Math 113 Homework # 7, due 3/2/01 at 5:00 PM

1. Section 4.3, problems 14, 15, 16. Use at least two methods to compute each determinant. (No MATLAB.)

2. (a) Define an n-cycle $f : \{1, ..., n\} \to \{1, ..., n\}$ by $f(i) = i + 1$ for $i < n$, and $f(n) = 1$. Show that f can be expressed as the product (composition) of $n - 1$ transpositions.

   (b) Define the "perfect shuffle" permutation $\sigma : \{1, \ldots, 52\} \to \{1, \ldots, 52\}$ by $\sigma(i) = 2i - 1$ for $i = 1, \ldots, 26$, and $\sigma(i) = 2(i - 26)$ for $i = 27, \ldots, 52$. Express σ as a product of disjoint cycles, and check that $\sigma^8 = 1$.

3. Let A be an $n \times n$ matrix. Suppose that A is upper triangular, i.e. $A_{ij} = 0$ for $i > j$. Give two proofs that $\det(A) = A_{11}A_{22} \cdots A_{nn}$:

   (a) using row reduction,

   (b) using the sum over permutations formula.

4. Let V be a finite dimensional vector space, $T : V \to V$ a linear transformation, and β a basis for V. Then $[T]_\beta^\beta$ is a square matrix. Define

$$\det(T) = \det\left([T]_\beta^\beta\right).$$

   (a) Show that this does not depend on the choice of basis β.

   (b) Suppose V has an inner product, let $W \subset V$ be a subspace, and let $R_W : V \to V$ be the reflection across W. Show that

$$\det(R_W) = (-1)^{\dim(V) - \dim(W)}.$$

5. We asserted in class that if U is a (bounded measurable) set in $\mathbb{R}^n$ and A is an $n \times n$ real matrix, then $\text{volume}(A(U)) = |\det(A)| \, \text{volume}(U)$. Prove this statement when $n = 2$ (so that volume=area) and U is the unit square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$.