Math H53 homework #2, suggested due date 9/26 discussion session

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. (In this exercise you will complete the key step in the proof of the "rank-nullity theorem".) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Let $\{v^1, \ldots, v^k\}$ be a basis for $\mathrm{Ker}(T)$. (Here the superscript is an index, so these are k different vectors.) Let $v^{k+1}, \ldots, v^n$ be vectors in $\mathbb{R}^n$ such that $\{v^1, \ldots, v^n\}$ is a basis for $\mathbb{R}^n$. Show that $\{T(v^{k+1}), \ldots, T(v^n)\}$ is a basis for $\mathrm{Im}(T)$.

2. Check directly that the area of the parallelogram in $\mathbb{R}^2$ generated by the vectors (a, c) and (b, d) is the absolute value of the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (If you like you can just prove this in the special case when $a, b, c, d > 0$ and $c/a < d/b$, so that the vectors (a, c) and (b, d) are in the first quadrant and the former vector points to the right of the latter vector.)

3. If a, b are vectors in $\mathbb{R}^3$, their **cross product** is another vector in $\mathbb{R}^3$ defined by

$$a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_2, a_1b_2 - a_2b_1).$$

   (*Warning:* The cross product is only defined in three dimensions.)

   (a) Show that the cross product is anti-commutative: $a \times b = -b \times a$.

   (b) Show that $a \times b$ is perpendicular to both a and b.

   (c) Show that if c is a third vector in $\mathbb{R}^3$, then $(a \times b) \cdot c$ equals the determinant of the 3×3 matrix whose rows are a, b, c in that order.

   (d) Show that

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$

   where $\theta \in [0, \pi]$ is the angle between a and b.

4. Recall that if A is an $n \times n$ matrix, its **determinant** is defined by

$$\det(A) = \sum_{\sigma} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

Here the sum is over permutations $\sigma : \{1, \ldots, n\} \circlearrowleft$, and $\epsilon(\sigma) \in \{0, 1\}$ denotes the parity of σ. Show that the determinant behaves as follows under row operations:

(a) Multiplying one row of A by a scalar c multiplies the determinant by c.

(b) Switching two rows of A multiplies the determinant by -1.

(c) Adding a multiple of one row to another row does not change the determinant.

5. An **orthogonal** matrix is an $n \times n$ matrix A such that $(Av) \cdot (Aw) = v \cdot w$ for every pair of vectors $v, w \in \mathbb{R}^n$.

   (a) Check that the 2×2 rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix.

   (b) Show that an $n \times n$ matrix A is an orthogonal matrix if and only if $A^T A = I$. Here A^T denotes the **transpose** of A defined by $A^T_{ij} = A_{ji}$, and I denotes the $n \times n$ identity matrix.

   (c) Show that if A is an orthogonal matrix, then $\det(A) \in \{\pm 1\}$.

6. Let A be an $n \times n$ matrix (of real numbers) and let λ be a real number. We say that λ is an **eigenvalue** of A if there exists a nonzero[1] vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In this case the nonzero vector v is called an **eigenvector** of A.

   (a) Show that λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. (*Hint:* Start by showing that λ is an eigenvalue of A if and only if the matrix $A - \lambda I$ has nontrivial kernel. Use the fact that the determinant of a matrix is zero if and only if the matrix is not invertible.)

   (b) Find all of the eigenvalues of the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. For each eigenvalue, find a corresponding eigenvector.

[1] It is important that the vector v is nonzero, because the equation $Av = \lambda v$ is trivially true if v is the zero vector.

7. Let (X,d) and (Y,e) be metric spaces. Let $X \times Y$ denote the **Cartesian product** consisting of ordered pairs (x,y) where $x \in X$ and $y \in Y$. Define a distance function $D : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$D((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)^2 + d(y_1,y_2)^2}.$$

Show that $(X \times Y, D)$ is a metric space.

