$ are bimodule homomorphisms, then $f \boxtimes g$ is a bimodule homomorphism of $(A \otimes_k C, B \otimes_k D)$-modules:

$$\begin{tikzcd} A \otimes_k C & & B \otimes_k D. \arrow[ll, bend right, "M\boxtimes N"'] \arrow[ll, bend left, "M'\boxtimes N'"] \arrow[ll, phantom, "\Downarrow f\boxtimes g"] \end{tikzcd}$$

For k-algebras A and B, the twist $\tau_{A,B}$, which gives k-Alg$_2$ the structure of a symmetric monoidal category, is $A \otimes_k B$ as a $(B \otimes_k A, A \otimes_k B)$-module.

Let k-Alg$_2(A, B)$ denote the morphism set of all (A, B)-modules. This forms a monoid under direct sums, where the additive unit is the zero (A, B)-module. The monoid structure is compatible with the 2-morphisms, which is to say that, for bimodule homomorphisms $f : {}_AM_B \to {}_AM'_B$ and $g : {}_AN_B \to {}_AN'_B$, there is a bimodule homomorphism $f \oplus g$ as follows:

$$\begin{tikzcd} A & & B \arrow[ll, bend right, "M\oplus N"'] \arrow[ll, bend left, "M'\oplus N'"] \arrow[ll, phantom, "\Downarrow f\oplus g"] \end{tikzcd}$$

It should be noted that the monoid axioms are only true up to coherence, for example $M_1 \oplus (M_2 \oplus M_3) \approx (M_1 \oplus M_2) \oplus M_3$.

Thinking of k-$\mathrm{Alg}_2(A,B)$ as a symmetric monoidal category, the objects are (A,B)-modules and the morphisms are bimodule homomorphisms. The hom sets of k-$\mathrm{Alg}_2(A,B)$, comprising of k-linear maps, are k-modules.

For A a k-algebra, let A^{op} be the same algebra but with opposite multiplication $\mu^{\mathrm{op}} = \mu \circ \tau$. For a (A,B)-module M, there is an opposite $(B^{\mathrm{op}}, A^{\mathrm{op}})$-module M^{op} with the opposite action:

$$A \xleftarrow{M} B \qquad \Rightarrow \qquad A^{\mathrm{op}} \xrightarrow{M^{\mathrm{op}}} B^{\mathrm{op}}.$$

In general, if M is a $(A \otimes_k B, C)$-module, we may treat it as a $(A, B^{\mathrm{op}} \otimes_k C)$-module. A special case is when M is an A-bimodule, where M can be thought of as a $(k, A^{\mathrm{op}} \otimes_k A)$-module, with $A^{\mathrm{op}} \otimes_k A$ being known as the *enveloping algebra* of A.

Let ev_A be A as a $(k, A^{\mathrm{op}} \otimes_k A)$-module and let coev_A be A as a $(A \otimes_k A^{\mathrm{op}}, k)$-module. These bimodules give A as a right dual of A^{op} in k-Alg_2, and similarly A^{op} as a right dual of A. Hiding some isomorphisms (such as $A \otimes_k k \cong A$) this says that, up to some 2-isomorphism, the compositions satisfy

$$A \xleftarrow{A \boxtimes \mathrm{ev}_A} A \otimes_k A^{\mathrm{op}} \otimes_k A \xleftarrow{\mathrm{coev}_A \boxtimes A} A \qquad = \qquad A \xleftarrow{A} A$$

and

$$A^{\mathrm{op}} \xleftarrow{\mathrm{ev}_A \boxtimes A^{\mathrm{op}}} A^{\mathrm{op}} \otimes_k A \otimes_k A^{\mathrm{op}} \xleftarrow{A^{\mathrm{op}} \boxtimes \mathrm{coev}_A} A^{\mathrm{op}} \qquad = \qquad A^{\mathrm{op}} \xleftarrow{A^{\mathrm{op}}} A^{\mathrm{op}}.$$

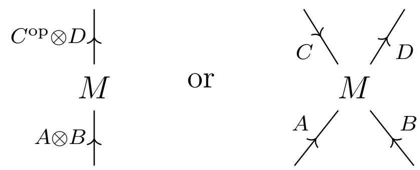
This will be illuminated in Section 4.1.1, where we will also see, for instance, how M^{op} may be formed from M through compositions with ev and coev.

Trace diagrams of bimodules

In this section, we recall trace diagrams (also known as tensor networks, Penrose notation, string diagrams, etc.), but as applied to objects and 1-morphisms of k-Alg_2, giving a convenient notation for writing tensor products of bimodules. Our temporary convention in this section is that tensor products over k are left-to-right and morphisms go bottom-to-top, to match Appendix A.2, and when we work with 2-morphisms we will rotate these diagrams 90 degrees counterclockwise.

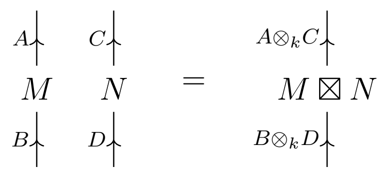
We write an (A,B)-module M as

For bimodules over tensor products of algebras, we may use multiple arrows to indicate this. For example, if M is a $(A \otimes B, C^{\mathrm{op}} \otimes D)$-module, we may write either

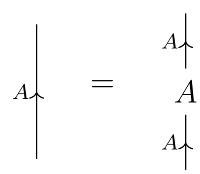


where downward-oriented arrows correspond to the opposite algebra.

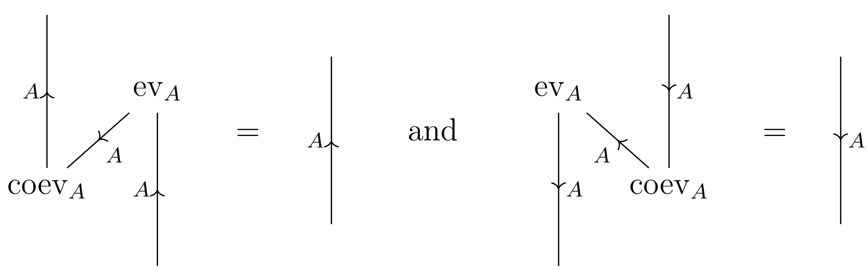
The $\boxtimes$ tensor product of bimodules is given by juxtaposition:



Since for an (A, B)-module M we have $A \otimes_A M \otimes_B B \cong M$ (that is, A and B as bimodules over themselves are the identities for k-Alg_2), vertical lines without a bimodule can be thought of as being the identity:

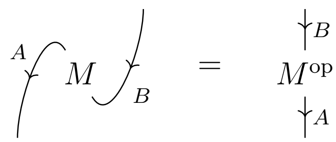


Connecting bimodules together corresponds to composition. For example, the relations that the ev_A and coev_A bimodules satisfy can be written as



Seeing that this is a "topological" identity, we will henceforth omit ev_A and coev_A on bends in the arrows. For example, if M is an (A, B)-module, the opposite $(B^{\mathrm{op}}, A^{\mathrm{op}})$-module M^{op}

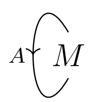
is given by



Arrows can cross, and the interpretation is to use the twist bimodules $\tau_{A,B}$. The way in which arrows are immersed ends up being immaterial: all that matters is what is connected to what and with what orientation.

For an A-bimodule M, the ev_A and $\mathrm{coev}_{A^{\mathrm{op}}}$ bimodules let one define a *trace* $\mathrm{tr}_A(M)$, which is a k-module:

$$\mathrm{tr}_A(M) =$$



$$= M/(am \sim ma : a \in A \text{ and } m \in M).$$

The trace $\mathrm{tr}_A(A)$ is also known as $A/[A,A]$, where $[A,A]$ is the k-module $\{ab - ba \mid a, b \in A\}$.

A *trace* $f : A \to k$ for a k-algebra A is a k-linear map that factors through $\mathrm{tr}_A(A)$, which is to say $f(ab) = f(ba)$ for all $a, b \in A$. From the point of view of the double commutant theorem for A as an (A,A)-module, a way to state the Artin–Weddernburn theorem is that if A is semisimple, then $A \cong \bigoplus_\lambda V'_\lambda \otimes_{D_\lambda} V_\lambda$ where λ ranges over isomorphism classes of simple A-modules (V_λ being a representative right A-module, $V'_\lambda = \mathrm{Hom}_A(V_\lambda, A)$ the corresponding left A-module, and $D_\lambda = \mathrm{End}_A(V_\lambda)$ a division ring). Each direct summand may be identified with the matrix ring $M_{\dim(V_\lambda)}(D_\lambda)$. With this decomposition, we calculate

$$\mathrm{tr}_A(A) = \bigoplus_\lambda \mathrm{tr}_A(V'_\lambda \otimes_{D_\lambda} V_\lambda) = \bigoplus_\lambda \mathrm{tr}_{D_\lambda}(V_\lambda \otimes_A V'_\lambda) = \bigoplus_\lambda \mathrm{tr}_{D_\lambda}(D_\lambda) = \bigoplus_\lambda D_\lambda.$$

Hence $\mathrm{tr}_A(A)$ may be identified with the center $Z(A)$ when A is semisimple. For example, every trace on $M_n(k)$ must be of the form $m \mapsto c \sum_i m_{ii}$ for some scalar $c \in k$.

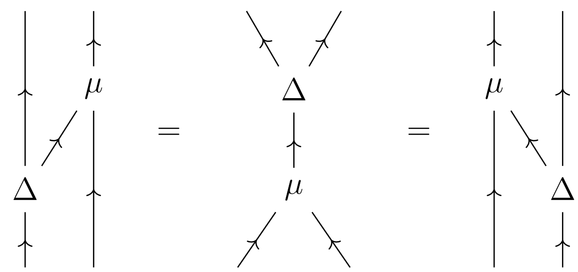
We mention that if A and B are bialgebras, then the coinvariants of an (A,B)-module M come from another trace diagram, using k as a trivial module (whose module structure depends on a counit):

$$= M/(am \sim \epsilon(a)m \text{ and } mb \sim m\epsilon(b) : a \in A,\ b \in B,\ \text{and } m \in M).$$

Frobenius objects

In this section, we give the definition of a Frobenius object in a category. There are some cases for which the $\otimes$ bimodules for a k-module A define a Frobenius object.

Definition 4.1.6. In a monoidal category $\mathcal{C}$ with monoidal unit I, a *Frobenius object* is an object A with both a monoid structure $(A, \eta : I \to A, \mu : A \otimes A \to A)$ and a comonoid structure $(A, \epsilon : A \to I, \Delta : A \to A \otimes A)$ that satisfy the *Frobenius axiom*:



If $\mathcal{C}$ is a symmetric monoidal category, we call the object *symmetric* if $\epsilon \circ \mu\tau = \epsilon \circ \mu$ and *commutative* if $\mu \circ \tau = \mu$. $\Diamond$

There are many consequences to the Frobenius axiom. For example, $\epsilon \circ \mu$ and $\Delta \circ \eta$ are pairings that give A as a self-dual object. Another is that any planar tree-like composition of ϵ, μ, η, and Δ (in that the diagram contains neither loops nor crossings) is equal to any other such composition with the same number of top and bottom arrows. This gives a well-defined family of morphisms $\mu_n^m : A^{\otimes n} \to A^{\otimes m}$ for $n, m \in \mathbb{Z}_{\geq 0}$ that may be recursively defined by the formulas

1. $\mu_0^0 = \epsilon \circ \eta$,
2. $\mu_0^1 = \eta$,
3. $\mu_1^0 = \epsilon$,
4. $\mu_{n+1}^m = \mu_n^m \circ (\mu \otimes \mathrm{id}^{\otimes(n-1)})$ for $n \geq 1$, and
5. $\mu_n^{m+1} = (\Delta \otimes \mathrm{id}^{\otimes(m-1)}) \circ \mu_n^m$ for $m \geq 1$.

In particular, $\mu_2^1 = \mu$ and $\mu_1^2 = \Delta$.

An *invariant pairing* $\sigma : A \otimes A \to I$ for a monoid A is a morphism satisfying $\sigma \circ (\mu \otimes \mathrm{id}_A) = \sigma \circ (\mathrm{id}_A \otimes A)$. It is called nondegenerate if it has an inverse $c : I \to A \otimes A$ satisfying the sort of relation ev and coev satisfy, and in particular these give A as a self-dual object. For a Frobenius object, the composition $\epsilon \circ \mu$ is a nondegenerate invariant pairing.

Proposition 4.1.7. *Suppose $(A, \eta : I \to A, \mu : A \otimes A \to A)$ is a monoid in a monoidal category $\mathcal{C}$. If there exists a nondegenerate invariant pairing $\sigma : A \otimes A \to I$ or if there is a morphism $\epsilon : A \to I$ such that $\sigma = \epsilon \circ \mu$ is a nondegenerate pairing, then A is a Frobenius object with $\epsilon = \sigma \circ (\eta \otimes \mathrm{id}_A)$ and $\Delta = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes c)$, with $c : I \to A \otimes A$ being the inverse for σ.*

Frobenius algebras

A *Frobenius k-algebra* is a Frobenius object in the category of k-modules. A Frobenius algebra is a kind of algebra but with additional structure.

***Warning* 4.1.8.** A Frobenius algebra is an algebra with specific additional structure (the choice of Δ and ϵ). Algebras that admit a Frobenius algebra structure almost never do so in a unique way. $\lozenge$

***Warning* 4.1.9.** The comonoid structure of a Frobenius algebra A does not generally give A a bialgebra structure. Many Frobenius algebras are simultaneously bialgebras (like the group algebra of a finite group), but the comultiplications and counits do not coincide. $\lozenge$

Example 4.1.10. If k is a field, then for every element $c \in k$ there is a commutative Frobenius algebra defined by

$$\begin{aligned} \mu(a \otimes b) &= ab & \qquad i(a) &= a \\ \Delta(a) &= c^{-1} \otimes a & \qquad \epsilon(a) &= ca. \end{aligned}$$

$\lozenge$

Example 4.1.11. The prototypical example of a noncommutative Frobenius algebra is the matrix ring $A = \mathrm{Mat}_{n\times n}(k)$ with k a field. Let $E_{ij} \in A$ denote the matrix with a 1 in the (i, j) entry and zeros elsewhere, and let $\delta_{ij} \in k$ denote the Kronecker delta, which is 1 if $i = j$ and otherwise 0. The Frobenius algebra is defined by

$$\begin{aligned} \mu(E_{ij} \otimes E_{k\ell}) &= \delta_{jk} E_{i\ell} & \qquad i(1) &= \mathrm{id} = \sum_{i=1}^{n} E_{ii} \\ \Delta(E_{ij}) &= \sum_{k=1}^{n} E_{ik} \otimes E_{kj} & \qquad \epsilon(E_{ij}) &= \delta_{ij}. \end{aligned}$$

That is, the multiplication is the standard matrix multiplication $\mu(A \otimes B) = AB$, and the counit is the trace $\epsilon(A) = \operatorname{tr} A$. As the counit is the trace, this Frobenius algebra is commutative. $\lozenge$

Example 4.1.12. A commutative Frobenius algebra that shows up in Khovanov homology is $A = R[x]/(x^2)$ is defined by

$$\begin{aligned} \mu(a \otimes b) &= ab & \qquad i(1) &= 1 \\ \Delta(1) &= 1 \otimes x + x \otimes 1 & \qquad \epsilon(1) &= 0 \\ \Delta(x) &= x \otimes x & \qquad \epsilon(x) &= 1. \end{aligned}$$

This is a non-semisimple algebra, unlike the previous examples. ◊

Since we are discussing Frobenius algebras, we give some results on the structure of Frobenius algebras over a field. Theorem 4.1.29 gives a decomposition of a Frobenius algebra into a semisimple and a "radicular" part using an element, the distinguished element, that is simple to define. We also characterize symmetric Frobenius algebras up to *graphical equivalence*, which is whether Frobenius algebras give the same interpretation to trace diagrams with no inputs or outputs, and in Theorem 4.1.35 we give a classification for symmetric Frobenius algebras over algebraically closed fields up to graphical equivalence.

For the rest of the section, assume that k is a field. We let A be a k-algebra for the following definitions.

For a left or right A-module M, the *socle* $\operatorname{soc}(M)$ is the sum of the simple submodules of M, and it is the largest semisimple submodule. With A as an A-bimodule, if $\operatorname{soc}({}_AA) = \operatorname{soc}(A_A)$, we will write $\operatorname{soc}(A)$ for the socle of A, which is a two-sided ideal in this case. The *Jacobson radical* $\operatorname{rad}(A)$ is the intersection of all the maximal left (or right) ideals of A, which is a two-sided ideal such that the *cosocle* $\operatorname{cosoc}(A) = A/\operatorname{rad}(A)$ is the largest semisimple quotient algebra.

Given a subset $S \subseteq A$, let $\operatorname{ann}^{\mathrm{L}}(S) = \{x \in A : xS = 0\}$ and $\operatorname{ann}^{\mathrm{R}}(S) = \{y \in A : Sy = 0\}$ be the *left* and *right annihilators* of S, respectively, and $\operatorname{ann}^{\mathrm{L}}(S)$ is a left ideal and $\operatorname{ann}^{\mathrm{R}}(S)$ a right ideal. If $I \subseteq A$ is a left ideal, then $\operatorname{ann}^{\mathrm{L}}(I)$ is a right ideal, and if $I \subseteq A$ is a right ideal, then $\operatorname{ann}^{\mathrm{R}}(I)$ is a left ideal.

