Math 113 Homework # 2, due 1/26/01 at 5:00 PM

1. Find all x such that the vectors $(1,1,1)$, $(1,2,3)$, and $(1,4,x)$ are linearly dependent in $\mathbf{R}^3$.

2. Prove or give a counterexample: Given three vectors in a vector space, if every pair is linearly independent, then all three are linearly independent.

3. Let W_1 and W_2 be subspaces of a vector space V. Prove that the intersection $W_1 \cap W_2$ is a subspace of V.

4. Section 1.4, problem 14.

5. Let $k_1, \ldots, k_n$ be distinct real numbers. Show that $e^{k_1 t}, \ldots, e^{k_n t}$ are linearly independent, as functions from $\mathbf{R}$ to $\mathbf{R}$. Hint: suppose that

$$c_1 e^{k_1 t} + \cdots + c_n e^{k_n t} = 0$$

for all t, and not all c_i's are zero. WLOG, $c_1 \neq 0$, and $k_1 > k_2 > \cdots > k_n$. (Why?) Show that if t is sufficiently large, then

$$\left| c_1 e^{k_1 t} \right| > \left| c_2 e^{k_2 t} + \cdots + c_n e^{k_n t} \right|,$$

giving a contradiction.

6. Let V be a finite dimensional vector space, let W be a subspace of V, and suppose that $\dim(W) = \dim(V)$. Show that $W = V$.

7. Let P_n denote the space of degree n polynomials with coefficients in F. Let $a_1, \ldots, a_n$ be *distinct* elements of F. Let $f = (x - a_1) \cdots (x - a_n) \in P_n$, and for $i = 1, \ldots, n$, define $f_i = f/(x - a_i) \in P_{n-1}$.

   (a) Show that $f_1, \ldots, f_n$ are linearly independent. Hint: suppose $c_1 f_1 + \cdots + c_n f_n = 0$, and plug in $x = a_i$.

   (b) Let $g \in P_{n-1}$. Show that there exist constants $c_1, \ldots, c_n$ such that

$$\frac{g}{f} = \frac{c_1}{x - a_1} + \cdots + \frac{c_n}{x - a_n}.$$

   Where do you need the assumption that $a_1, \ldots, a_n$ are distinct?

8. Find a basis for the subspace $\{x \in \mathbf{R}^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.