Math 113 Homework # 8, due 3/9/01 at 5:00 PM

0. (optional, don't hand in) If you haven't seen eigenvectors and eigenvalues before, section 5.1, problem 3 is good practice.

1. Section 5.1 problem 4.

2. Let A be an $n \times n$ matrix. Recall that $\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Show that

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n, \qquad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

3. Let $\mathrm{O}(n)$ denote the set of invertible $n \times n$ real matrices A such that $A^t = A^{-1}$.

   (a) Show that if $A \in \mathrm{O}(n)$ then $\det(A) = \pm 1$.

   (b) Show that if $A \in \mathrm{O}(n)$, then $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all $x \in \mathbb{R}^n$. Deduce that $\|Ax\| = \|x\|$.

   (c) Show that if $A \in \mathrm{O}(n)$, then the columns of A are orthonormal.

4. Suppose $A \in \mathrm{O}(2)$ and $\det(A) = -1$.

   (a) Let $p(z) = a_n z^n + \cdots + a_0$ be a polynomial with $a_0, \ldots, a_n$ real and $a_n \neq 0$. Show that if $z \in \mathbb{C}$ and $p(z) = 0$, then $p(\overline{z}) = 0$ also.

   (b) Show that the eigenvalues of A are real.

   (c) Show that the eigenvalues of A have absolute value 1.

   (d) Show that A has eigenvalues 1 and -1.

   (e) Show that if v and w are eigenvectors with eigenvalues 1 and -1 respectively, then $v \perp w$.

   (f) Show that A is the reflection across a line in $\mathbb{R}^2$.

5. (a) Let $A = \begin{pmatrix} 5/2 & -3 & 3/2 \\ 3 & -5 & 3 \\ 4 & -8 & 5 \end{pmatrix}$. Find an invertible matrix B and a diagonal matrix Λ such that $A = B \Lambda B^{-1}$.

   (b) Show that $\lim_{n \to \infty} A^n$ exists, and calculate it without using a computer.