If A is a Frobenius algebra, let $(-,-)$ denote the nondegenerate bilinear form $\epsilon \circ \mu$. For a subset $S \subseteq A$, let ${}^{\perp}S = \{x \in A : (x, S) = 0\}$ and $S^{\perp} = \{y \in A : (S, y) = 0\}$ and be the *left* and *right orthogonal complements* of S, respectively (where, for example, $(S, y) = \{(x, y) : x \in S\}$). With $V \subseteq A$ a subspace, ${}^{\perp}(V^{\perp}) = V = ({}^{\perp}V)^{\perp}$.

The *Nakayama automorphism* $\Phi : A \to A$ is defined by $(x, y) = (y, \Phi(x))$ for all $x, y \in A$. With $V \subseteq A$ a subspace, $(V^{\perp})^{\perp} = \Phi(V)$. A *symmetric* Frobenius algebra is one whose bilinear form is symmetric, in which case ${}^{\perp}(-) = (-)^{\perp}$ and $\Phi = \mathrm{id}$.

Lemma 4.1.13. *Let A be a Frobenius algebra. If $I \subseteq A$ is a left ideal, then ${}^{\perp}I$ is a right ideal. If $I \subseteq A$ is a right ideal, then $I^{\perp}$ is a left ideal.*

Proof. Suppose $I \subseteq A$ is a left ideal, hence $AI = I$. Then $({}^{\perp}IA)I = {}^{\perp}I(AI) = {}^{\perp}II = 0$, so ${}^{\perp}IA = {}^{\perp}I$, thus ${}^{\perp}I$ is a right ideal. The second statement is similar. □

Lemma 4.1.14 ([Jan59, Lemma 4.1])**.** *Let A be a Frobenius algebra. If $I \subseteq A$ is a left ideal, then $I^{\perp} = \operatorname{ann}^{\mathrm{R}}(I)$ is a right ideal. If $I \subseteq A$ is a right ideal, then ${}^{\perp}I = \operatorname{ann}^{\mathrm{L}}(I)$ is a left ideal.*

Proof. For all $y \in A$, we have the following chain of equivalences:

$$Iy = 0 \quad \Leftrightarrow \quad (A, Iy) = 0 \quad \Leftrightarrow \quad (AI, y) = 0 \quad \Leftrightarrow \quad (I, y) = 0,$$

where the first equivalence is due to nondegeneracy, the second is due to associativity, and the third is due to I being a left ideal. Therefore, $\operatorname{ann}^{\mathrm{R}}(I) = I^{\perp}$. □

Lemma 4.1.15 ([Jan59, Theorem 6.4])**.** *Let A be a Frobenius algebra. Then* $\operatorname{soc}({}_AA) = \operatorname{soc}(A_A)$ *and* $\operatorname{soc}(A) = \operatorname{rad}(A)^\perp = {}^\perp\operatorname{rad}(A)$.

Proof. Right complementation bijectively carries minimal left ideals to maximal right ideals, hence

$$\operatorname{soc}({}_AA)^\perp = \left(\sum_{\mathfrak{m}} \mathfrak{m}\right)^\perp = \bigcap_{\mathfrak{m}} \mathfrak{m}^\perp = \operatorname{rad}(A),$$

where the sum and intersection range over minimal left ideals. Similarly, by considering minimal right ideals, $\operatorname{soc}(A_A)^\perp = \operatorname{rad}(A)$, hence $\operatorname{soc}({}_AA) = \operatorname{soc}(A_A)$. □

Definition 4.1.16. Let $e_1, \ldots, e_n \in A$ be a basis for a Frobenius algebra A, and let $e^1, \ldots, e^n \in A$ be a dual basis, in the sense that $(e_i, e^j) = \delta_i^j$ for all $1 \le i, j \le n$. The *distinguished element* is $\omega = \sum_i e^i e_i \in A$. ◊

Remark 4.1.17. The distinguished element has numerous properties. It is nearly central (Lemma 4.1.19), it is a sort of dual to the trace form (Lemma 4.1.21), and $A\omega^2$ is the largest semisimple subalgebra that is a direct summand (Theorem 4.1.29). ■

Remark 4.1.18. Graphically, $\omega =$  $= (\mu \circ \tau \circ \Delta)(1)$, where $\tau : A^{\otimes 2} \to A^{\otimes 2}$ is the transposition map defined by $x \otimes y \mapsto y \otimes x$. This is because $\Delta(1) = \sum_i e_i \otimes e^i$. One can also check that $\omega = (\mu \circ (\operatorname{id} \otimes \Phi) \circ \Delta)(1)$. Equivalently, ω is either of the two partial traces of the comultiplication.[2]

In [LP], they describe a projector that is ω times a suitable unit with the same diagram. ■

Lemma 4.1.19. *If A is a Frobenius algebra and $a \in A$, then $a\omega = \omega\Phi(a)$. Furthermore, $\Phi(\omega) = \omega$.*

Proof. Using a diagram for $\mu \circ (\operatorname{id} \otimes \omega)$, one drags the id around ω, which yields a twist on its string that is equal to an application of Φ. □

Lemma 4.1.20 (Dickson's criterion)**.** *Let A be a finite-dimensional algebra over a field k of characteristic* 0, *and let* $\operatorname{tr}_A : A \to k$ *denote the trace of the action of A on its left regular representation. Then* $\operatorname{rad}(A) = \operatorname{rad}(\operatorname{tr}_A)$, *where* $\operatorname{rad}(\operatorname{tr}_A) = \{r \in A : \operatorname{tr}_A(rA) = 0\}$ *is the radical of the symmetric bilinear form* $a \otimes b \mapsto \operatorname{tr}_A(ab)$.

Proof. Let $r \in \operatorname{rad}(A)$. Nakayama's lemma implies $\operatorname{rad}(A)$ is a nilpotent ideal, hence for all $s \in A$ the element rs is nilpotent. Similarly, left multiplication by rs is a nilpotent linear operator, thus $\operatorname{tr}_A(rs) = 0$. Hence $\operatorname{rad}(A) \subseteq \operatorname{rad}(\operatorname{tr}_A)$.

[2]Given a linear map $f : U \otimes V \to U \otimes W$ with U a finite-dimensional vector space, the *partial trace* of f with respect to U is a map $V \to W$ defined by $v \mapsto \sum_i \langle u^i, f(u_i, v)_{(1)} \rangle f(u_i, v)_{(2)}$ in sumless Sweedler notation, where $\{u_i\}_i$ is a basis of U with a corresponding dual basis $\{u^i\}_i$ of U^*.

Conversely, let $r \in \operatorname{rad}(\operatorname{tr}_A)$. For all $s \in A$ and $n \geq 1$, $\operatorname{tr}_A((rs)^n) = \operatorname{tr}_A(r(s(rs)^{n-1})) = 0$. Hence left multiplication by rs is nilpotent, so the right ideal rA is a nil ideal, and thus nilpotent since A is artinian. Hence $\operatorname{rad}(\operatorname{tr}_A) \subseteq \operatorname{rad}(A)$. □

Lemma 4.1.21. *Let A be a Frobenius algebra over k. Then the function $(-, \omega) : A \to k$ is equal to $\operatorname{tr}_A(-)$.*

Proof. This is due to the fact that for $x \in A$, (x, ω) is the trace of $(\mu(x, -) \circ (\mathrm{coev} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{ev})$, which is equal to $\mu(x, -)$. □

The following lemma appears in [Abr96] for the special case of A being a commutative Frobenius algebra.

Lemma 4.1.22. *If A is a Frobenius algebra over a field k of characteristic 0 then $\operatorname{soc}(A) = A\omega$.*

Proof. Using Lemma 4.1.21 for the characterization of tr_A, then by Lemma 4.1.20, $\operatorname{rad}(A)$ is $\{r \in A : (rA, \omega) = 0\}$. By associativity, this is $\{r \in A : (r, A\omega) = 0\} = (A\omega)^\perp$. Hence, by Lemma 4.1.15, $\operatorname{soc}(A) = \operatorname{rad}(A)^\perp = A\omega$. □

Lemma 4.1.23. *Let A be a Frobenius algebra and $\mathfrak{m} \subseteq A$ be a minimal left ideal. The following are equivalent: (1) $\omega\mathfrak{m} = 0$, (2) $\mathfrak{m} \subseteq \operatorname{rad}(A)$, and (3) $\mathfrak{m}^2 = 0$.*

Proof. The set $\omega\mathfrak{m}$ is a left ideal since $\omega\mathfrak{m} = \omega A\mathfrak{m} = A\omega\mathfrak{m}$. Then,

$$\omega\mathfrak{m} = 0 \quad \Leftrightarrow \quad A\omega\mathfrak{m} = 0 \quad \Leftrightarrow \quad \mathfrak{m} \subseteq \operatorname{ann}^{\mathrm{R}}(A\omega) = (A\omega)^\perp = \operatorname{rad}(A).$$

Since $\mathfrak{m}$ is a minimal left ideal, $\mathfrak{m} \subseteq \operatorname{soc}({}_AA)$. If $\mathfrak{m} \subseteq \operatorname{rad}(A)$, then $\mathfrak{m}^2 = 0$ since the radical annihilates the socle. Conversely, for every simple A-module M, there is a descending chain of modules $M \supseteq \mathfrak{m}M \supseteq \mathfrak{m}^2M = 0$, hence $\mathfrak{m}M = 0$ by simplicity. Since this is true for all such M, $\mathfrak{m} \subseteq \operatorname{rad}(A)$. □

Corollary 4.1.24. *Let A be a Frobenius algebra and $I \subseteq A$ a minimal two-sided ideal. Then either (1) $\omega I = 0$ and $I \subseteq \operatorname{rad}(A)$ or (2) $\omega I = I$ and $I \cap \operatorname{rad}(A) = 0$.*

Lemma 4.1.25. *For A a Frobenius algebra and $I \subseteq A$ a minimal two-sided ideal with $\omega I = I$, then I is a unital subalgebra of A.*

Proof. Since $\omega I = I$, there is an $e \in I$ such that $\omega e = \omega$. For each $x \in I$, there is an $x' \in I$ such that $x = \omega x'$, so $ex = e\omega x' = \Phi(\omega e)x' = \Phi(\omega)x' = \omega x' = x$. Similarly, for each $x \in I$, there is an $x' \in I$ such that $x = x'\omega$, so $xe = x'\omega e = x'\omega = x$. Therefore, $e \in I$ is a unit. □

Lemma 4.1.26 ([Nak41, Abr96])**.** *Suppose A, A', and A'' are k-algebras with $A = A' \oplus A''$, and let $\epsilon : A \to k$, $\epsilon' : A' \to k$, and $\epsilon'' : A'' \to k$ with $\epsilon = \epsilon' + \epsilon''$. Then ϵ gives A the structure of a Frobenius algebra iff ϵ' and ϵ'' respectively give A' and A'' the structures of Frobenius algebras.*

Definition 4.1.27. A Frobenius algebra A is *radicular* if $\operatorname{soc}(A) \subseteq \operatorname{rad}(A)$. ◊

Remark 4.1.28. In [Hal40], an algebra A is *bound to its radical* (or, is a *bound algebra*) if $\operatorname{ann}^{\mathrm{L}}(\operatorname{rad}(A)) \cap \operatorname{ann}^{\mathrm{R}}(\operatorname{rad}(A)) \subseteq \operatorname{rad}(A)$. In the case of a Frobenius algebra, this is exactly that $\operatorname{soc}(A) \subseteq \operatorname{rad}(A)$. This is equivalent to the definition in [Jan59], where a bound Frobenius algebra is one where every minimal two-sided ideal squares to zero. ■

The following is largely [Jan59, Theorem 6.5], but strengthened to given an explicit splitting, using the distinguished element. One can also view it to be a specialization and refinement of [Hal40, Theorem 2.2], which shows that every k-algebra is uniquely decomposable as a semisimple algebra and a bound algebra.

Theorem 4.1.29. *Let A be a Frobenius algebra. Then $A = \omega^2 A + (\omega^2 A)^\perp$ is an algebra direct sum decomposition, where $\omega^2 A$ is semisimple and $(\omega^2 A)^\perp$ is radicular.*

Proof. The socle decomposes as a direct sum $\operatorname{soc}(A) = \omega^2 A + \operatorname{soc}(A) \cap \operatorname{rad}(A)$, with $\omega^2 A$ being the sum of the minimal two-sided ideals $I \subseteq A$ for which $\omega I = I$.

The intersection $\omega^2 A \cap (\omega^2 A)^\perp$ is a two-sided ideal. Since $(\omega^2 A)^\perp = \operatorname{ann}^{\mathrm{R}}(\omega^2 A)$, any minimal two-sided ideal contained in the intersection squares to zero, but since $\omega^2 A$ is the sum of minimal two-sided ideals that square to themselves, the intersection is trivial, and hence $A = \omega^2 A + (\omega^2 A)^\perp$ is a direct sum as ideals.

Let $e \in \omega^2 A$ be the unit. Then since $1 - e$ annihilates $\omega^2 A$, $1 - e \in (\omega^2 A)^\perp$. Hence $(\omega^2 A)^\perp$ is a unital subalgebra, so the direct sum is as a direct sum of algebras.

The subalgebra $\omega^2 A$ is semisimple since it is a sum of minimal left ideals. The subalgebra $(\omega^2 A)^\perp$ is radicular since $\operatorname{rad}(A) = \operatorname{rad}((\omega^2 A)^\perp)$ and, if there were a minimal left ideal $\mathfrak{m} \subseteq (\omega^2 A)^\perp$, then $\mathfrak{m}^2 = 0$ since $\omega^2 A \mathfrak{m} = 0$. □

Definition 4.1.30. A *graphical element* of A is an element composed out of μ, Δ, ϵ, 1, and transpositions. In other words, graphical elements are elements obtained from Frobenius algebra trace diagrams with zero inputs and one output. ◊

Lemma 4.1.31. *Graphical elements of a symmetric Frobenius algebra A are in the center. For example, the distinguished element $\omega \in A$ is central.*

Definition 4.1.32. Two Frobenius k-algebras A_1 and A_2 are *graphically equivalent* if for every Frobenius algebra trace diagram with no inputs or outputs, interpreting the diagram with respect to A_1 or A_2 yields the same value. Equivalently, it is if for every graphical element $a_1 \in A_1$, then, defining $a_2 \in A_2$ to be the element taking a trace diagram for a_1 and interpreting it to be a trace diagram with respect to A_2, we have $\epsilon_1(a_1) = \epsilon_2(a_2)$. ◊

Lemma 4.1.33. *With $\tau : A^{\otimes 2} \to A^{\otimes 2}$ being the transposition $x \otimes y \mapsto y \otimes x$, the linear map $\Phi = \mu \circ \tau \circ \Delta : A \to A$ commutes with multiplication by graphical elements and has $\varphi(\varphi(x)) = \varphi(x)\omega$ for all $x \in A$.*

Theorem 4.1.34. *If A is a symmetric radicular Frobenius algebra, then it is graphically equivalent to $A' = k[x]/(x^n)$ with $n = \dim \operatorname{rad}(A)$ and any functional $\epsilon' : A' \to k$ with $\epsilon'(1) = \epsilon(1)$ and $\epsilon'(x^{n-1}) = 1$.*

Proof. Given a Frobenius algebra trace diagram G with no inputs or outputs, let $Z(G) \in A$ denote its interpretation. We can put it into normal form where $Z(G) = \epsilon(\omega^a \gamma^b)$ where $a, b \geq 0$ and $\gamma = \mu \circ (\Phi \otimes \mathrm{id}) \circ \Delta \circ 1 \in A$ is the "genus element" (by thinking of a trace diagram as a ribbon graph). If $a > 1$, then $Z(G) = 0$ since $\omega^2 = 0$. If $a = 1$, then $Z(G) = \dim(A)$, and if $a = 0$, then $Z(G) = \epsilon(1)$.

Suppose $g \in A$ is a graphical element, and let $f(x) = \Phi(gx)$ be a linear map $N \to N$. By the previous lemmas, $f^3(x) = g^3\Phi(x)\omega^2 = 0$ for all $x \in N$, hence f is nilpotent and has trace 0. Since $\operatorname{tr}(f) = \epsilon(g\gamma)$, we have that if $b > 0$, $Z(G) = 0$.

