

Math H53 homework #5, suggested due date 11/24/25

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. Calculate the iterated integral

$$\int_0^1 \int_{1-x}^1 e^{y^2} dy dx$$

by changing the order of integration.

2. Rewrite the triple integral

$$\int_0^1 \int_0^{\sqrt{y}} \int_y^1 f dz dx dy$$

in the other five possible orders.

3. (a) Ignoring issues of convergence, prove that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Hint: By Fubini's theorem, the square of the left hand side equals $\int_{\mathbb{R}^2} e^{-x^2-y^2}$. Evaluate this using polar coordinates.

- (b) *Extra credit:* More rigorously, let us define

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \lim_{a \rightarrow \infty} \int_{-a}^a e^{-x^2} dx.$$

Prove that this limit exists and equals $\sqrt{\pi}$.

4. Let E be the region in \mathbb{R}^3 defined by the inequalities

$$x \geq 0, \quad 0 \leq z \leq \sqrt{x^2 + y^2}, \quad x^2 + y^2 + z^2 \leq 4.$$

Use spherical coordinates to calculate $\int_E xz$.

5. Calculate the area of the region in the \mathbb{R}^2 enclosed by the curve

$$(x + y)^2 + (x - 2y)^2 = 1.$$

6. Use a suitable change of variables to calculate the integral

$$\int_D \sin(9x^2 + 4y^2)$$

where D is the region in the plane bounded by the ellipse $9x^2 + 4y^2 = 1$.

7. Calculate the iterated integral

$$\int_0^{1/2} \int_x^{1-x} (x+y)^{17} (x-y)^{17} dy dx.$$

8. Let $a, b, c > 0$. Use a suitable change of variables to show that the solid ellipsoid

$$\left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right\} \leq 1.$$

has volume $4\pi abc/3$.

9. Let n be any positive integer.

- (a) Use induction on n to show that the unit simplex

$$\{x \in \mathbb{R}^n \mid x_1, \dots, x_n \geq 0, \quad x_1 + \dots + x_n \leq 1\}$$

has volume $1/n!$.

- (b) Show that the unit ball in \mathbb{R}^{2n} has volume $\pi^n/n!$. *Hint:* Use the change of variables replacing (x_1, \dots, x_{2n}) by $(r_1, \theta_1, \dots, r_n, \theta_n)$, where (r_i, θ_i) are the polar coordinates corresponding to (x_{2i-1}, x_{2i}) .

10. Find the area of the part of the plane

$$2x + 3y + 5z = 7$$

that lies in the first octant (where $x, y, z \geq 0$).

11. Find the area of the part of the hyperbolic paraboloid $z = x^2 - y^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

12. Calculate the $(n-1)$ -dimensional volume of the hypersurface

$$\{x \in \mathbb{R}^n \mid x_1, \dots, x_n \geq 0, \quad x_1 + \dots + x_n = 1\}.$$