Math H53 homework #4, suggested due date 11/3

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. Let (X, d) be a metric space. Recall that a function $g : X \to X$ is a **contraction** if there exists a constant $C < 1$ such that $d(g(x), g(x')) \leq Cd(x, x')$ for all $x, x' \in X$. Recall that the **contraction mapping theorem** asserts that every contraction on a nonempty complete metric space has a unique fixed point.

   (a) Show that if we replace the contraction condition by

   (*) $d(g(x), g(x')) < d(x, x')$ for all $x, x' \in X$ with $x \neq x'$,

   then g might not have a fixed point. *Hint:* Find a differentiable function $g : \mathbb{R} \to \mathbb{R}$ such that $|g'(x)| < 1$ for all x and the graph of g does not touch the line $y = x$.

   (b) Let (X, d) be a nonempty complete metric space and let $f : X \to X$ and $g : X \to X$ be two contractions. Show that there is a unique pair of points $p, q \in X$ such that $f(p) = q$ and $g(q) = p$.

2. Let $U \subset \mathbb{R}^n$ be an open set, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, and let $p \in U$. Recall that f is **differentiable** at p if there exists an $m \times n$ matrix A such that if $h \in \mathbb{R}^n$ is sufficiently small that $p + h \in U$, then

$$f(p + h) = f(p) + Ah + e(h) \tag{1}$$

   where

$$\lim_{h \to 0} \frac{\|e(h)\|}{\|h\|} = 0. \tag{2}$$

   (a) Show that if f is differentiable at p, then f is continuous at p, i.e.

$$\lim_{h \to 0} f(p + h) = f(p).$$

   (b) Show that if f is differentiable at p, then the matrix A is unique, i.e. there is only one matrix A for which (1) and (2) hold.

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$. Show that if the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are defined and continuous in a ball of radius δ containing 0, then f is differentiable at 0. *Hint:* Let $h = (h_1, h_2)$ with $\|h\| < \delta$. Use the mean value theorem to show that there are real numbers t_1, t_2 with $0 \le t_i \le h_i$ such that

$$f(h_1, 0) - f(0,0) = h_1 \frac{\partial f}{\partial x}(t_1, 0),$$
$$f(h_1, h_2) - f(h_1, 0) = h_2 \frac{\partial f}{\partial y}(h_1, t_2).$$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$. The **second partial derivatives** are defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

when the corresponding limit exists. If $i = j$ we denote this by $\frac{\partial^2 f}{\partial x_i^2}$. For example, when $n = 2$, we have

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

when this limit exists.

Clairaut's theorem asserts that if the second partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ are defined and continuous in a neighborhood of a point, then they are equal.

Prove this theorem in the two-dimensional case, i.e. if $B \subset \mathbb{R}^2$ is an open ball centered at the origin, if $f : B \to \mathbb{R}$, and if $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are defined and continuous in B, then

$$\frac{\partial^2 f}{\partial x \partial y}(0) = \frac{\partial^2 f}{\partial y \partial x}(0).$$

Moreover, both of these second partial derivatives agree with the limit

$$\lim_{h \to 0} \frac{f(h,h) - f(h,0) - f(0,h) + f(0,0)}{h^2}.$$

Hint: Assume that $h > 0$ is sufficiently small so that the rectangle with vertices $(0,0)$, $(h,0)$, $(0,h)$, and (h,h) is contained in B. For $0 \le t \le h$ define

$$g(t) = f(h,t) - f(0,t).$$

Use the mean value theorem to show that there exists $t \in [0, h]$ such that

$$g(h) - g(0) = h\left(\frac{\partial f}{\partial y}(h, t) - \frac{\partial f}{\partial y}(0, t)\right).$$

Use the mean value theorem again to show that there exists $s \in [0, h]$ such that

$$\frac{\partial f}{\partial y}(h, t) - \frac{\partial f}{\partial y}(0, t) = h\frac{\partial^2 f}{\partial x \partial y}(s, t).$$

4. Let p be a real number, and define a function $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ by

$$f(x, y, z) = (x^2 + y^2 + z^2)^p.$$

Find p such that f satisfies the **Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

5. Let a be a constant, and define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t, x) = \frac{1}{\sqrt{t}} e^{-x^2/(at)}.$$

Find a such that f satisfies the **heat equation**

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

6. Show that if $f, g : \mathbb{R} \to R$ are differentiable functions, and if a is a constant, then the function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(t, x) = f(x + at) + g(x - at)$$

is a solution of the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

7. Suppose that x, y, z are related by the equation

$$z = xy.$$

Use this equation to regard z as a function of x and y, to regard y as a function of x and z, and to regard x as a function of y and z. Show that

$$\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1.$$

8. Let $\gamma : [0, 1] \to \mathbb{R}^3$ be a smooth function (a parametrized curve), and write $\gamma(t) = (x(t), y(t), z(t))$. Suppose that the equation

$$xx'(t) + yy'(t) + zz'(t) = 0$$

holds for all t. If $x(0) = y(0) = z(0) = 3$ and $x(1) = y(1) = 2$, find $|z(1)|$.

9. A particle moves along the intersection of the surfaces

$$x^2 + y^2 + 2z^2 = 4, \qquad z = xy$$

in $\mathbb{R}^3$. Let $(x(t), y(t), z(t))$ denote the location of the particle at time t, and assume that this is a differentiable function of t. Suppose that $(x(0), y(0), z(0)) = (1, 1, 1)$ and $x'(0) = 1$. Calculate $y'(0)$ and $z'(0)$.

10. Suppose that z is implicitly defined as a function of x and y by the equation

$$xyz + z^3 = 33.$$

in a neighborhood of the point $(1, 2, 3)$. Show that z is differentiable at $(x, y) = (1, 2)$, and calculate $\partial z/\partial x$ and $\partial z/\partial y$ at $(x, y) = (1, 2)$.

11. Suppose we are given some data points $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane. We would like to find a "best fit" line $y = ax + b$ approximately going through these points. The **method of least squares** is to find a and b minizing the total squared error

$$e(a, b) = \sum_{i=1}^{n} (y_i - (ax_i + b))^2 .$$

Show that the minimum is given by a and b solving the two linear equations

$$a \sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i,$$
$$a \sum_{i=1}^{n} x_i^2 + b \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i.$$

12. (a) Find the point on the plane $\{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + 3z = 4\}$ which minimizes the distance to the point $(5, 6, 7)$, assuming that a distance minimizer exists.

(b) Find the point or points on the surface

$$\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2 - 3/2\}$$

that minimize the distance to the origin, assuming that a distance minimizer exists.

(c) Prove that the distance minimizers in (a) and (b) exist.

13. Find the minimum and maximum values of the function

$$f(x, y, z) = x + y + z$$

subject to the constraint

$$x^2 + y^2 + 2z^2 \leq 10.$$

14. Let $v \in \mathbb{R}^n$ be a nonzero vector. Use Lagrange multipliers to show that if $w \in \mathbb{R}^n$ is a unit vector, then $v \cdot w$ is maximized when w is a positive scalar multiple of v, and $v \cdot w$ is minimized when w is a negative scalar multiple of v. This gives another proof of the Cauchy-Schwarz inequality.

15. Show that if $x_1, \ldots, x_n$ are positive real numbers, then

$$(x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}$$

with equality if and only if $x_1 = \cdots = x_n$. *Hint:* Use Lagrange multipliers to maximize the left hand side while fixing the right hand side.