Thus, $Z(G)$ depends only on $\dim(A)$ and $\epsilon(1)$. Therefore A is graphically equivalent to $A' = k[x]/(x^n)$ with any $\epsilon' : A' \to k$ satisfying $\epsilon'(1) = \epsilon(1)$ and $\epsilon'(x^{n-1}) = 1$. □

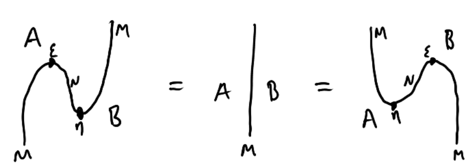
Theorem 4.1.35. *If A is a symmetric Frobenius algebra over an algebraically closed field, then it is graphically equivalent to $\mathrm{Mat}_{n_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{n_m}(k) \oplus k[x]/(x^n)$ for $m \geq 0$, $n_i \geq 1$ for all i, and $n \geq 0$, with some choice of counit. The matrix rings are from the Artin–Wedderburn decomposition of $\operatorname{cosoc}(A)$, and $n = \dim \operatorname{rad}(A)$.*

Proof. By Theorem 4.1.29, $A = A' \oplus N$ with A' semisimple and N radicular. By the Artin–Wedderburn theorem, A is a direct sum of matrix rings over division rings over k; since k is algebraically closed, these division rings are just k. Applying Theorem 4.1.34, N is graphically equivalent to $k[x]/(x^n)$. □

Morita contexts

Suppose A and B are two k-algebras. If A and B are isomorphic in k-Alg_2, then they are said to be *Morita equivalent*. In particular, this is saying that exist bimodules ${}_AM_B$ and ${}_BN_A$ such that ${}_AM \otimes_B N_A$ is isomorphic to ${}_AA_A$ and ${}_BN \otimes_A M_B$ is isomorphic to ${}_BB_B$.

A refinement of this is a *Morita context*, which is an adjunction $({}_AM_B, {}_BN_A, \eta : {}_BB_B \to {}_BN \otimes_A M_B, \epsilon : {}_AM \otimes_B N_A \to {}_AA_A)$ where η and ϵ are invertible. We may depict the equations for an adjunction in the following way using string diagrams, reading bottom-to-top:



Since the morphisms are invertible, then the 180 degree rotation of these equations also holds. Morita contexts are the adjoint 1-equivalences in k-Alg_2. (Note Morita contexts are more general, and we are stating the version in which the Morita theorems apply. See [Lam99, 18C–D]. The present version appears in [SP09] and [HSV16].)

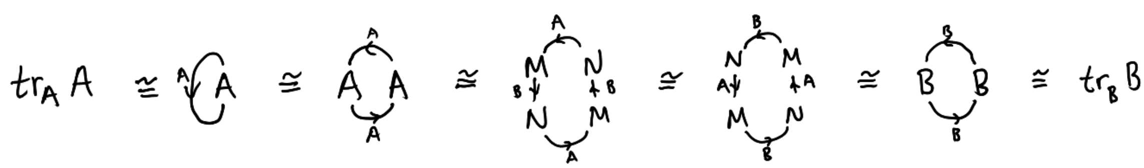
A morphism of Morita contexts $({}_AM_B, {}_BN_A, \eta, \epsilon)$ and $({}_AM'_B, {}_BN'_A, \eta', \epsilon')$ consists of a morphism $f : M \to M'$ of (A, B)-modules and a morphism $g : N \to N'$ of (B, A)-modules such that the following diagrams commute:

$$\begin{tikzcd} & B \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ {}_BN \otimes_A M_B \arrow[rr, "g\otimes f"] & & {}_BN' \otimes_A M'_B \end{tikzcd} \qquad \begin{tikzcd} {}_AM \otimes_B N_A \arrow[rr, "f\otimes g"] \arrow[dr, "\epsilon"'] & & {}_AM' \otimes_B N'_A \arrow[dl, "\epsilon"] \\ & A & \end{tikzcd}$$

Morita contexts between A and B form a category in this way.

We now follow [HSV16] for some facts about Morita contexts between Frobenius algebras.

For A a k-algebra, recall that we defined the trace of an (A, A)-module $\mathrm{tr}_A(A)$, which is also known as $A \otimes_{A\otimes A^{\mathrm{op}}} A$ or $A/[A, A]$. Suppose B is another algebra and $({}_AM_B, {}_BN_A, \eta, \epsilon)$ is a Morita context. Through "proof by rotation" we get a canonical isomorphism $\mathrm{tr}_A A \cong \mathrm{tr}_B B$:



Definition 4.1.36 ([HSV16])**.** Let A and B be two symmetric Frobenius k-algebras and let $({}_AM_B, {}_BN_A, \eta, \epsilon)$ be a Morita context and let $f : \mathrm{tr}_A A \to \mathrm{tr}_B B$ be the canonical isomorphism. Since the Frobenius algebras are symmetric, their counits ϵ_A and ϵ_B factor through $A \to \mathrm{tr}_A A \to k$ and $B \to \mathrm{tr}_B B \to k$. We call the Morita context *compatible with the Frobenius algebras* if the following diagram commutes:

$$\begin{tikzcd} \mathrm{tr}_A A \arrow[rr, "f"] \arrow[dr] & & \mathrm{tr}_B B \arrow[dl] \\ & k & \end{tikzcd}$$

The diagonal arrows are the factored Frobenius algebra counits. $\Diamond$

Note that [SP09] has a second compatibility condition, but [HSV16] points out that this definition is sufficient. The main point is that $\mathrm{tr}_A A$ inherits a Frobenius algebra structure from A, and we wish for the canonical isomorphism to be an isomorphism of Frobenius algebras.

Theorem 4.1.37 ([HSV16])**.** *Let A and B be semisimple symmetric Frobenius algebras over an algebraically closed field k and suppose $({}_AM_B, {}_BN_A, \eta, \epsilon)$ is a Morita context. Let $(S_1, \dots, S_r)$ be the simple modules of A and $(T_1, \dots, T_r)$ of B. After a permutation of the simple modules,*

$$ {}_AM_B \approx \bigoplus_{i=1}^{r} S_i \otimes_k T_i \qquad\qquad {}_BN_A \approx \bigoplus_{i=1}^{r} T_i \otimes_k S_i. $$

Furthermore, if we write $A = \bigoplus_{i=1}^{r} \mathrm{Mat}_{c_i}(k)$ *and* $B = \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ *with respect to the lists of simple modules (using the Artin–Wedderburn theorem) then the Morita context is compatible with the Frobenius algebras if and only if for all* i, $\epsilon_A(\mathrm{id}_{\mathrm{Mat}_{c_i}}) = \epsilon_B(\mathrm{id}_{\mathrm{Mat}_{d_i}})$.

This theorem is implicitly making use of the following lemma, seen in [Koc04] for instance:

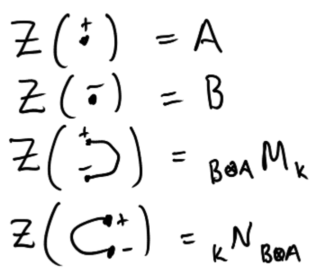
Lemma 4.1.38. *With* $A = \mathrm{Mat}_n(k)$*, then if* $\epsilon : A \to k$ *is a counit giving a symmetric Frobenius algebra structure,* $\epsilon = c\,\mathrm{tr}_{k^n}$ *for some nonzero* $c \in k$.

Proof. Recall that, since tr_{k_n} and ϵ both give Frobenius algebra structures, there is a unit $u \in A$ such that $\epsilon(x) = \mathrm{tr}_{k^n}(xu)$ for all $x \in A$. For ϵ to give a symmetric Frobenius algebra, the Nakayama automorphism must be the identity, which implies u is in the center of A. Therefore, since the center of A is isomorphic to k, we have our result. □

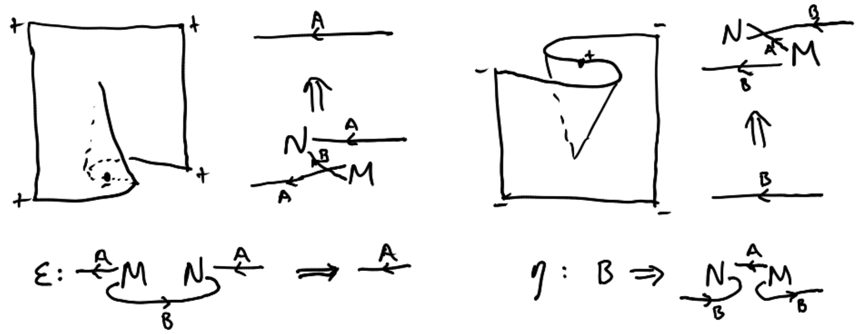
4.1.2 Extended 2D TQFTs

In [SP09], Schommer-Pries classified k-Alg_2-valued TQFTs for the extended cobordism category for oriented 2D surfaces, providing the oriented version of the 2D cobordism hypothesis. We give a simplified version here for the case of an algebraically closed field k. Define Frob to be the symmetric monoidal 2-category whose objects are Frobenius algebras, whose 1-morphisms are Morita contexts compatible with Frobenius algebras, and whose 2-morphisms are isomorphisms of Morita contexts. Then the symmetric monoidal 2-category of oriented 2D TQFTs $\mathrm{Fun}_\otimes(\mathsf{Cob}_2^{\mathrm{or}}, k\text{-}\mathrm{Alg}_2)$ is equivalent to Frob via the weak 2-functor that sends Z to $Z(+)$, where $+$ denotes the positively oriented point.

Here is a brief overview of how some of this works. Suppose Z is such a TQFT. We get some algebras and modules from the 0- and 1-morphisms:

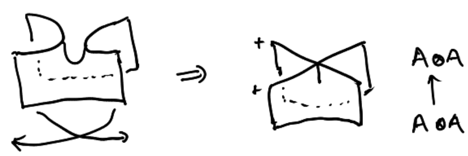


Then from the cusps we can get a Morita context on A and B^{op}:



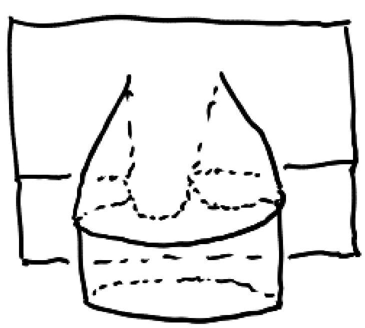
Cusps are certainly invertible, and the swallowtail singularity provides the adjunction relation.

One can observe that the Morse cap provides a function $\mathrm{tr}_A A \to k$ and that the saddle provides a function $s : A \otimes A \to A \otimes A$:



One can check that $A \to \mathrm{tr}_A A \to k$ gives a counit and the function $A \to A \otimes A$ defined by $a \mapsto as(1 \otimes 1)$ defines a comultiplication, giving A the structure of a symmetric Frobenius algebra. A reason the Morita context is compatible with the Frobenius algebras is that one can turn a Morse cap inside-out (this is, in some sense, what the "proof by rotation" from earlier was doing, but for a cylinder).

Furthermore, there is a map from $\mathrm{tr}_A A \otimes A$ to A:



It gives a map from $\mathrm{tr}_A A \to Z(A)$, and it is an isomorphism. With this it is possible to deduce that A is semisimple. By symmetry, B is also a semisimple Frobenius algebra.

Once one works out that a pair of semisimple symmetric Frobenius algebras with a compatible Morita context is sufficient to define a TQFT, then Schommer-Pries uses a careful analysis of the TQFT category and a Whitehead's theorem for symmetric monoidal 2-categories to prove the equivalence.

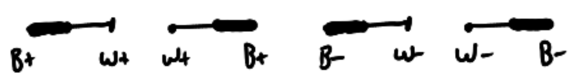
4.1.3 The data of a black-white TQFT

We now give a description of the data that goes into a black-white TQFT. Suppose that Z is an k-Alg_2-valued TQFT for $\mathsf{BW2Cob}$, with k an algebraically closed field. There are four algebras involved associated to the black and white points with positive and negative orientation:

$$\begin{aligned} A_b &= Z(+_b) \qquad & A_w &= Z(+_w) \\ B_b &= Z(-_b) \qquad & B_w &= Z(-_w) \end{aligned}$$

Since the uncolored cobordism category includes into $\mathsf{BW2Cob}$ as both monochromatic black and monochromatic white surfaces, then it follows from Schommer-Pries's theorem that all four algebras have the structure of semisimple symmetric Frobenius algebras, and the pairs (A_b, B_b) and (A_w, B_w) have associated Morita contexts compatible with the Frobenius algebras.

There are also four modules associated to the 1-morphisms for color changes:



These have adjunctions from all the 2-morphisms of the following type:

In the terminology of [Lau], these are ambidextrous adjunctions, and as such their pairwise compositions define four Frobenius algebras — it should be noted that these are Frobenius algebras over one of the four point algebras rather than over k.

We leave the full analysis of black-white TQFTs for future work.

4.2 Surface graph invariants

In this section, we give a way to represent surface graph invariants from combinatorics as extended black-white TQFTs with additional codimension-2 defects to represent edges. (As a side note, explaining this carefully was the original goal of this thesis, but the detour through the land of singularity theory turned into a longer adventure than anticipated!)

We give background on graphs, surface graphs, and their invariants from the combinatorics literature, and in Section 4.2.6 we give an extended TQFT that computes a normalization of the Krushkal polynomial.

4.2.1 Conventions

A *graph* $G = (V, E)$ is a set V of *vertices* and a set E of *edges* with a map $E \to V^{\times 2}/S_2$ sending an edge to the unordered pair of its *endpoints*. That is, our notion of a graph allows loops and multiple edges. Unless otherwise specified, our vertex and edge sets will be finite. An *oriented graph* instead has a map $E \to V^{\times 2}$ assigning an ordered pair of endpoints to each edge.

Each graph has a topological realization as a CW complex. Vertices and edges correspond to 0-cells and 1-cells, respectively, and the map $E \to V^{\times 2}/S_2$ describes the attachment maps for each edge. We will tend not to make a distinction between an abstract graph and its topological realization. We presume we are working in an unspecified stratified smooth category, where an embedding of a graph requires that all pairs of edges incident to a vertex meet transversely.

A *surface graph* is an embedding $G \hookrightarrow \Sigma$ of the topological realization of a graph G into the interior of a compact oriented surface Σ. Two surface graphs $i : G \hookrightarrow \Sigma$ and $i' : G \hookrightarrow \Sigma'$ are equivalent if there is an orientation-preserving diffeomorphism $h : \Sigma \to \Sigma'$ such that $i' = h \circ i$. Isotopies of maps $G \hookrightarrow \Sigma$ extend to equivalences.

A *cellular* or *combinatorial* embedding (or *cellulation*) $G \hookrightarrow \Sigma$ is a surface graph with Σ closed such that the complement of a regular neighborhood of G in Σ is a disjoint union of closed disks, called *faces*.

Up to equivalence, every cellular embedding $G \hookrightarrow \Sigma$ has a well-defined (*geometric* or *Poincaré*) *dual* cellular embedding $G^* \hookrightarrow \Sigma$ obtained from the dual cell structure, where the faces of G correspond to the vertices of G^*, and the edges of G and G^* are in one-to-one correspondence. The surface Σ can be given a handle decomposition where the 0-handles and 2-handles respectively correspond to the vertices of G and G^* and where the 1-handles correspond to the edges of both G and G^*, as illustrated in Figure 4.3.

A *ribbon graph* is a surface graph $i : G \hookrightarrow \Sigma$ such that i is a homotopy equivalence. A ribbon graph can be thought of as a handle decomposition of a surface, all of whose components have nonempty boundary, where the 0-handles and 1-handles correspond to the vertices and edges, respectively. (Recall: a k-handle for a 2-manifold is $D^k \times D^{2-k}$. The $S^0 \times D^1$ boundary of a 1-handle is glued into the $D^0 \times S^1$ boundary of the 0-handles.) Figure 4.4 is an example of a ribbon graph with a handle decomposition.

Given a surface graph $i : G \hookrightarrow \Sigma$, the restriction $G \hookrightarrow \overline{\nu(i(G))}$ to the closure of a regular neighborhood of G gives a ribbon graph. For cellular embeddings, this map has an inverse given by gluing disks into the boundary components of the ribbon graph. Thus, when $G \hookrightarrow \Sigma$ is either a cellular embedding or ribbon graph, we may refer to the *faces* $F(G)$ and *boundary components* ∂G for either case. As a small extension to this, for every surface graph $G \hookrightarrow \Sigma$ we may refer to ∂G for the boundary of a ribbon graph restriction.

Given a subset $A \subseteq E(G)$ of edges for a surface graph $G \hookrightarrow \Sigma$, $G - A$ denotes *edge deletion*, which is the restriction $(V(G), E(G) - A) \hookrightarrow \Sigma$. If $e \in E(G)$ is not a loop, then G/e denotes *edge contraction*, which is from collapsing e and its two endpoints into a single vertex.

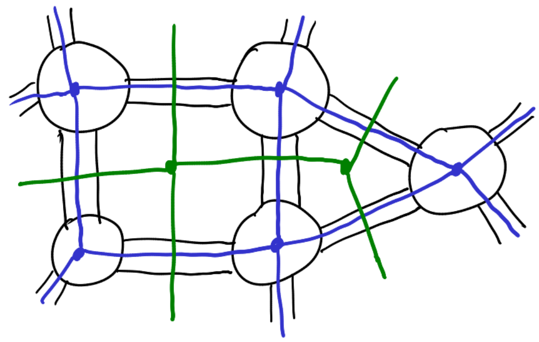


Figure 4.3: A portion of a graph and its dual, with a compatible handle decomposition.

