Math H53 homework #6, suggested due date 12/12/25

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. (a) Find all values of the constant c such that vector field

$$\mathbf{F} = (y + cz, x + 2z, 2y + 3x)$$

on $\mathbb{R}^3$ is conservative.

(b) For each such value of c, find a function f with $\nabla f = \mathbb{F}$.

2. Consider the vector field $\mathbf{F}$ on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$\mathbf{F} = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Is $\mathbf{F}$ conservative?

3. Let x, y, z be differentiable functions of $t \in [0, 1]$ such that

$$yx'(t) + (x + z)y'(t) + yz'(t) = 0$$

for all $t \in [0, 1]$. Suppose that $x(0) = y(0) = z(0) = 1$ and $x(1) = 2$ and $y(1) = 3$. Find $z(1)$. *Hint:* Find a way to use the Fundamental Theorem of Line Integrals.

4. Let S be the upper hemisphere

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\},$$

oriented upward. Calculate the flux through S of the vector field $\mathbf{F} = (1, 1, 0)$.

5. (a) Prove that if f is a differentiable function on $\mathbb{R}^3$ and $\mathbf{F}$ is a differentiable vector field on $\mathbb{R}^3$, then

$$\operatorname{div}(f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f \operatorname{div}(\mathbf{F}).$$

and

$$\operatorname{curl}(f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f \operatorname{curl}(\mathbf{F}).$$

(b) Prove that if f and g are twice differentiable functions on $\mathbb{R}^3$ then

$$\operatorname{div}(\nabla f \times \nabla g) = 0.$$

6. Let C be the curve of intersection in $\mathbb{R}^3$ of the cylinder $x^2 + y^2 = 1$ and the plane $z = x + y$, oriented counterclockwise when viewed from above. Let

$$\mathbf{F} = (x^{100} - y, x + y^{100}, z^{100}).$$

Use Stokes theorem to calculate $\int_C \mathbf{F} \cdot d\mathbf{s}$.

7. Let C be a simple closed smooth curve in the plane $x + y + z = 1$, oriented counterclockwise when viewed from above. Show that

$$\int_C (z\,dx + 2x\,dy + 3y\,dz)$$

depends only on the area of the region in the plane $x + y + z = 1$ enclosed by C. What is the relation?

8. Define a vector field $\mathbf{F}$ on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ by

$$\mathbf{F} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}.$$

(a) Show that $\operatorname{div}(\mathbf{F}) = 0$.

(b) Calculate the flux of $\mathbf{F}$ through the unit sphere $x^2 + y^2 + z^2 = 1$, oriented outward.

(c) Calculate the flux of $\mathbf{F}$ through the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1,$$

oriented outward.

9. Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented upward. Let

$$\mathbf{F} = (y^4 z, x^4 z, z).$$

Calculate the flux of $\mathbf{F}$ through S. *Hint:* Use the Divergence Theorem to replace S by a simpler surface.

10. Let E be a compact subset of $\mathbb{R}^3$ whose boundary is a smooth surface S, and orient S pointing outwards. Show that the volume of E is given by

$$\int_S x\,dy\,dz = -\int_S y\,dx\,dz = \int_S z\,dx\,dy.$$

11. Let S be the torus in $\mathbb{R}^3$ obtained by rotating the circle $(x-2)^2+z^2=1$ in the x, z plane around the z axis. Oriented S outward from the solid region that it encloses.

    (a) Compute the area of S.

    (b) Compute $\int_S (z^2+1)$.

    (c) Compute $\int_S z\,dx\,dy$.

12. (a) Let $\mathbf{F}$ be a differentiable vector field on $\mathbb{R}^3$ such that $\operatorname{div}(\mathbf{F}) = 0$. Show that there exists a differentiable vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\operatorname{curl}(\mathbf{G}) = \mathbf{F}$.

    (b) Show that (a) is false if one replaces $\mathbb{R}^3$ by $\mathbb{R}^3 \setminus \{0\}$.