Math 53 Homework #1, suggested due date 9/15/25 discussion session

The following exercises are suggested to help you understand the material. This homework will not be collected or graded.

1. Use vectors and dot products to prove the **law of cosines**: given a triangle in $\mathbb{R}^2$ with side lengths a, b, c, we have

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

   where θ is the angle between the sides of length a and b.

2. Use vectors and dot products to prove the **parallelogram law**: given a parallelogram in $\mathbb{R}^2$, the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of the four sides.

3. A list of vectors $v^1, \ldots, v^k$ in $\mathbb{R}^n$ is called **orthonormal** if $\|v^i\| = 1$ for each i, and $v^i \cdot v^j = 0$ for all $i \neq j$.

   (a) Show that if the vectors $v^1, \ldots, v^k$ are orthonormal, then they are linearly independent. *Hint:* The zero vector has length zero.

   (b) Find an orthonormal basis for the subspace of $\mathbb{R}^3$ defined by

$$\{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

   (c) (a bit tricky) Prove that every subspace of $\mathbb{R}^n$ has an orthonormal basis. *Hint:* Start with any basis, and use an inductive process to modify it to make it orthonormal without changing its span. If you get stuck, look up "Gram-Schmidt orthogonalization".

4. Let $S \subset \mathbb{R}^n$ be a subspace. Define the **orthogonal complement**

$$S^\perp = \{x \in \mathbb{R}^n \mid x \cdot y = 0 \text{ for all } y \in S\}.$$

   (a) Prove that $S^\perp$ is a subspace of $\mathbb{R}^n$.

   (b) Prove that every vector $x \in \mathbb{R}^n$ can be *uniquely* written as $x = y + z$ where $y \in S$ and $z \in S^\perp$. (The vector y is called the **orthogonal projection** of x onto S.) *Hint:* For existence, use the fact that S has an orthonormal basis.

(c) Let S be the subspace of $\mathbb{R}^3$ defined in the previous problem. Compute the orthogonal projections of the standard basis vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ onto S.

5. Let A be an $m \times n$ matrix. Define its **kernel** (or "nullspace")

$$\operatorname{Ker}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

Define its **image** (or "range" or "column space")

$$\operatorname{Im}(A) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n : Ax = y\}.$$

Show that $\operatorname{Ker}(A)$ is a subspace of $\mathbb{R}^n$, and $\operatorname{Im}(A)$ is a subspace of $\mathbb{R}^n$. Also, describe these subspaces as the span or the orthogonal complement of the span of the rows or the columns of the matrix.

6. Prove that matrix multiplication is associative: if A is an $m \times n$ matrix, if B is an $n \times p$ matrix, and if C is a $p \times q$ matrix, then we have an equality of $m \times q$ matrices

$$(AB)C = A(BC).$$

7. Let $S : \mathbb{R}^n \to \mathbb{R}^p$ and $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear maps.

(a) Check that the composition $S \circ T : \mathbb{R}^m \to \mathbb{R}^p$ is a linear map.

(b) Show that if A is the matrix corresponding to S, and if B is the matrix corresponding to T, then AB is the matrix corresponding to $S \circ T$.

8. Fix a positive integer n. A **permutation** is a bijection[1] σ from the set $\{1, \ldots, n\}$ to itself. A **transposition** is a permutation τ which switches two of the numbers $1, \ldots, n$ and fixes all of the others. That is, there are distinct numbers $i, j \in \{1, \ldots, n\}$ such that $\tau(i) = j$, $\tau(j) = i$, and $\tau(k) = k$ for $k \neq i, j$. The goal of this exercise is to prove that *every permutation can be expressed as a composition of either an even number of transpositions or an odd number of transpositions but not both.*

If σ is a permutation, define $\epsilon(\sigma) \in \{0, 1\}$ to be the parity[2] of the number of pairs i, j with $1 \leq i < j \leq n$ such that $\sigma(i) > \sigma(j)$.

[1] A **bijection** is a function which is both injective and surjective. Here this means that the list of numbers $\sigma(1), \ldots, \sigma(n)$ contains each of the numbers $1, \ldots, n$ exactly once.

[2] The **parity** of an integer is 0 if it is even and 1 if it is odd.

(a) Prove that every permutation can be expressed as a composition of at most $n-1$ transpositions. *Hint:* Use induction on n.

(b) Prove that if τ is a transposition, then $\epsilon(\tau) = 1$.

(c) (a bit tricky) Prove that if σ and ψ are permutations, then

$$\epsilon(\sigma \circ \psi) \equiv \epsilon(\sigma) + \epsilon(\psi) \mod 2.$$