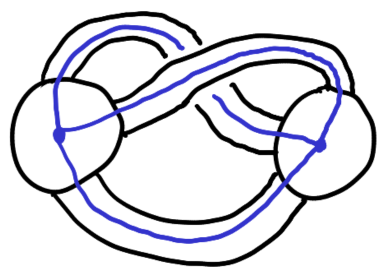


Figure 4.4: A genus-1 ribbon graph whose underlying topological graph is a *theta graph*. Contrary to its portrayal, this is an abstract surface, and the crossing is merely an artifact of its non-planarity.

For G a graph, the ith *Betti number* is $b_i(G) = \operatorname{rank} H_i(G)$, and we will elide parenthesis and write $b_i(V(G), E(G))$ for $b_i((V(G), E(G)))$. Given a surface graph $G \hookrightarrow \Sigma$, the *genus* $g(G)$ is $g(\overline{\nu(i(G))})$, the genus of the corresponding ribbon graph, where for a disconnected surface the genus is the sum of the genera of each component, and for a non-closed surface the genus is the genus of the surface obtained by gluing disks into each boundary component. In general, $g(G) \le g(\Sigma)$ with equality iff $G \hookrightarrow \Sigma$ is a ribbon graph or cellular embedding.

At least for ribbon graphs and combinatorial embeddings, the relationship between the notation commonly found in graph polynomial literature and this notation from algebraic topology is given in Table 4.1. The correspondence comes from linear algebra of the incidence matrix. If A is the oriented incidence matrix for an oriented graph G, with the columns and rows of A corresponding to the edges and vertices, respectively, then

$$0 \to \mathbb{Z}^{|E(G)|} \xrightarrow{A} \mathbb{Z}^{|V(G)|} \to 0$$

is a cellular chain complex for G as a CW complex. Then with $n(A)$ being the nullity of A and

Table 4.1: Correspondence between the combinatorial notation and the topological notation we use, where $G \hookrightarrow \Sigma$ is a ribbon graph or cellular embedding.

Combinatorial	Topological
$v(G)$	$\lvert V(G)\rvert$
$e(G)$	$\lvert E(G)\rvert$
$c(G)$	$b_0(G) = b_0(\Sigma)$
$r(G)$	$\lvert V(G)\rvert - b_0(G)$
$n(G)$	$b_1(G)$
$bc(G)$	$b_0(\partial G) = \lvert F(G)\rvert = b_0(\partial\Sigma)$
$c(G) + n(G) - bc(G)$	$2g(G)$

$r(A)$ being the rank of A, we have $b_1(G) = \operatorname{rank}\ker A = n(A)$ and $b_0(G) = \operatorname{rank}\operatorname{coker} A = |V(G)| - r(A)$. By definition, $n(G) = n(A)$ and $r(G) = r(A)$.

The *complementary genus* $g^\perp(G)$ for $i : G \hookrightarrow \Sigma$ any surface graph is $g(\Sigma - \nu(i(G)))$. This is an abuse of notation, with $g^\perp(G \hookrightarrow \Sigma)$ being more precise. A cellular embedding is characterized by $g^\perp(G) = 0$ with Σ being a closed surface. Similarly, the ith *complementary Betti number* is $b_i^\perp(G) = \operatorname{rank} H_i(\Sigma - \nu(i(G)))$, by a similar abuse of notation.

Some useful identities between all these graph invariants are given in Appendix A.3.

4.2.2 Partial duality

In [Chm09], Chmutov defines general edge contraction for ribbon graphs and cellular embeddings by generalizing the fact for planar graphs that edge contraction corresponds to deleting the edge from the dual graph. The main ingredient is a factorization of the graph dualization operation. For an edge e of a ribbon graph G, the *partial dual* $\delta_e G$ is a ribbon graph with the same edge set, but the handle attachments for the ribbon graph is manipulated like so:

The data for the handle attachments is known as an *arrow presentation*. This operation is an involution in that $\delta_e\delta_e G = G$. For e and f distinct edges, $\delta_e\delta_f G = \delta_f\delta_e G$, and $G^* = \delta_{E(G)} G$, where $\delta_{E(G)}$ is the composition of all δ_e for $e \in E(G)$.

The general edge contraction is defined to be $G/e = \delta_e G - e$, which has the following illustrated effect, depending on whether the edge is a loop:

Given distinct edges e and f,

$$\begin{aligned}(G/e)/f &= \delta_f(\delta_e G - e) - f = \delta_f\delta_e G - e - f \\ &= \delta_e\delta_f G - e - f = (G/f)/e.\end{aligned}$$

Therefore G/A for $A \subseteq E(G)$ is well-defined.

We provide another explanation for Chmutov's edge contraction. The following is a generalization of the notion of a surface graph.

Definition 4.2.1. An *archipelago* on a closed oriented surface Σ is a collection of disjoint *islands*, which are compact connected submanifolds of Σ, and *edges*, which are disjoint paths in the complement of the islands, the endpoints of which intersect the boundaries of the islands transversely. The union of the interiors of the islands is called *land*, the boundaries of the islands are collectively the *coast*, and the complement of the islands is called *sea*. $\Diamond$

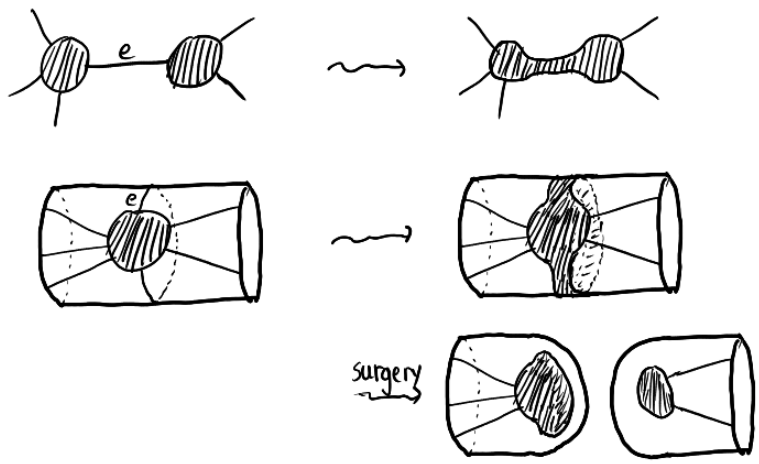
Remark 4.2.2. Surface graphs are in correspondence with archipelagos whose islands are all disks. ■

There are two operations one may perform to an edge e in an archipelago $G \hookrightarrow \Sigma$. The first is edge deletion $G - e \hookrightarrow \Sigma$, and the second is landfill $G/e \hookrightarrow \Sigma$, where a regular neighborhood of the edge is replaced by land. If e is not a loop, then this joins two islands into one island, and if e is a loop, this adds a handle to the island. For example, a disk island becomes an annulus.

Another operation one may apply to an archipelago is surgery in the land or sea, modifying the surface while leaving the coast and edges the same. Any time there is an essential loop in the land or sea disjoint from the edges, 1-surgery (which involves cutting along that loop then gluing in two disks) reduces the genus of the surface. We can also add or remove closed surfaces that are entirely land or sea. An archipelago from a surface graph, when taken up to 1-surgery in the sea and up to removal of sea spheres, has a representative that is a cellular embedding. The equivalence class for a surface graph is known as a *virtual graph* in [MSM20].

Consider archipelagos up to 1-surgery in the land and up to removal of land spheres. Then there is a representative archipelago that comes from a surface graph. Given a ribbon graph G with a cellular embedding $G \hookrightarrow \Sigma$, edge contraction G/e for $e \in E(G)$ is from taking

$G/e \hookrightarrow \Sigma$ up to 1-surgery in the land. G/e is the ribbon graph from the representative that is a surface graph, as illustrated below:



Since $(G/e)/f = (G/f)/e$ as archipelagos, we have another proof that $(G/e)/f = (G/f)/e$ for ribbon graphs.

4.2.3 The Krushkal polynomial and other graph polynomials

Many interesting ring-valued graph invariants f satisfy a *deletion-contraction relation*, which for fixed constants a and b is a linear relation of the form

$$f(G) = af(G-e) + bf(G/e)$$

that holds for all graphs G and edges e of a particular type, such as non-loop or non-bridge edges. Examples include the chromatic polynomial [Bir13, Whi32], the flow polynomial [Tut47], the partition function of the Q-states Potts model in statistical physics [Wu82], and the Jones polynomial of an alternating knot from its Tait graph [Thi87]. Tutte initiated the study of graph invariants satisfying deletion-contraction relations [Tut47, Tut54], and the *Tutte polynomial* $T_G(x,y)$ for a graph $G = (V,E)$ is the "universal" polynomial invariant that satisfies, for each edge $e \in E$,

- if e is a bridge, $T_G(x,y) = xT_{G/e}(x,y)$; and
- if e is neither a loop nor bridge, $T_G = T_{G-e} + T_{G/e}$.

With the normalization that $T_G(x,y) = y^i$ if all i edges of G are loops, then it can be given as a sum over all subsets of edges of $G = (V,E)$:

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{b_0(V\cup A)-b_0(G)}(y-1)^{b_1(V\cup A)}.$$

When $G \hookrightarrow \Sigma$ is a planar cellular embedding, $T_G(x,y) = T_{G^*}(y,x)$.

The *Bollobás–Riordan polynomial* $BR_G(x,y,z)$ for an (orientable) ribbon graph $G = (V,E)$ is the most-general polynomial invariant that satisfies the same two relations as the Tutte polynomial but for ribbon graphs [BR01, BR02]. It, too, can be given as a sum:

$$\begin{aligned} BR_G(x,y,z) &= \sum_{A \subseteq E} (x-1)^{b_0(V\cup A)-b_0(G)} y^{b_1(V\cup A)} z^{b_0(V\cup A)+b_1(V\cup A)-b_0(\partial(V\cup A))} \\ &= \sum_{A \subseteq E} (x-1)^{b_0(V\cup A)-b_0(G)} y^{b_1(V\cup A)} z^{2g(V\cup A)}. \end{aligned}$$

Notably, $BR_G(x,y,1) = T_G(x,y)$.

The *Krushkal polynomial* $P_{G\hookrightarrow\Sigma}(x,y,a,b)$ is an invariant of surface graphs $G \hookrightarrow \Sigma$ that extends the Bollobás–Riordan polynomial to have a duality relation [Kru11], and it is defined by the sum

$$P_{G\hookrightarrow\Sigma}(x,y,a,b) = \sum_{A\subseteq E} x^{b_0(V\cup A)-b_0(G)} y^{k(V\cup A)} a^{g(V\cup A)} b^{g^\perp(V\cup A)},$$

where for a surface graph $H \hookrightarrow \Sigma$ there is a homological invariant defined by

$$k(H) = \dim(\ker(i_* : H_1(H;\mathbb{R}) \to H_1(\Sigma;\mathbb{R}))),$$

which satisfies $k(H) = b_1(H) - g(\Sigma) - g(H) + g^\perp(H)$ [Kru11, 4.7]. Given a cellular embedding $G \hookrightarrow \Sigma$, there is a duality relation

$$P_G(x,y,a,b) = P_{G^*}(y,x,b,a).$$

The relationship to the Bollobás–Riordan polynomial is that for a ribbon graph or cellular embedding $G \hookrightarrow \Sigma$,

$$BR_G(x,y,z) = y^{g(\Sigma)} P_{G\hookrightarrow\Sigma}(x-1,y,yz^2,y^{-1}).$$

Furthermore, $T_G(x,y) = y^{g(\Sigma)} P_{G\hookrightarrow\Sigma}(x,y,y,y^{-1})$. Using the identities from Proposition A.3.3,

$$P_{G\hookrightarrow\Sigma}(x,y,a,b) = x^{-b_0(G)} y^{-b_0(\Sigma)} \sum_{A\subseteq E} x^{b_0(V\cup A)} y^{b_0^\perp(V\cup A)} a^{g(V\cup A)} b^{g^\perp(V\cup A)}.$$

In this form, the duality relation for a cellular embedding is obvious.

Substituting out genus using those identities, this is

$$\begin{aligned} P_{G\hookrightarrow\Sigma}(x,y,a,b) = {} & x^{-b_0(G)} y^{-b_0(\Sigma)} (a^{-1}b)^{\frac{1}{2}|V|} b^{g(\Sigma)-b_0(\Sigma)} \\ & \sum_{A\subseteq E} (xa)^{b_0(V\cup A)} (yb)^{b_0^\perp(V\cup A)} (ab^{-1})^{\frac{1}{2}|A|} (ab)^{-\frac{1}{2}b_0(\partial(V\cup A))}. \end{aligned}$$

We use the following as our version of the Krushkal polynomial:

Proposition 4.2.3. *For $G \hookrightarrow \Sigma$ a surface graph with Σ a closed surface, the Krushkal polynomial is a renormalization of*

$$P'_{G\hookrightarrow\Sigma}(x,a,b,c) := \sum_{A\subseteq E} x^{|A|} a^{b_0(V\cup A)} b^{b_0^{\perp}(V\cup A)} c^{b_0(\partial(V\cup A))} \tag{4.2.1}$$

in the sense that

$$P_{G\hookrightarrow\Sigma}(x,y,a,b) = x^{-b_0(G)} y^{-b_0(\Sigma)} (a^{-1}b)^{\frac{1}{2}|V|} b^{g(\Sigma)-b_0(\Sigma)} P'_{G\hookrightarrow\Sigma}((ab^{-1})^{1/2}, xa, yb, (ab)^{-1/2}).$$

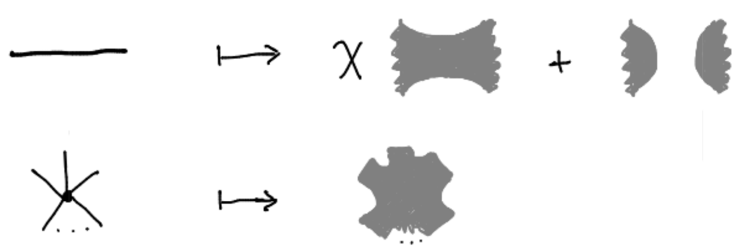
4.2.4 An invariant from a TQFT

Let $G \hookrightarrow \Sigma$ be a surface graph with Σ a closed surface and $G = (V,E)$. Given a TQFT $Z : \mathsf{BW2Cob} \to k\text{-Mod}$ (or an extended TQFT $Z : \mathsf{BW2Cob} \to k\text{-Alg}_2$), we can construct a graph invariant with values in k that has a "universal deletion-contraction rule." If Z is an extended TQFT, the invariant takes graphs-with-boundary and outputs bimodule intertwiners.

By $Z(\Sigma, G)$ we will mean the value $Z((\Sigma, \nu(G))) \in k$ for $(\Sigma, \nu(G)) \in \mathrm{Mor}(\varnothing, \varnothing)$, which is from interpreting the given graph as the black region. Define

$$P^Z_{G\hookrightarrow\Sigma}(x) = \sum_{A\subseteq E} x^{|A|} Z(\Sigma, V\cup A),$$

which comes from taking the following linear combination of elements along each edge of G:



More precisely, the edge expansion is an element of $k\,\mathrm{Mor}(\varnothing, X_2)$ which is plugged into an element of $\mathrm{Mor}(X_2^{\amalg|E|}, \varnothing)$ that comes from the rest of the graph.

Remark 4.2.4. This has a close connection to the medial construction (described in [EMM13] for surface graphs). The complement in Σ of the union of a graph and its dual is a collection of quadrilateral faces, two vertices of each coming from G and G^*. By taking the vertices that do not come from either G or G^* then taking edges through each quadrilateral that connect them, we get a 4-regular graph called the *medial graph*. The medial graph's faces can be checkerboard colored: faces containing a G vertex are colored black and faces containing a G^* vertex are colored white. This is gives a "singular" morphism. The singularities may be resolved in two ways to get an actual morphism, corresponding to either including or excluding the edge from the set A. See Figure 4.5 for an illustration. ■

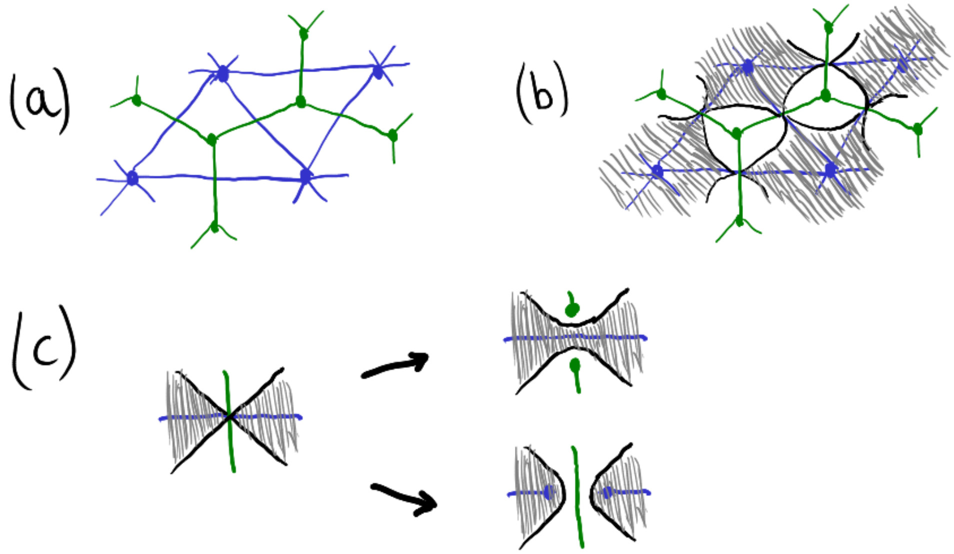


Figure 4.5: (a) A local picture of a graph and its dual. (b) The medial construction, giving a "singular morphism." (c) The two ways to resolve the singularities so that the resulting graphs are the black and white graphs for the resulting morphism.