8. Let (X,d) be a metric space. Let $(x_1,x_2,\ldots)$ be a sequence in X with $\lim_{i\to\infty} x_i = L$. Let $(y_1,y_2,\ldots)$ be a sequence in X with $\lim_{i\to\infty} y_i = M$. Show that

$$\lim_{i\to\infty} d(x_i,y_i) = d(L,M).$$

9. Let $(x^1,x^2,\ldots)$ be a sequence of vectors in $\mathbb{R}^n$ with $\lim_{i\to\infty} x^i = v$. Let $(y^1,y^2,\ldots)$ be a sequence of vectors in $\mathbb{R}^n$ with $\lim_{i\to\infty} y^i = w$. Show that

$$\lim_{i\to\infty} x^i \cdot y^i = v \cdot w.$$

10. (As in the previous homework, this last problem is more of a "deep dive" and you don't have to do all of it.) Let (X,d) be a metric space. Recall that a **Cauchy sequence** is a sequence $(x_i)_{i\geq 1}$ in X such that for every $\epsilon > 0$, there exists a positive integer N such that $d(x_i,x_j) < \epsilon$ for all $i,j > N$. If $(y_i)_{i\geq 1}$ is another Cauchy sequence in X, let us define

$$(x_i)_{i\geq 1} \sim (y_i)_{i\geq 1}$$

to mean that for all $\epsilon > 0$, there exists a positive integer N such that $d(x_i,y_i) < \epsilon$ for all $i > N$.

   (a) Show that if $\lim_{i\to\infty} x_i = L$ and $\lim_{i\to\infty} y_i = M$, then $(x_i)_{i\geq 1} \sim (y_i)_{i\geq 1}$ if and only if $L = M$.

   (b) Show that $\sim$ is an equivalence relation[2] on the set of Cauchy sequences in X.

[2]If S is a set, a **relation** $\sim$ on S is a definition of when $a \sim b$ is true for $a,b \in S$. (More formally it is a subset $R \subset S \times S$, and $a \sim b$ means that $(a,b) \in R$.) The relation $\sim$ is an **equivalence relation** if for all $a,b,c \in S$ we have $a \sim a$ (reflexivity); $a \sim b$ if and only if $b \sim a$ (symmetry); and if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity). In this case, S is partitioned into **equivalence classes**. An equivalence class is a subset $A \subset S$ such that $a \sim b$ for all $a,b \in A$, while $a \not\sim b$ for all $a \in A$ and $b \in S \setminus A$.

(c) If $(x_i)_{i\geq 1}$ and $(y_i)_{i\geq 1}$ are two Cauchy sequences, define

$$\overline{d}((x_i)_{i\geq 1}, (y_i)_{i\geq 1}) = \lim_{i\to\infty} d(x_i, y_i).$$

Show that this limit exists. (You will need to use the fact that the real numbers are complete, i.e. every Cauchy sequence of real numbers has a limit.) Also show that this limit depends only on the equivalence classes of the Cauchy sequences $(x_i)_{i\geq 1}$ and $(y_i)_{i\geq 1}$. That is, if $(x_i)_{i\geq 1} \sim (x'_i)_{i\geq 1}$ and $(y_i)_{i\geq 1} \sim (y'_i)_{i\geq 1}$, then

$$\overline{d}((x_i)_{i\geq 1}, (y_i)_{i\geq 1}) = \overline{d}((x'_i)_{i\geq 1}, (y'_i)_{i\geq 1}).$$

(d) Define the **completion** of X, denoted by $\overline{X}$, to be the set of equivalence classes of Cauchy sequences in X. Show that the pair $(\overline{X}, \overline{d})$ is a metric space.

(e) Define a function $\imath : X \to \overline{X}$ by sending $x \in X$ to the constant sequence $(x_i)_{i\geq 0}$ where $x_i = x$ for all i. Show that the function $\imath$ is injective.

(f) Show that the metric space $(\overline{X}, \overline{d})$ is complete, i.e. every Cauchy sequence in $\overline{X}$ has a limit in $\overline{X}$. (*Hint:* This requires a clever trick called "diagonalization".)