The edge expansion should be thought of as being a projective linear combination that we have immediately renormalized to eliminate an unnecessary variable. This particular normalization is also analogous to the partition function of the Q-states Potts model.

Remark 4.2.5. Notice that this polynomial immediately generalizes to archipelagos (Definition 4.2.1). For an archipelago $G \hookrightarrow \Sigma$, let $\operatorname{land}(G)$ be the land. Then we may define

$$P^Z_{G\hookrightarrow\Sigma}(x) = \sum_{A\subseteq E} x^{|A|} Z(\Sigma, \operatorname{land}(G/A)),$$

which is a sum over all choices of landfilling or deleting edges from the archipelago. ■

Proposition 4.2.6. *The polynomial P^Z satisfies, for $e \in E$ a non-loop edge,*

$$P^Z_{G\hookrightarrow\Sigma}(x) = xP^Z_{G/e\hookrightarrow\Sigma}(x) + P^Z_{G-e\hookrightarrow\Sigma}(x).$$

If $G_1 \hookrightarrow \Sigma_1$ and $G_2 \hookrightarrow \Sigma_2$ are surface graphs, then

$$P^Z_{G_1\amalg G_2\hookrightarrow\Sigma_1\amalg\Sigma_2}(x) = P^Z_{G_1\hookrightarrow\Sigma_1}(x)P^Z_{G_2\hookrightarrow\Sigma_2}(x).$$

And for the empty surface graph, $P^Z_{\varnothing\hookrightarrow\varnothing}(x) = 1$.

Proposition 4.2.7. *Letting $Z' : \mathsf{BW2Cob} \to k\text{-Mod}$ be the TQFT coming from reversing the roles of the black and white regions, there is a duality relation for cellular embeddings $G \hookrightarrow \Sigma$ given by*

$$P^Z_{G\hookrightarrow\Sigma}(x) = x^{|E|}P^{Z'}_{G^*\hookrightarrow\Sigma}(x^{-1}).$$

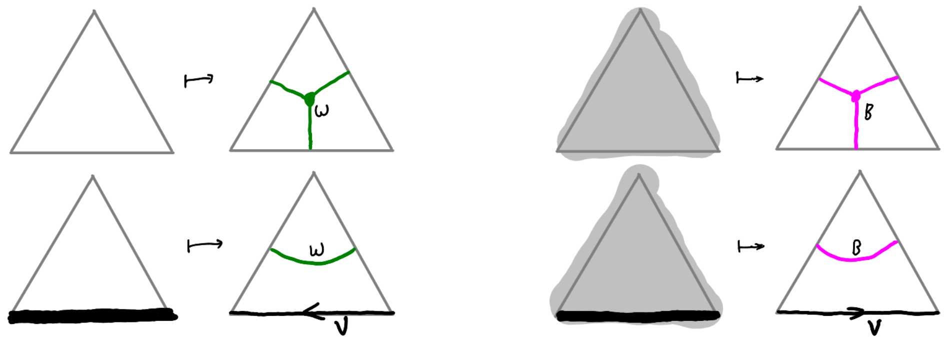


Figure 4.6: Replacements for black and white triangles. Triangles without a boundary edge receive a (co)multiplication for the corresponding Frobenius algebra, triangles with boundary edge get two strands, one for the algebra and one for the boundary vector space

4.2.5 The Krushkal polynomial via a trace diagram

We can create a classic state sum TQFT (as in [FHK] or [LP]) to calculate the Krushkal polynomial.

Let Q_B, Q_W, and Q_∂ be non-negative natural numbers. We give the field k a Frobenius algebra structure with $\epsilon(1) = 1$. Then, let B be the Frobenius algebra k^{Q_B} and W be the Frobenius algebra k^{Q_W}, both as direct sums of the Frobenius algebra k. Let V be a vector space of dimension Q_∂.

The B Frobenius algebra has the property that trace diagrams D with no boundary strands evaluate to $Q_B^{b_0(D)}$, where $b_0(D)$ is the number of connected components in D when thought of as a graph. Similarly for W (with $Q_W^{b_0(D)}$) and V (with $Q_\partial^{b_0(D)}$). In particular, the Frobenius algebras B and W are *special*, that $\mu \circ \Delta = \mathrm{id}$, which implies that in any trace diagram we can delete non-bridge edges without changing the value — hence, the diagrams reduce to trees, and using the identity axiom to contract edges one is left with a disjoint union of vertices, each of which evaluates to the dimension of the Frobenius algebra due to the choice of counit.

Given a black-white surface Σ, there exists a triangulation of it for which the boundary of the black region is contained in the 1-skeleton and every triangle has at most one edge from the boundary. If Σ has boundary components, then we can arrange for there to be no contiguous edges of the same color. Then, using the replacements in Figure 4.7, we construct a trace diagram associated to the triangulation. Note:

- The Frobenius algebras B and W do not depend in any way on the orientations, so the trace diagram makes sense no matter how the pieces are connected up. An equivalent construction here is to associate to each edge of the triangulation either B, W, or V depending on whether it is an edge in the black region, an edge in the white region, or an edge on the boundary of the regions, then for each face we associate a Kronecker delta with indices for each edge of the faces color, and finally for each vertex on the boundary we associate a Kronecker delta with indices for the two incident edges.

- The black region induces an orientation on the boundary, which is why the boundary vector spaces do not need an inner product for contraction — we can use the dual pairing.
- The choice of triangulation does not matter since the Frobenius algebras are special and since subdividing a boundary edge does not change what the V part of the trace diagram evaluates to.
- Each boundary component yields a vector space of the form B, W, or $(B \otimes V \otimes W \otimes V^*)^{\otimes n}$, where n is the number of incident black regions.

Hence, we may define $Z : \mathsf{BW2Cob} \to k$-Mod to be the state sum TQFT constructed in this way.

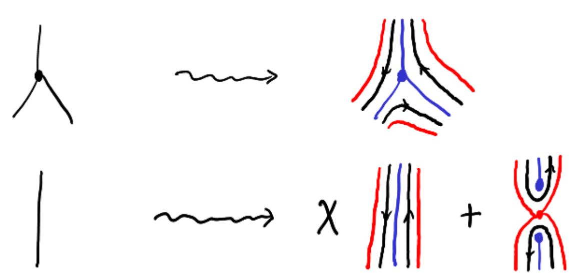
These replacements give

$$P^Z_{G\hookrightarrow\Sigma}(x) = \sum_{A\subseteq E(G)} x^{|A|} Q_B^{b_0(V\cup A)} Q_W^{b_0^\perp(V\cup A)} Q_\partial^{b_0(\partial(V\cup A))} = P'_{G\hookrightarrow\Sigma}(x, Q_B, Q_W, Q_\partial),$$

which is the version of the Krushkal polynomial in Proposition 4.2.3.

Remark 4.2.8. The TQFT extends the Krushkal polynomial to archipelagos. ■

Remark 4.2.9. The P^Z polynomial also can be constructed for ribbon graphs by cellularly embedding them. Because the white region is a collection of disks, we can be more economical with our replacement scheme and create a trace diagram in the neighborhood of the graph itself. In the following, the black curves correspond to V, the blue graphs to B, and the red graphs to W:



Alternatively, we could place a red vertex in the center of each complementary cell in the cellular embedding and run edges through the opening is the second term of the edge expansion. With this version we can see the duality relation for the Krushkal polynomial in a very direct way. ■

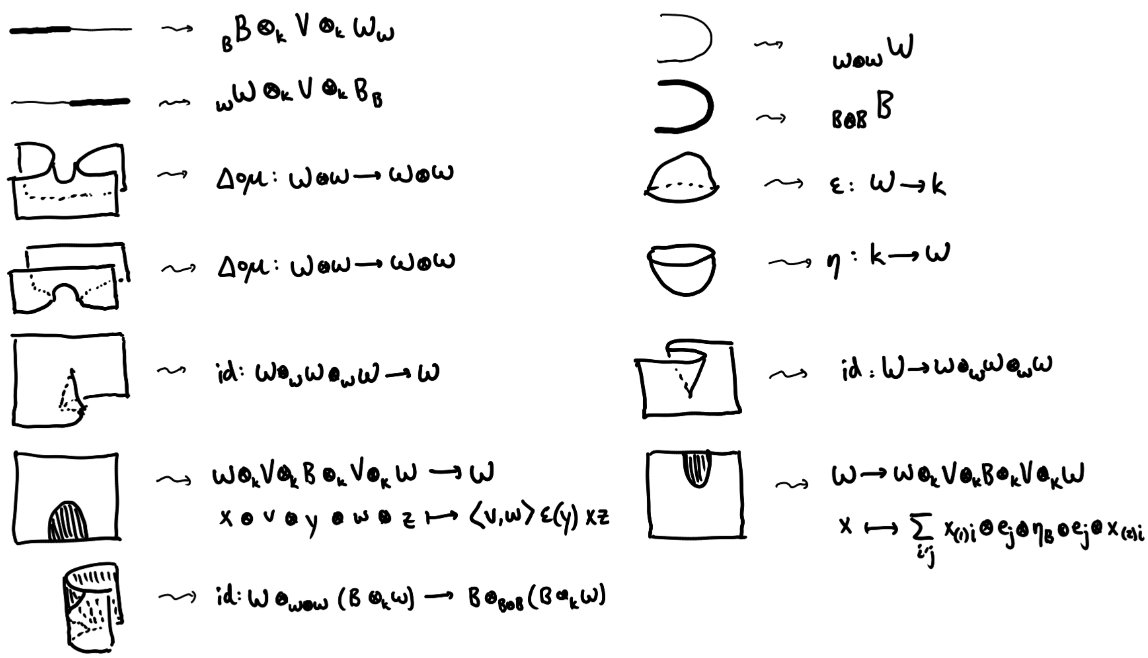


Figure 4.7: The values of the TQFT Z for which P^Z gives our normalization of the Krushkal polynomial from Proposition 4.2.3. Morphisms labeled "id" stand for the obvious isomorphism. The missing generators from this list are all obtained from swapping the roles of black and white.

4.2.6 An extended TQFT for the Krushkal polynomial

In this final section, we categorify the construction from Section 4.2.5 to show that there exists some extended 2D TQFT that computes the Krushkal polynomial. As a quick remark about what it means for the TQFT to compute the polynomial. In principle, one could define a version of $\mathsf{BW2Cob}$ with 0-dimensional singularities between the black and white regions, like in Figure 4.5, and then one can associate to this singularity a morphism that expands to the P^Z state sum that we have been working with.

Let B and W be the Frobenius algebras from the previous section of respective dimensions Q_B and Q_W, and let V be a vector space of dimension Q_∂. For simplicity in presentation, assume that V is an inner product space, and let $e_1, \ldots, e_{Q_\partial}$ denote an orthogonal basis. Recall *Sweedler notation*, where if $x \in B$, we write $\Delta(x) = \sum_i x_{(1)i} \otimes x_{(2)i}$.

Both B and W are commutative Frobenius algebras, so they are equal to their opposites. We will construct a TQFT for *unoriented* surfaces since the construction does not depend on orientation, and then that way we can assign the same value to generators that are different only in orientation.

With that said, define an TQFT $Z : \mathsf{BW2Cob} \to k$-Alg according to Figure 4.7. Checking that these satisfy the relations is mostly trivial. First, we have defined this using a pair of semisimple symmetric Frobenius algebras, and so from the Schommer-Pries classification we only need to check relations involving the boundary. Going through relations in Figure 4.2:

- The "adjoint cancelation" corresponds to absorbing a counit for B, a unit for A, and canceling the coeval/eval for V.
- The "boundary cusp slide" and "boundary fold slide" are trivial.
- The "boundary saddle slide" is just like the adjoint cancelation, but without needing to change anything for V.
- The "boundary cup/cap slide" is similar, but the cup/cap is also a unit/counit.

Another argument we could make is that since we are tensoring the boundary in over k, we can cut the surfaces open along the boundary and homotope each region to the boundary without changing the assigned intertwiner.

Thus, we have a TQFT. With a little thought, we can see that on closed surfaces it computes the same value as the construction from Section 4.2.5. A way to see it is using the above observation about cutting the surface open along the boundary — the boundary itself contributes a factor of Q_∂ per component, and for a monochromatic component we can puncture it without changing the value, and by puncturing it in the center of each vertex of a triangulation we can view the component as being a trace diagram for the corresponding Frobenius algebra.

Bibliography

[Abr96] Lowell Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (1996), no. 5, 569–587. DOI:10.1142/S0218216596000333, MR 1414088

[AGZV12] V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Volume 1*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012, Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition. MR 2896292

[Arn76] V. I. Arnold, *Wave front evolution and equivariant Morse lemma*, Comm. Pure Appl. Math. **29** (1976), no. 6, 557–582. DOI:10.1002/cpa.3160290603, MR 436200

[Ber08] Pierre Berger, *Structural stability of attractor-repellor endomorphisms with singularities*. arXiv:0809.0277

[Bir13] George D. Birkhoff, *A determinant formula for the number of ways of coloring a map*, Ann. of Math. (2) **14** (1912/13), no. 1-4, 42–46. DOI:10.2307/1967597, MR 1502436, https://doi.org/10.2307/1967597

[BL98] John C. Baez and Laurel Langford, *Higher-dimensional algebra iv: 2-tangles*, Adv. Math. 180 (2003), 705-764. (1998). arXiv:math/9811139

[Boa67] J. M. Boardman, *Singularities of differentiable maps*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 33, 21–57. MR 231390, http://www.numdam.org/item?id=PMIHES_1967__33__21_0

[Bou98] Nicolas Bourbaki, *General topology. Chapters 1–4*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR 1726779

[BR01] Béla Bollobás and Oliver Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 513–531. DOI:10.1112/plms/83.3.513, MR 1851080

[BR02] ———, *A polynomial of graphs on surfaces*, Mathematische Annalen **323** (2002), no. 1, 81–96. DOI:10.1007/s002080100297, MR 1906909

[Buc77] Michael A. Buchner, *Stability of the cut locus in dimensions less than or equal to* 6, Invent. Math. **43** (1977), no. 3, 199–231. DOI:10.1007/BF01390080, MR 482816

[Cer70] Jean Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 39, 5–173. MR 292089, http://www.numdam.org/item?id=PMIHES_1970__39__5_0

[Chm09] Sergei Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, J. Combin. Theory Ser. B **99** (2009), no. 3, 617–638. arXiv:0711.3490v3 [math.CO], MR 2507944

[CRS97] J. Scott Carter, Joachim H. Rieger, and Masahico Saito, *A combinatorial description of knotted surfaces and their isotopies*, Adv. Math. **127** (1997), no. 1, 1–51. DOI:10.1006/aima.1997.1618, MR 1445361

[Dam84] James Damon, *The unfolding and determinacy theorems for subgroups of A and K*, Mem. Amer. Math. Soc. **50** (1984), no. 306, x+88. DOI:10.1090/memo/0306, MR 748971

[Dor18] Christoph Dorn, *Associative n-categories*. arXiv:1812.10586

[Duf75] Jean-Paul Dufour, *Déploiements de cascades d'applications différentiables*, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), no. 1, Aii, A31–A34. MR 385916

[Duf77] ———, *Sur la stabilité des diagrammes d'applications différentiables*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 153–174. MR 448395, http://www.numdam.org/item?id=ASENS_1977_4_10_2_153_0

[Duf83] J.-P. Dufour, *Familles de courbes planes différentiables*, Topology **22** (1983), no. 4, 449–474. DOI:10.1016/0040-9383(83)90037-X, MR 715250

[EMM13] Joanna A. Ellis-Monaghan and Iain Moffatt, *A Penrose polynomial for embedded graphs*, European Journal of Combinatorics **34** (2013), no. 2, 424–445. DOI:10.1016/j.ejc.2012.06.009, MR 2994409

[FHK] M. Fukuma, S. Hosono, and H. Kawai, *Lattice topological field theory in two dimensions*. DOI:10.1007/BF02099416, arXiv:hep-th/9212154v1 [hep-th]

[FNWW] M. Freedman, C. Nayak, K. Walker, and Z. Wang, *On picture (2+1)-tqfts*. arXiv:0806.1926v2 [math.QA]

[GG73] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York-Heidelberg, 1973, Graduate Texts in Mathematics, Vol. 14. MR 0341518

[Hal40] Marshall Hall, *The position of the radical in an algebra*, Trans. Amer. Math. Soc. **48** (1940), 391–404. DOI:10.2307/1990089, MR 2855, https://doi.org/10.2307/1990089

[Hir76] Morris W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, No. 33, Springer-Verlag, New York-Heidelberg, 1976. MR 0448362

[HSV16] Jan Hesse, Christoph Schweigert, and Alessandro Valentino, *Frobenius algebras and homotopy fixed points of group actions on bicategories*, Theory Appl. Categ. 32 (2017), No. 18, p. 652-681 (2016). arXiv:1607.05148

[Isa02] Daniel C. Isaksen, *Calculating limits and colimits in pro-categories*, Fund. Math. **175** (2002), no. 2, 175–194. DOI:10.4064/fm175-2-7, arXiv:math/0106094, MR 1969635

[J̈68] Klaus Jänich, *On the classification of $O(n)$-manifolds*, Math. Ann. **176** (1968), 53–76. DOI:10.1007/BF02052956, MR 226674

[Jan59] J. P. Jans, *On Frobenius algebras*, Ann. of Math. (2) **69** (1959), 392–407. DOI:10.2307/1970189, MR 104711, https://doi.org/10.2307/1970189

[Jon] Vaughan F. R. Jones, *Planar algebras, I.* arXiv:math/9909027v1 [math.QA]

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301

[KÏ0] Balázs Kőműves, *On computing Thom polynomials*, Ph.D. thesis, Central European University, 2010.

[KMS93] Ivan Kolář, Peter W. Michor, and Jan Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993. DOI:10.1007/978-3-662-02950-3, MR 1202431

[Koc04] Joachim Kock, *Frobenius algebras and 2D topological quantum field theories*, London Mathematical Society Student Texts, vol. 59, Cambridge University Press, Cambridge, 2004. MR 2037238

[KPS] Vijay Kodiyalam, Vishwambhar Pati, and V. S. Sunder, *Subfactors and 1+1-dimensional tqfts*. DOI:10.1142/S0129167X07003923, arXiv:math/0507050v1 [math.QA]

[Kru11] Vyacheslav Krushkal, *Graphs, links, and duality on surfaces*, Combin. Probab. Comput. **20** (2011), 267–287. arXiv:0903.5312v3 [math.CO], MR 2769192

[Lam99] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. DOI:10.1007/978-1-4612-0525-8, MR 1653294

[Lau] Aaron D. Lauda, *Frobenius algebras and ambidextrous adjunctions*. arXiv:math/0502550v2 [math.CT]

[Lau00] Gerd Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5667–5688. DOI:10.1090/S0002-9947-00-02676-3, MR 1781277

[Lee13] John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013. MR 2954043

[Lei03] Tom Leinster, *Higher operads, higher categories*. arXiv:math/0305049

[LP] Aaron D. Lauda and Hendryk Pfeiffer, *State sum construction of two-dimensional open-closed topological quantum field theories*. DOI:10.1142/S0218216507005725, arXiv:math/0602047v2 [math.QA]

[LP08] ———, *Open-closed strings: two-dimensional extended TQFTs and Frobenius algebras*, Topology Appl. **155** (2008), no. 7, 623–666. DOI:10.1016/j.topol.2007.11.005, arXiv:math/0510664v3 [math.AT], MR 2395583, https://doi.org/10.1016/j.topol.2007.11.005

[Lur18] Jacob Lurie, *Ultracategories*, October 2018. https://www.math.ias.edu/~lurie/papers/Conceptual.pdf

[Mat68] John N. Mather, *Stability of C^∞ mappings. III. Finitely determined mapgerms*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 279–308. MR 275459, http://www.numdam.org/item?id=PMIHES_1968__35__279_0

[Mat69a] ———, *Stability of C^∞ mappings. II. Infinitesimal stability implies stability*, Ann. of Math. (2) **89** (1969), 254–291. DOI:10.2307/1970668, MR 259953

[Mat69b] ———, *Stability of C^∞ mappings. IV. Classification of stable germs by R-algebras*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 37, 223–248. MR 275460, http://www.numdam.org/item?id=PMIHES_1969__37__223_0

[Mat71] J. N. Mather, *Stability of C^∞ mappings. VI: The nice dimensions*, Proceedings of Liverpool Singularities-Symposium, I (1969/70), Springer, Berlin, 1971, pp. 207–253. Lecture Notes in Math., Vol. 192. MR 0293670

[Mat73] John N. Mather, *On Thom-Boardman singularities*, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), 1973, pp. 233–248. MR 0353359

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872

[Mor65] Bernard Morin, *Formes canoniques des singularités d'une application différentiable*, C. R. Acad. Sci. Paris **260** (1965), 5662–5665. MR 180982

[Mor07] Jeffrey Morton, *Extended tqft's and quantum gravity*. arXiv:0710.0032

[MS] Gregory W. Moore and Graeme Segal, *D-branes and k-theory in 2d topological field theory*. arXiv:hep-th/0609042v1 [hep-th]

[MSM20] Calvin McPhail-Snyder and Kyle A. Miller, *Planar diagrams for local invariants of graphs in surfaces*, J. Knot Theory Ramifications **29** (2020), no. 1, 1950093, 49. DOI:10.1142/S0218216519500937, arXiv:1805.00575v2 [math.GT], MR 4079619

[MW12] Scott Morrison and Kevin Walker, *The blob complex*, Geom. Topol. **16** (2012), 1481–1607. DOI:10.2140/gt.2012.16.1481, arXiv:1009.5025v3 [math.AT]

[Nak41] Tadasi Nakayama, *On Frobeniusean algebras. II*, Ann. of Math. (2) **42** (1941), no. 1, 1–21. DOI:10.2307/1968984, MR 4237, https://doi.org/10.2307/1968984

[Nak89] Isao Nakai, *Topological stability theorem for composite mappings*, Ann. Inst. Fourier (Grenoble) **39** (1989), no. 2, 459–500. MR 1017287, http://www.numdam.org/item?id=AIF_1989__39_2_459_0

[Nes20] Jet Nestruev, *Smooth manifolds and observables*, Graduate Texts in Mathematics, vol. 220, Springer, Cham, 2020, Second edition. DOI:10.1007/978-3-030-45650-4, MR 4221224

[Poé74] Valentin Poénaru, *Analyse différentielle*, Lecture Notes in Mathematics, Vol. 371, Springer-Verlag, Berlin-New York, 1974. MR 0353360

[Por71] I. R. Porteous, *Simple singularities of maps*, Proceedings of Liverpool Singularities Symposium, I (1969/70), 1971, pp. 286–307. Lecture Notes in Math., Vol. 192. MR 0293646

[Por72] Ian R. Porteous, *The second-order decomposition of* Σ^2, Topology **11** (1972), 325–334. DOI:10.1016/0040-9383(72)90028-6, MR 312521

[Por83] ———, *Probing singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R.I., 1983, pp. 395–406. MR 713263

[Ros98] Dennis Roseman, *Reidemeister-type moves for surfaces in four-dimensional space*, Knot theory (Warsaw, 1995), Banach Center Publ., vol. 42, Polish Acad. Sci. Inst. Math., Warsaw, 1998, pp. 347–380. MR 1634466

[Sel] Peter Selinger, *A survey of graphical languages for monoidal categories*. DOI:10.1007/978-3-642-12821-9_4, arXiv:0908.3347v1 [math.CT]

[SP09] Christopher J. Schommer-Pries, *The classification of two-dimensional extended topological field theories*, Ph.D. thesis, 2009. arXiv:1112.1000v2 [math.AT]

[Sta13] Michael Stay, *Compact closed bicategories*, Theory and Applications of Categories, 31 (2016), 755–798 (2013). arXiv:1301.1053

[Thi87] Morwen B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology. An International Journal of Mathematics **26** (1987), no. 3, 297–309. DOI:10.1016/0040-9383(87)90003-6, MR 899051

[Tho56] R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier (Grenoble) **6** (1956), 43–87. MR 87149, http://aif.cedram.org/item?id=AIF_1955__6__43_0

[Tri10] Todd Trimble, *Surface diagrams*, October 2010. https://ncatlab.org/toddtrimble/published/Surface+diagrams

[Tut47] W. T. Tutte, *A ring in graph theory*, Mathematical Proceedings of the Cambridge Philosophical Society **43** (1947), no. 1, 26–40. DOI:10.1017/s0305004100023173

[Tut54] ———, *A contribution to the theory of chromatic polynomials*, Canadian Journal of Mathematics **6** (1954), 80–91. DOI:10.4153/cjm-1954-010-9

[Was75] Gordon Wassermann, *Stability of unfoldings in space and time*, Acta Math. **135** (1975), no. x, 57–128. DOI:10.1007/BF02392016, MR 433497

[Whi32] Hassler Whitney, *The coloring of graphs*, Ann. of Math. (2) **33** (1932), no. 4, 688–718. DOI:10.2307/1968214, MR 1503085, https://doi.org/10.2307/1968214

[Wu82] F. Y. Wu, *The Potts model*, Rev. Modern Phys. **54** (1982), no. 1, 235–268. DOI:10.1103/RevModPhys.54.235, MR 641370, https://doi.org/10.1103/RevModPhys.54.235

Appendix A

Additional material

A.1 Adequate homomorphisms

There is a version of Nakayama's Lemma for families of modules over families of rings. The lemma is essentially due to Mather, but other than some elements in [Mat69a] it went unpublished (and then rewritten by Baas and left unpublished yet again), and then published in some form by others, for example [Duf75], [Buc77] and [Dam84]. Our proof is adapted from the one found in the appendix of [Ber08].

Damon has a theory of adequate systems of rings [Dam84], but it is universally quantified over all modules, rather than the modules present in the family, which made it difficult for this author to see where it was safe to relax the adequacy property without reproving things from scratch as we do here. We remove generality that does not pertain to its use to analyze cascades, and we expose internal details that have been relaxed as far as possible. This was in an attempt to get a concrete bound on finite determinacy for singularities with a foliated codomain; given the difficulty of finding a complete proof of this lemma, we include it here. For convenience, we first state Nakayama's lemma and a corollary.

Lemma A.1.1 (Nayakama's Lemma)**.** *Let R be a unital commutative ring, and let $I \subseteq R$ be an ideal that is a subset of every maximal ideal of R (I is called a* Jacobson ideal*). Suppose M a finitely generated R-module. We give a few formulations:*

1. *If $N \subseteq M$ is a submodule such that $M = N + IM$, then $M = N$.*
2. *If $IM = M$, then $M = 0$.*
3. *If N and N' are submodules of M and $N \subseteq N' + IN$, then $N \subseteq N'$.*

Proof. For the third, we have $N + N' \subseteq IN + N'$, so applying the second for the quotient module M/N' gives $N + N' \subseteq N'$, hence $N \subseteq N'$. $\square$

Corollary A.1.2. *Let A be a unital commutative algebra over $\mathbb{R}$ and let $I \subseteq A$ be a Jacobson ideal. Suppose M is a finitely generated A-module, $N \subseteq M$ is a submodule, and $d, k \in \mathbb{N}$ are such that*

$$d = \dim_{\mathbb{R}} M/(I^{k+1}M + N) \leq k.$$

Then $I^d M \subseteq N$.

Proof. Let $B = M/(I^{k+1}M + N)$, and consider the descending chain

$$B \supseteq IB \supseteq I^2 B \supseteq I^3 B \supseteq \cdots$$

Since B is a finite-dimensional vector space of dimension d, there must be some $0 \leq i \leq d$ such that

$$I^i B = I^{i+1} B.$$

Since this means $I(I^i B) = I^i B$, then $I^i B = 0$ by Nakayama's Lemma (Lemma A.1.1), which implies $I^d B = 0$. Thus, $I^d M \subseteq I^{k+1}M + N$. Applying the third formulation of Nakayama's Lemma using the Jacobson ideal I^{k+1-d}, we get $I^d M \subseteq N$. □

Lemma A.1.3. *Let $s \in \mathbb{N}$, let $R_1, \ldots, R_s$ be unital commutative rings, and for each $1 \leq i < s$ let $\varphi_i : R_i \to R_{i+1}$ be a ring homomorphism. Let $I_1 \subseteq R_1$, $\ldots$, $I_s \subseteq R_s$ be Jacobson ideals (see Lemma A.1.1) such that $\varphi_i(I_i) \subseteq I_{i+1}$ for all $1 \leq i < s$. For each $1 \leq i \leq s$, let M_i and N_i be finitely generated R_i-modules and let $\alpha_i : M_i \to N_i$ be an R_i-module homomorphism. For each $1 \leq i < s$, let $\beta_i : M_i \to N_{i+1}$ be an R_i-module homomorphism over φ_i (that is, $\beta_i(rm) = \varphi_i(r)\beta_i(m)$ for all $r \in R_i$ and $m \in M_i$).*

$$\begin{tikzcd} R_1 \arrow[r, "\varphi_1"] & R_2 \arrow[r, "\varphi_2"] & \cdots \arrow[r, "\varphi_{s-1}"] & R_s \end{tikzcd}$$

$$\begin{tikzcd} M_1 \arrow[d, "\alpha_1"] \arrow[dr, "\beta_1"] & M_2 \arrow[d, "\alpha_2"] \arrow[dr, "\beta_2"] & \cdots \arrow[dr, "\beta_{s-1}"] & M_s \arrow[d, "\alpha_s"] \\ N_1 & N_2 & \cdots & N_s \end{tikzcd}$$

Let $M = \bigoplus_i M_i$ and $N = \bigoplus_i N_i$, and define $f : M \to N$ by

$$f(m_1, \ldots, m_s) = (\alpha_1(m_1), \beta_1(m_1) + \alpha_2(m_2), \ldots, \beta_{s-1}(m_{s-1}) + \alpha_s(m_s)).$$

Suppose this system of rings and modules satisfies an adequacy condition*: For each $2 \leq i \leq s$, let $N_i' = \{n \in N_i \mid n \in f(\sum_{i \leq j \leq s} M_j)\}$, and suppose that whenever*

$$N_i = N_i' + \beta_{i-1}(M_{i-1}) + I_i \beta_{i-1}(M_{i-1})$$

then

(1) $R_i \beta_{i-1}(M_{i-1}) \subseteq N_i' + \beta_{i-1}(M_{i-1})$ and

(2) $I_i \beta_{i-1}(M_{i-1}) \subseteq I_i N_i' + \beta_{i-1}(I_{i-1} M_{i-1})$.

If $N = f(M) + \sum_i I_i N_i$*, then* (i) $N = f(M)$ *and* (ii) $\sum_i I_i N_i \subseteq f(\sum_i I_i M_i)$.

Keeping everything else the same, if $N_1 = 0$ *and adequacy condition (2) does not necessarily hold for* $i = 2$*, then* (i) $N = f(M)$ *and* (ii') $\sum_{2\le i\le s} I_i N_i \subseteq I_2\beta_1(M_1) + f(\sum_{2\le i\le s} I_i M_i)$.

Proof. If $s = 0$ this is trivial, and if $s = 1$ this reduces to Nakayama's lemma. We induct on the number of nonzero modules in the diagram when $s \geq 2$. If both $M_1 = 0$ and $N_1 = 0$, then we can remove them and reindex everything, and otherwise we consider two cases: whether $N_1 \neq 0$ or $N_1 = 0$.

First suppose that $N_1 \neq 0$. Let $N' = \bigoplus_{2\le i\le s} N_s$, and let $f' : M \to N'$ be the composition of f and the projection onto N' (or equivalently $f' = f - \alpha_1$). Since we assume $f(M) + \sum_i M_i = M$, we have that $f'(M) + \sum_{2\le i\le s} I_i N_i = N'$. The adequacy condition is still satisfied by this f' system since it does not involve α_1, hence we may apply the induction hypothesis to get

$$\text{(i') } N' = f'(M) \quad \text{and} \quad \text{(ii') } \textstyle\sum_{2\le i\le s} I_i N_i \subseteq f'(\sum_{1\le i\le s} I_i M_i).$$

With $\iota : N_2 \hookrightarrow N'$ the natural inclusion, let $M_1' = (\iota \circ \beta_1)^{-1} f(\sum_{2\le i\le s} M_i)$, which is equivalently

$$M_1' = \left\{ m_1 \in M_1 \;\middle|\; \beta_1(m_1) \in f\left(\sum_{i=2}^{s} M_i\right)\right\}.$$

We have $\alpha_1(M_1') \subseteq f(M)$, since for all $m' \in M_1'$ there is an $m \in \sum_{2\le i\le s} M_i$ with $\beta_1(m') = f(m)$, and $\alpha_1(m') = f(m' - m)$. Also, we have $\alpha_1(I_1 M_1') \subseteq f(\sum_{1\le i\le s} I_i M_i)$, since for all $r \in I_1$ and $m' \in M_1'$, because $\beta_1(rm') = \varphi(r)\beta_1(m_1)$ and $\varphi(r) \in I_2$, then by (ii') there is an $m \in \sum_{2\le i\le s} I_i M_i$ such that $\varphi(r)\beta_1(m') = f'(m)$, so $\alpha_1(rm') = f(rm' - m)$.

We claim that it suffices to show $N_1 = \alpha_1(M_1')$. We see $f'(M) \subseteq f(M)$ since for $m \in M$, $f'(m) - f(m) \in N_1 = \alpha_1(M_1) \subseteq f(M)$, thus $f'(m) \in f(m) + f(M) = f(M)$. For (i), we see by (i') that

$$N = N_1 + N' = \alpha_1(M_1') + f'(M) \subseteq f(M),$$

and thus $N = f(M)$. For (ii): Since $I_1 N_1 = \alpha_1(I_1 M_1')$, we see that $f'(\sum_{1\le i\le s} I_i M_i) \subseteq f(\sum_{1\le i\le s} I_i M_i)$ because for all $m \in \sum_{1\le i\le s} I_i M_i$, then $f'(m) - f(m) \in I_1 N_1 = \alpha(I_1 M_1') \subseteq f(\sum_{1\le i\le s} I_i M_i)$, so $f'(m) \in f(\sum_{1\le i\le s} I_i M_i)$. Therefore, applying (ii'),

$$\sum_{1\le i\le s} I_i N_i = I_1 N_1 + \sum_{2\le i\le s} I_i N_i \subseteq \alpha_1(I_1 M_1') + f'\left(\sum_{1\le i\le s} I_i M_i\right) \subseteq f\left(\sum_{1\le i\le s} I_i M_i\right).$$

We will show that $N_1 = \alpha_1(M_1')$ by showing $N_1 = \alpha_1(M_1') + I_1 N_1$, which suffices due to Nakayama's lemma since N_1 is a finitely generated R_1-module and I_1 is a Jacobson ideal.

Recalling the assumption $N = f(M) + \sum_i I_i N_i$, then by (ii'),

$$\begin{aligned} N &= f(M) + I_1 N_1 + f'\left(\sum_{1\le i\le s} I_i M_i\right) \\ &= f(M) + I_1 N_1 + \beta_1(I_1 M_1) + f\left(\sum_{2\le i\le s} I_i M_i\right) \\ &= f(M) + I_1 N_1 + \beta_1(I_1 M_1). \end{aligned}$$

Projecting onto the N_1 component, we have $N_1 \subseteq \alpha_1(M_1) + I_1 N_1$. Letting $n_1 \in N_1$ be arbitrary, then there are some $m_i \in M_i$ for $1 \le i \le s$ as well as an $n_1' \in I_1 N_1$ and an $m_1' \in I_1 M_1$ such that $n_1 = f(m_1 + \cdots + m_s) + n_1' + \beta_1(m_1')$. Collecting the N_1 and N' components, we have $n_1 = \alpha_1(m_1) + n_1'$ and $0 = \beta_1(m_1) + f(m_2 + \cdots + m_s) + \beta_1(m_1')$. From the second equation, we deduce $m_1 + m_1' \in M_1'$. Thus, $n_1 = \alpha_1(m_1 + m_1') + n_1' - \alpha_1(m_1')$ has $\alpha_1(m_1 + m_1') \in \alpha_1(M_1')$ and $n_1' - \alpha_1(m_1') \in I_1 N_1$, hence $n_1 \in \alpha_1(M_1) + I_1 N_1$. Therefore $N_1 = \alpha_1(M_1) + I_1 N_1$, completing the $N_1 \neq 0$ case.

We now continue with the second case for the induction, which is that $N_1 = 0$. Consider the induced R_2-module $R_2 \otimes_{R_1} M_1$, where R_2 is an R_1-module via φ_1. This is a finitely generated R_2 module, since if $\{a_i\}_i$ is a finite generating set for M_1 as an R_1-module, then $\{1 \otimes a_i\}_i$ is a finite generating set for $R_2 \otimes_{R_1} M_1$. We define $\gamma_1 : R_2 \otimes_{R_1} M_1 \to N_2$ by $r \otimes m \mapsto r\beta_1(m)$, which is well-defined because for $r_1 \in R_1$, $r_2 \in R_2$, and $m \in M_1$,

$$\gamma_1(r_2 \otimes r_1 m) = r_2 \beta_1(r_1 m) = r_2 \varphi_1(r_1)\beta_1(m) = \gamma_1(r_2\varphi_1(r_1) \otimes m).$$

We use this to construct a new system of rings and modules with fewer nonzero modules:

$$\begin{tikzcd} R_2 \arrow[r, "\varphi_2"] & R_3 \arrow[r, "\varphi_3"] & \cdots \arrow[r, "\varphi_{s-1}"] & R_s \\ (R_2 \otimes_{R_1} M_1) \oplus M_2 \arrow[d, "\gamma_1+\alpha_1"'] \arrow[dr, "0+\beta_2"] & M_3 \arrow[d, "\alpha_3"] \arrow[dr, "\beta_3"] & \cdots \arrow[dr, "\beta_{s-1}"] & M_s \arrow[d, "\alpha_s"] \\ N_2 & N_3 & \cdots & N_s \end{tikzcd}$$

We show that this satisfies the adequacy condition involving the N_i' modules for $3 \le i \le s$, which are the same for this modified system. The property still holds for $4 \le i \le s$, so we just have to check $i = 3$. Since the relevant diagonal homomorphism is $0 + \beta_2$, what we need to show is exactly our assumed N_i' hypothesis, so the condition is satisfied.

Let $M' = (R_2 \otimes_{R_1} M_1) \oplus \bigoplus_{2\le i\le s} M_i$, and let $f' : M' \to N$ be defined by $f'|_{R_2 \otimes_{R_1} M_1} = \gamma_1$ and $f'|_{M_i} = f|_{M_i}$ for $2 \le i \le s$, which is the map analogous to f for this diagram. Then, since $\operatorname{im}\gamma_1 = \operatorname{im}\beta_1 = f(M_1)$, we see $f'(M') = \operatorname{im}\gamma_1 + \sum_{2\le i\le s} f(M_i) = f(M)$, and thus $f'(M') + \sum_i I_i N_i = N$ by the assumption that $f(M) + \sum_i I_i N_i = N$. Therefore we may apply the induction hypothesis to get that

$$\text{(i')}\ N = f'(M') \quad \text{and} \quad \text{(ii')}\ \textstyle\sum_{2\le i\le s} I_i N_i \subseteq f'\left(I_2 \otimes_{R_1} M_1 + \sum_{2\le i\le s} I_i M_i\right).$$

Note that we can equivalently write these as

(i') $N = R_2\beta_1(M_1) + f(\sum_{2\le i\le s} M_i)$ and (ii') $\sum_{2\le i\le s} I_iN_i \subseteq I_2\beta_1(M_1) + f(\sum_{2\le i\le s} I_iM_i)$.

Recall that we have defined in the lemma statement the R_2-submodule

$$N_2' = N_2 \cap f\left(\sum_{i=2}^{s} M_i\right) = \left\{ n_2 \in N_2 \;\middle|\; n_2 \in f\left(\sum_{i=2}^{s} M_i\right)\right\}.$$

We claim it is sufficient to prove that $N_2 = N_2' + \beta_1(M_1) + I_2\beta_1(M_1)$. Given this, by the adequacy condition for N_2' we get

$$(1)\ R_2\beta_1(M_1) \subseteq N_2' + \beta_1(M_1) \quad \text{and} \quad (2)\ I_2\beta_1(M_1) \subseteq I_2N_2' + \beta_1(I_1M_1).$$

Since $N_2' \subseteq f(M)$, by (i') and (1),

$$N = R_2\beta_1(M_1) + f\left(\sum_{i=2}^{s} M_i\right) \subseteq N_2' + \beta_1(M_1) + f\left(\sum_{i=2}^{s} M_i\right) \subseteq f(M),$$

and hence (i) holds. Since N_2' is an R_2-submodule and $N_2' \subseteq f(\sum_{i=2}^{s} M_i)$, then $I_2N_2' \subseteq f\left(\sum_{i=2}^{s} I_iM_i\right)$ because $(\varphi_j \circ \cdots \circ \varphi_3 \circ \varphi_2)(I_2) \subseteq I_j$ for all $3 \le j \le s$. Thus, using (ii') and (2), (ii) also holds:

$$\sum_{i=2}^{s} I_iN_i \subseteq I_2\beta_1(M_1) + f\left(\sum_{i=2}^{s} I_iM_i\right) \subseteq I_2N_2' + f\left(\sum_{i=1}^{s} I_iM_i\right) \subseteq f\left(\sum_{i=1}^{s} I_iM_i\right).$$

Now we will show that we indeed have $N_2 = N_2' + \beta_1(M_1) + I_2\beta_1(M_1)$ and $\sum_{3\le j\le s} N_j = f(\sum_{3\le j\le s} M_j)$. By assumption and by (ii') we have that

$$\begin{aligned} N = f(M) + \sum_{i=2}^{s} I_iN_i &= f(M) + I_2\beta_1(M_1) + f\left(\sum_{i=2}^{s} I_iM_i\right) \\ &= f(M) + I_2\beta_1(M_1) \\ &= f\left(\sum_{i=2}^{s} M_i\right) + \beta_1(M_1) + I_2\beta_1(M_1). \end{aligned}$$

Intersecting each side with N_2, we get $N_2 = N_2' + \beta_1(M_1) + I_2\beta_1(M_1)$. This completes the $N_1 = 0$ case, and therefore the induction argument. $\square$

A.2 Symmetric monoidal categories

This section is a brisk review of symmetric monoidal categories, compact closed categories, and symmetric monoidal functors. See [Sel] for a more in-depth overview of these concepts and graphical notations.

A *monoidal category* is a category $\mathcal{C}$ along with (1) a *horizontal composition* bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, (2) a *unit object* $1 \in \mathcal{C}$, (3) a natural isomorphism $a_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ called the *associator*, and (4) natural isomorphisms $\lambda_x : 1 \otimes x \to x$ and $\rho_x : x \otimes 1 \to x$ respectively called the *left unitor* and the *right unitor*. These must satisfy two conditions. The first is the *pentagon identity*

$$\begin{tikzcd}
((w \otimes x) \otimes y) \otimes z \arrow[r, "a_{w\otimes x,y,z}"] \arrow[d, "a_{w,x,y}\otimes \mathrm{id}_z"] & (w \otimes x) \otimes (y \otimes z) \arrow[r, "a_{w,x,y\otimes z}"] & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "a_{w,x\otimes y,z}"] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "\mathrm{id}_w \otimes a_{x,y,z}"']
\end{tikzcd}$$

The second is the *triangle identity*

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "a_{x,1,y}"] \arrow[dr, "\rho_x \otimes \mathrm{id}_y"'] & & x \otimes (1 \otimes y) \arrow[dl, "\mathrm{id}_x \otimes \lambda_y"] \\
& x \otimes y &
\end{tikzcd}$$

The point is that the set of objects for $\mathcal{C}$ has the structure of a monoid, but all the axioms of the monoid are represented as isomorphisms in the category. The pentagon and triangle identities guarantee that any sequence of re-parenthesizations and introductions or removals of identity objects in a horizontal composition results in the same object, rather than an object that is merely isomorphic (this is known as "coherence").

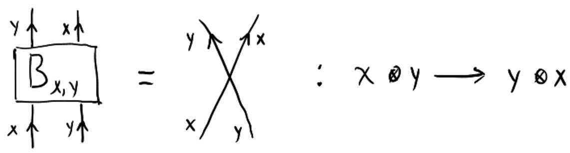
A *strict monoidal category* is a monoidal category whose a, λ, and ρ morphisms are identities. Every monoidal category is equivalent through monoidal functors (defined below) to a strict monoidal category.

A *symmetric monoidal category* $\mathcal{C}$ is a monoidal category with a natural isomorphism $B_{x,y} : x \otimes y \to y \otimes x$ called the *braiding* satisfying $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}$ and the *hexagon identity*

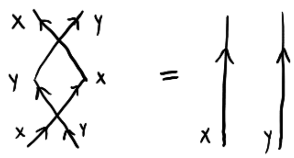
$$\begin{tikzcd}
(x \otimes y) \otimes z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x,y}\otimes \mathrm{id}_z"] & x \otimes (y \otimes z) \arrow[r, "B_{x,y\otimes z}"] & (y \otimes z) \otimes x \arrow[d, "a_{y,z,x}"] \\
(y \otimes x) \otimes z \arrow[r, "a_{y,x,z}"] & y \otimes (x \otimes z) \arrow[r, "\mathrm{id}_y \otimes B_{x,z}"] & y \otimes (z \otimes x)
\end{tikzcd}$$

An important example of a symmetric monoidal category is R-Mod, the category of modules over a commutative ring R. The braiding $B_{M,N} : M \otimes N \to N \otimes M$ is given by $m \otimes n \mapsto n \otimes m$. A special case is Vect_k, the category of vector spaces over a field k.

A graphical calculus for working with (symmetric) monoidal categories is *string diagrams*. Horizontal compositions of objects and morphisms by horizontal juxtaposition, and morphism composition is by vertical concatenation. Identity objects are suppressed, and identity morphisms are represented by vertical lines. We take the convention that morphisms go from bottom to top. The braiding morphisms can be represented as follows:



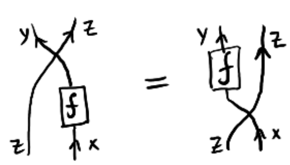
The left-hand representation is as a "coupon," and the right-hand representation suggests certain valid topological manipulations. For example, the equation $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x\otimes y}$ corresponds to a homotopy of an immersion of strings in the plane:



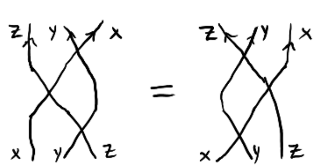
Naturality of the braiding is that, for every morphism $f : x \to y$, the following diagram commutes:

$$\begin{tikzcd} z \otimes x \arrow[r, "\mathrm{id}_z \otimes f"] \arrow[d, "B_{z,x}"] & z \otimes y \arrow[d, "B_{z,y}"] \\ x \otimes z \arrow[r, "f \otimes \mathrm{id}_z"] & y \otimes z \end{tikzcd}$$

A graphical representation of this is



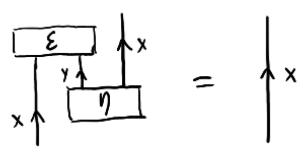
Combining the hexagon identity with naturality applied to $B_{y,z}$ yields another homotopy of immersions of strings:

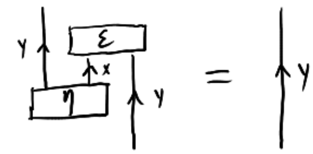


Therefore, the braiding yields representations of the symmetric group $S_n \to \mathrm{End}_{\mathcal{C}}(x^{\otimes n})$ for every object $x \in \mathcal{C}$ and $n \in \mathbb{Z}_{\geq 0}$, where $x^{\otimes(n+1)} = x \otimes x^{\otimes n}$ and $x^{\otimes 0} = 1$. This is what is "symmetric" about a symmetric monoidal category. For an element $\sigma \in S_n$, we denote the induced endomorphism as $\sigma_* : x^{\otimes n} \to x^{\otimes n}$.

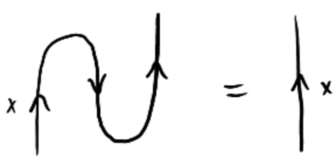
An *exact pairing* between two objects x and y in a monoidal category $\mathcal{C}$ is given by a pair of morphisms $\epsilon : x \otimes y \to 1$ and $\eta : 1 \to x \otimes y$ such that $(\epsilon \otimes \mathrm{id}_x) \circ (\mathrm{id}_x \otimes \eta) = \mathrm{id}_x$ and

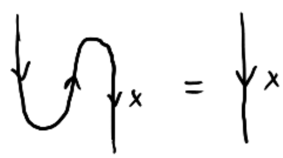
$(\mathrm{id}_y \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_y) = \mathrm{id}_y$. In such an exact pairing, y is called the *left dual* of x and y is called the *right dual* of x. (That is, x is left dual to y if the functor $x \otimes -$ is left adjoint to $y \otimes -$, which is to say $\mathrm{Hom}(x \otimes z, w) \cong \mathrm{Hom}(z, y \otimes w)$ naturally in z and w.) Graphically,



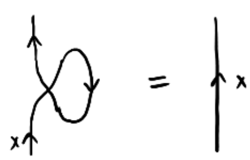
and


An *autonomous category* (or *rigid category*) is a monoidal category such that every object has both a right and left dual, and a *compact closed category* is an autonomous symmetric monoidal category. The graphical language for compact closed categories allows for strings to reverse direction, with the understanding that a portion of a string labeled x in the reverse direction corresponds to the dual object x^*, which is a well-defined notion since x and x^{**} are naturally isomorphic. "Cups" and "caps" from backtracking correspond to the respective η or ϵ for the exact pairing. Hence, the equations for duals are represented as



and


Furthermore, the relationship between the braiding, cups, and caps is that twists may be removed:



Therefore, diagrams for compact closed categories are invariant under planar isotopies and arbitrary string rerouting moves — that is, all that matters is what is connected by strings, but not the route by which a connection takes place — where invariance means such moves correspond to well-formed equations. This is in contrast with general braided monoidal categories, where the strings might be knotted.

Example A.2.1. The category of finite-dimensional vector spaces over a field k is a compact closed category. So is the category of finite-dimensional super vector spaces, which are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces with braiding $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ for homogeneous vectors v and w. $\lozenge$

A *(strong) monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor along with an isomorphism $\epsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and a natural isomorphism $\mu_{x,y} : F(x) \otimes F(y) \to$

$F(x \otimes y)$ subject to two conditions. The first is *associativity*

$$\begin{tikzcd}
(F(x) \otimes F(y)) \otimes F(z) \arrow[r, "\mu_{x,y}\otimes \mathrm{id}"] \arrow[d, "a^{\mathcal{D}}_{F(x),F(y),F(z)}"] & F(x \otimes y) \otimes F(z) \arrow[r, "\mu_{x\otimes y,z}"] & F((x \otimes y) \otimes z) \arrow[d, "F(a^{\mathcal{C}}_{x,y,z})"] \\
F(x) \otimes (F(y) \otimes F(z)) \arrow[r, "\mathrm{id}\otimes \mu_{y,z}"] & F(x) \otimes F(y \otimes z) \arrow[r, "\mu_{x,y\otimes z}"] & F(x \otimes (y \otimes z))
\end{tikzcd}$$

The second is *unitality*

$$\begin{tikzcd}
1_{\mathcal{D}} \otimes F(x) \arrow[r, "\epsilon\otimes \mathrm{id}"] \arrow[d, "\lambda^{\mathcal{D}}_{F(x)}"] & F(1_{\mathcal{C}}) \otimes F(x) \arrow[d, "\mu_{1_{\mathcal{C}},x}"] \\
F(x) & F(1 \otimes x) \arrow[l, "F(\lambda^{\mathcal{D}}_x)"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
F(x) \otimes 1_{\mathcal{D}} \arrow[r, "\mathrm{id}\otimes \epsilon"] \arrow[d, "\rho^{\mathcal{D}}_{F(x)}"] & F(x) \otimes F(1_{\mathcal{C}}) \arrow[d, "\mu_{x,1_{\mathcal{C}}}"] \\
F(x) & F(x \otimes 1) \arrow[l, "F(\rho^{\mathcal{D}}_x)"']
\end{tikzcd}$$

A *(strong) symmetric monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor such that the following diagram commutes

$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "B_{F(x),F(y)}"] \arrow[d, "\mu_{x,y}"] & F(y) \otimes F(x) \arrow[d, "\mu_{y,x}"] \\
F(x \otimes y) \arrow[r, "F(B_{x,y})"] & F(y \otimes x)
\end{tikzcd}$$

A *monoidal natural transformation* between two monoidal functors $F, G : \mathcal{C} \to \mathcal{D}$ is a natural transformation φ such that the following two diagrams commute:

$$\begin{tikzcd}
F(x) \otimes F(y) \arrow[r, "\varphi_x\otimes\varphi_y"] \arrow[d, "\mu^F_{x,y}"] & G(x) \otimes G(y) \arrow[d, "\mu^G_{x,y}"] \\
F(x \otimes y) \arrow[r, "\varphi_{x\otimes y}"] & G(x \otimes y)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 1_{\mathcal{D}} \arrow[dl, "\epsilon^{\mathcal{D}}"'] \arrow[dr, "\epsilon^{\mathcal{D}}"] & \\
F(1_{\mathcal{C}}) \arrow[rr, "\varphi_{1_{\mathcal{C}}}"] & & G(1_{\mathcal{C}})
\end{tikzcd}$$

Two monoidal categories are *monoidally equivalent* if there is an equivalence of categories using monoidal functors and monoidal natural transformations.

Example A.2.2. With R a commutative ring, let $F : \mathbf{Set} \to R$-Mod be the functor sending a set to a free R-module with that set as its basis.

The category **Set** is a symmetric monoidal category with disjoint unions as the horizontal composition and the empty set as the unit object, and the category R-Mod is a symmetric monoidal category with direct sums as the horizontal composition and the zero module as the unit object. With respect to these structures, F is a symmetric monoidal functor.

Also, the category **Set** is a symmetric monoidal category with cartesian products as the horizontal composition and the one-point set as the unit object, and the category R-Mod is a symmetric monoidal category with tensor products as the horizontal composition and R as the unit object. With respect to these structures, F is a symmetric monoidal functor.

This is akin to ring homomorphisms being monoid homomorphisms with respect to two sets of monoidal structures. $\Diamond$

A.3 Topological graph identities

Many computations with graph polynomials involve the many identities between the various topological invariants of the underlying spaces. As a reference, we give identities and generating sets of topological invariants for a few situations. With these we may tell whether two graph polynomials are renormalizations of one another by rewriting expressions in terms of a given generating set.

We will consider topological invariants to be functions $\mathcal{G} \to \mathbb{Q}$ with $\mathcal{G}$ the set of all combinatorial objects under consideration. These form a vector space over $\mathbb{Q}$, and it is in this sense we will speak of linearly independent graph invariants.

A.3.1 Graphs

Consider a graph $G = (V, E)$ and a subset $A \subseteq E$ of edges. We use Betti numbers $b_0(G)$ for the number of connected components and $b_1(G)$ for the dimension of the cycle space. By Euler characteristics, the principal identities are:

$$\begin{aligned} b_0(G) - b_1(G) &= |V| - |E| \\ b_0(V \cup A) - b_1(V \cup A) &= |V| - |A| \end{aligned}$$

Proposition A.3.1. *Consider a graph $G = (V, E)$ and a subset $A \subseteq E(G)$ of edges. The invariants $|V|$, $|E|$, $b_0(G)$, $|A|$, and $b_0(V \cup A)$ are independent, and we may write*

$$\begin{aligned} b_1(G) &= b_0(G) - |V| + |E| \\ b_1(V \cup A) &= b_0(V \cup A) - |V| + |A|. \end{aligned}$$

Proof. The vectors $(|V(G)|, |E(G)|, b_0(G))$ for G_1 a graph with a single vertex, G_2 a graph with a single vertex and a loop, and G_3 a graph with two vertices and an edge between them are $(1, 0, 1)$, $(1, 1, 1)$, and $(2, 1, 1)$, which are independent. $\square$

A.3.2 Compact subsurfaces

Let Σ be a closed orientable surface, $B \subseteq \Sigma$ a compact subsurface, $W = \mathrm{cl}(\Sigma - B)$, and $\sigma = B \cap W$. We use g for the genus of the given surface after being closed up.

- $b_0(\Sigma) = b_2(\Sigma)$ and $b_0(\sigma) = b_1(\sigma)$ by Poincaré duality.
- $2g(\Sigma) = b_1(\Sigma)$ by definition.
- $b_0(B) + b_1(B) - b_2(B) - b_0(\sigma) = 2g(B)$ by Euler characteristics.
- $b_0(W) + b_1(W) - b_2(W) - b_0(\sigma) = 2g(W)$ by Euler characteristics.
- $b_2(B) + b_2(W) - b_1(B) - b_1(W) + b_1(\Sigma) + b_0(B) + b_0(W) - 2b_0(\Sigma) = 0$ by the Euler characteristic of the Mayer–Vietoris sequence for $\Sigma = B \cup W$, simplified.

Krushkal in [Kru11] defines the invariant k of an embedding $B \hookrightarrow \Sigma$ to be

$$k(B \hookrightarrow \Sigma) = \dim(\ker(H_1(B;\mathbb{R}) \to H_1(\Sigma;\mathbb{R}))).$$

The discussion in [Kru11, Section 5.1] applies to arbitrary subsurfaces (not just regular neighborhoods of graphs) thus there are relations like [Kru11, (4.7)] for B and W:

$$\begin{aligned} k(B) &= -g(B) + g(W) + b_1(B) - g(\Sigma) \\ k(W) &= -g(W) + g(B) + b_1(W) - g(\Sigma). \end{aligned}$$

Proposition A.3.2. *Let Σ be a closed orientable surface, $B \subseteq \Sigma$ a compact subsurface, $W = \mathrm{cl}(\Sigma - B)$, and $\sigma = B \cap W$. Then $b_0(\Sigma)$, $g(\Sigma)$, $b_0(B)$, $b_0(W)$, $b_2(B)$, $b_2(W)$, $g(B)$, and $g(W)$ are independent with*

$$\begin{aligned} b_0(\sigma) &= g(\Sigma) - b_0(\Sigma) + b_0(B) - g(B) + b_0(W) - g(W) \\ b_1(B) &= g(\Sigma) - b_0(\Sigma) + b_2(B) + g(B) + b_0(W) - g(W) \\ b_1(W) &= g(\Sigma) - b_0(\Sigma) + b_2(W) + g(W) + b_0(B) - g(B) \\ k(B) &= -b_0(\Sigma) + b_2(B) + b_0(W) \\ k(W) &= -b_0(\Sigma) + b_2(W) + b_0(B). \end{aligned}$$

A.3.3 Surface graphs

Consider a surface graph $G \hookrightarrow \Sigma$ with Σ closed and $G = (V, E)$. Then with $B = \nu(G)$ a regular neighborhood of G, since $b_2(B) = 0$, some of the identities from Appendix A.3.2 look like

- $b_0(G) + b_1(G) - b_0(\partial G) = 2g(G)$
- $b_0^\perp(G) + b_1^\perp(G) - b_2^\perp(G) - b_0(\partial G) = 2g^\perp(G)$
- $b_2^\perp(G) - b_1(G) - b_1^\perp(G) + b_1(\Sigma) + b_0(G) + b_0^\perp(G) - 2b_0(\Sigma) = 0$
- $k(G) = -g(G) + g^\perp(G) + b_1(G) - g(\Sigma)$.

Proposition A.3.3. *Let Σ be a closed orientable surface, $G \hookrightarrow \Sigma$ a surface graph with $G = (V, E)$, and $A \subseteq E$. Then $b_0(\Sigma)$, $g(\Sigma)$, $|V|$, $b_2^\perp(G)$, $|A|$, $b_0(V \cup A)$, $b_0^\perp(V \cup A)$, $g(V \cup A)$ are independent with*

$$\begin{aligned} b_0(\partial(V \cup A)) &= -|V| + |A| + 2b_0(V \cup A) - 2g(V \cup A) \\ b_1(V \cup A) &= -|V| + |A| + b_0(V \cup A) \\ b_1^\perp(V \cup A) &= 2g(\Sigma) - 2b_0(\Sigma) + b_2^\perp(G) + |V| - |A| + b_0^\perp(V \cup A) \\ g^\perp(V \cup A) &= g(\Sigma) - b_0(\Sigma) + |V| - |A| - b_0(V \cup A) + g(V \cup A) + b_0^\perp(V \cup A) \\ k(V \cup A) &= -b_0(\Sigma) + b_0^\perp(V \cup A) \end{aligned}$$

Proof. Since $|V| - |A| = b_0(V \cup A) - b_1(V \cup A)$ and

$$b_0(V \cup A) - b_1(V \cup A) + b_2(V \cup A) + b_0(\partial(V \cup A)) = 2b_0(V \cup A) - 2g(V \cup A),$$

then using $b_2(V \cup A) = 0$ we can solve for $b_0(\partial(V \cup A))$. $\square$

A.3.4 Cellular embeddings and ribbon graphs

Proposition A.3.4. *Let $G \hookrightarrow \Sigma$ be a cellular embedding with $G = (V, E)$, and let $A \subseteq E$. Then $b_0(\Sigma)$, $g(\Sigma)$, $|V|$, $|A|$, $b_0(V \cup A)$, $b_0^\perp(V \cup A)$, $g(V \cup A)$ are independent with*

$$\begin{aligned}
b_0(\partial(V \cup A)) &= -|V| + |A| + 2b_0(V \cup A) - 2g(V \cup A)\\
b_1(V \cup A) &= -|V| + |A| + b_0(V \cup A)\\
b_1^\perp(V \cup A) &= 2g(\Sigma) - 2b_0(\Sigma) + |V| - |A| + b_0^\perp(V \cup A)\\
g^\perp(V \cup A) &= g(\Sigma) - b_0(\Sigma) + |V| - |A| - b_0(V \cup A) + g(V \cup A) + b_0^\perp(V \cup A)\\
k(V \cup A) &= -b_0(\Sigma) + b_0^\perp(V \cup A)
\end{aligned}$$

Notation Index

Symbols

$X \circ\!\!\to Y$ (local homeomorphism from X to Y), 38
$U|_u$ (restriction of pro-open set to open set), 27
$[\mathcal{C}, \mathcal{D}]$ (category of functors from $\mathcal{C}$ to $\mathcal{D}$), 14
$[f]_A$ (germ of f along A), 27
$[f]_{A,B}$ (germ of f in $C^\infty_A(M,N)_B$), 27
$f \pitchfork W$ (f is transverse to W), 34
$\downarrow S$ or $I \downarrow S$ (downward set generated by S), 59
$\downarrow^\circ S$ or $I \downarrow^\circ S$ (open downward set generated by S), 59
$x \in U$ (element of pro-open set), 24
$V^{\odot n}$ (nth symmetric power), 72
$W \odot W'$ (symmetric product), 72
$(X \downarrow j)$ (comma category), 18
$\top$ (maximum element), 58
$i \wedge j$ (meet of i and j), 58
$\wedge S$ (meet of every element in S), 58

A

k-Alg_2 (Morita 2-category), 178
α (source of k-jet), 8

B

β (target of k-jet), 8

C

Cob_m (m-dimensional cobordism category), 2
$\operatorname{cod}\varphi$ (codomain of definition), 38
$C^\infty(M)$ (ring of smooth functions $M \to \mathbb{R}$), 29
$C^\infty_A(M)$ or C^∞_A (ring of germs of smooth functions $M \to \mathbb{R}$ along A), 29
$C^\infty_p(M)$ or C^∞_p (ring of germs of smooth functions $M \to \mathbb{R}$ at p), 29
$C^\infty(M,N)$ (smooth maps $M \to N$), 25
$C^\infty_A(M,N)_B$ (smooth maps of smooth manifold germs from $\mathcal{N}(A)$ to $\mathcal{N}(B)$), 27
$C^\infty_A(M,N)$ (smooth map germs $M \to N$ along A), 27
$C^\infty_p(M,N)$ (germs of smooth maps $M \to N$ at p), 27
$C^\infty(M,N;f|C)$ (smooth maps that agree with f on C), 35

D

Δ^r_M (diagonal), 36
$\boldsymbol{\Delta}^r_M$ (fat diagonal), 36
Df_p (intrinsic derivative), 71
$\mathrm{Diff}(M)$ (diffeomorphism group), 39
$\mathrm{Diff}(M,N)$ (diffeomorphisms), 39
$\mathrm{Diff}_p(M)$ (group of diffeomorphism germs), 45
$\mathrm{Diff}_p(M)_q$ (diffeomorphism germs), 45
$\mathrm{Diff}_p(M;N)$ (group of diffeomorphism germs leaving N invariant), 144
$\operatorname{dom}\varphi$ (domain of definition), 38
$\operatorname{drk} a$ (dropped rank), 57

F

$F(k_1 \leq \cdots \leq k_n)$ (pseudogroup for

foliations), 40
$F(\{U_i\}_i)$ (pseudogroup for family of integrable distributions), 40
$\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (category of functors from $\mathcal{C}$ to $\mathcal{D}$), 14
$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ (category of left-exact functors), 14

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$\mathbb{G}$ (globe category), 159
Γ^k_G (local diffeomorphisms with differentials in G), 39
$\mathrm{GL}(V; \{U_i\}_i)$ (subgroup stabilizing subspaces), 39
$\mathrm{Gr}_n(V)$ (Grassmannian of n-dimensional subspaces of V), 58
gSet (category of globular sets), 159

H
$\mathrm{Hom}^r(V, W)$ (linear maps with drk $= r$), 57
$\mathrm{Hom}(V, W; \{r_i\}_i)$ (linear maps with drk conditions), 58

J
$j^k f_p$ (k-jet of f at p), 8
$J^k(M, N)$ (k-jet bundle), 8
$j^k f$ (k-jet extension), 8
$J^{\mathbf{k}}(M, N)$ (multijet bundle), 36
$j^{\mathbf{k}} f$ (multijet extension), 36
$J^k_p(M, N)$ (k-jets of maps at p), 8
$J^k_p(M, N)_q$ (k-jets of maps at p with target q), 8
$J^k f$ (iterated Jacobian of f), 71
$J(M, N)$ (infinite jet bundle), 12

K
$\ker U$ (kernel of a pro-open set), 22

M
$\mathfrak{m}_p(M)$ or $\mathfrak{m}_p$ (maximal ideal of C^∞_p), 30

N
$\mathcal{N}(A)$ (neighborhood pro-open set of A), 24

P
P^k_m (degree $\leq k$ polynomials in m variables), 10
$\mathrm{Pro}(\mathcal{C})$ (category of pro-objects), 17

S
$S(f)$ (S-type singularity points of f), 74

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$T^k f_p$ (kth order Taylor polynomial of f at p), 9
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$\mathcal{V}_n(V; \{V_i\}_i; \{k_i\}_i)$ (submanifold of Grassmannian), 58

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