Notes for Math H185

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1 Complex numbers

1.1 Basic definitions

Informally, the complex numbers are obtained by starting with the real numbers and introducing a new number i whose square is -1. More formally:

Definition 1.1. A **complex number** is an expression of the form $x + yi$ where x and y are real numbers[1]. Sometimes we write $x+iy$ instead to denote the same complex number. The set of all complex numbers is denoted by $\mathbb{C}$. We regard $\mathbb{R}$ as a subset of $\mathbb{C}$, where $x \in \mathbb{R}$ is identified with $x + 0i \in \mathbb{C}$. Similarly, when $x = 0$ we denote the complex number $0 + iy$ simply by iy. If $z = x + iy$ is a complex number, the **real part** of z is $\mathrm{Re}(z) := x$, and the **imaginary part** of z is $\mathrm{Re}(z) := y$. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, we define their sum and product by

$$\begin{aligned} z_1 + z_2 &:= (x_1 + x_2) + (y_1 + y_2)i, \\ z_1 z_2 &:= (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i. \end{aligned}$$

That is, we multiply by using the distributive law and replacing i^2 by -1.

The set $\mathbb{C}$, together with the addition and multiplication operations defined above, is a *field* (review the definition of this if necessary). One can verify most of the field axioms by straightforward calculations (however it is not always obvious from the above definition that these will work out!). The only axiom which is not straightforward to verify is that every nonzero complex number has a multiplicative inverse. To prove this, one observes that if $x + iy \neq 0$, then $(x + iy)(x - iy) = x^2 + y^2$ which is a nonzero real number, so

$$\frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i$$

[1] More formally, we could say that a complex number is an element of the set $\mathbb{R}^2$. We identify the ordered pair $(x, y) \in \mathbb{R}^2$ with the expression $x + yi$.

is a (the) multiplicative inverse of $x + iy$.

For an alternate proof that $\mathbb{C}$ is a field, you may recall a theorem from algebra stating that if F is a field and if $p \in F[t]$ is a nonconstant polynomial which is irreducible over F, then the quotient ring $F[t]/(p)$ is a field. Now take $F = \mathbb{R}$ and $p = t^2 + 1$. The polynomial $t^2 + 1$ is irreducible over $\mathbb{R}$ since -1 has no real square root, so by the above theorem $\mathbb{R}/(t^2 + 1)$ is a field. On the other hand you can check that every element of $\mathbb{R}/(t^2 + 1)$ has a unique representative of the form $x + yt$ with $x, y \in \mathbb{R}$. Replacing t by i, we obtain $\mathbb{C}$ as defined previously.

We represent complex numbers geometrically as points in the x, y plane. The x axis is called the **real axis**, and the y axis is called the **imaginary axis**. Addition of complex numbers is now given by the familiar addition of vectors.

To describe multiplication geometrically, we first make two additional definitions. If $z = x + iy$ is a complex number, define its **absolute value** by

$$|z| := \sqrt{x^2 + y^2}.$$

The terminology is justified because if z is real then $|z|$ as defined above agrees with the usual absolute value. More generally $|z|$ is always a nonnegative real number, namely the distance from the point (x, y) to the origin. If $z \neq 0$, then $(x/|z|, y/|z|)$ is a point on the unit circle, so it can be written as $(\cos\theta, \sin\theta)$ for some real number θ which is defined up to adding integer multiples of 2π. This number θ is called the **argument** of z, and denoted by $\arg z$, with the understanding that it is only defined mod 2π. Putting this together, we can write

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg z$. The numbers (r, θ) are just the usual polar coordinates of the point (x, y).

Now if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we calculate that

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)i\right) \\ &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \end{aligned} \tag{1.1}$$

Here we have used the angle addition formulas from trigonometry; a little later, using the exponential function, we will see an alternate proof of (1.1) which does not use (and instead implies) the angle addition formulas. In any case, the conclusion is that *to multiply two nonzero complex numbers, you multiply their absolute values and add their arguments*. Note in particular that

$$|z_1 z_2| = |z_1||z_2|.$$

(This also can be deduced without using trigonometry from (1.2) and (1.3) below).

We have one more definition to make. To motivate it, recall that the idea of the definition of $\mathbb{C}$ is to define i to be a square root of -1. One could then ask: "Which square root? Doesn't -1 have two square roots?" The point is that there is a symmetry which exchanges i with $-i$. Namely, if $z = x + iy \in \mathbb{C}$, define the **complex conjugate** of z by

$$\overline{z} = x - iy.$$

Note that z is real if and only if $\overline{z} = z$. It is easy to check that complex conjugation respects the field operations, i.e.

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}, \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2}. \end{aligned} \tag{1.2}$$

In fancier language, complex conjugation is an automorphism of the field extension $\mathbb{C} \supset \mathbb{R}$.

Note the useful identity

$$|z|^2 = z\overline{z}. \tag{1.3}$$

In particular, our previous calculation of the multiplicative inverse of z can now be written as

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

Also, the real and imaginary parts of a complex number can be recovered using complex conjugation via the formulas

$$\begin{aligned} \mathrm{Re}(z) &= \frac{z + \overline{z}}{2}, \\ \mathrm{Im}(z) &= \frac{z - \overline{z}}{2i}. \end{aligned}$$

1.2 Solving polynomial equations

The original motivation for introducing complex numbers was to find solutions to polynomial equations. Indeed, we have:

Theorem 1.2. *("Fundamental theorem of algebra") If $a_0, a_1, \ldots, a_n$ are complex numbers with $n > 0$ and $a_n \neq 0$, then the equation*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

We will prove this theorem later. Note that in fact there are exactly n solutions (some of which may be repeated). The reason is that if z_1 is a solution, and if we write $p(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[z]$, then we can divide the polynomial p by $z - z_1$ to obtain

$$p(z) = (z - z_1)q(z) + r \tag{1.4}$$

where $q(z)$ is a polynomial of degree $n-1$ and r is a constant. To be precise, equation (1.4) is an equality of polynomials, so it holds for all $z \in \mathbb{C}$. In particular, plugging in $z = z_1$ we find that $r = 0$. Now let z_2 be a zero of q and continue by induction to find that

$$p(z) = a_n(z - z_1)\cdots(z - z_n)$$

where $z_1, \ldots, z_n$ are complex numbers, some of which may be repeated. Then $p(z) = 0$ if and only if z is one of the numbers $z_1, \ldots, z_n$.

Of course it may be difficult to find the solutions $z_1, \ldots, z_n$ (or even impossible if one only allows certain standard operations). But let us consider some simple examples where we can work this out.

To start, suppose we want to find the square root of a nonzero complex number a, i.e. solve the equation

$$z^2 = a.$$

Writing $z = x + iy$ and $a = u + iv$, we need to solve the equations

$$\begin{aligned} x^2 - y^2 &= u, \\ 2xy &= v. \end{aligned}$$

If $v = 0$ then this is easy to solve; either $x = 0$ or $y = 0$ depending on the sign of u. If $v \neq 0$, one can solve this by using the second equation to eliminate y from the first equation. There are then four solutions for x, but only two of them are real. We omit further details; the important point here is that (when $a \neq 0$) you always get two solutions z, one of which is minus the other. Unlike with nonnegative real numbers, where given $x \geq 0$ we define $\sqrt{x}$ to be the nonnegative square root of x, for general complex numbers *there is no preferred square root*. More precisely, while one could of course pick some convention for selecting a square root of each complex number, there is no way to do so continuously. That is:

Proposition 1.3. *There does not exist a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = z$ for all $z \in \mathbb{C}$.*

Here "continuous" means with respect to the standard metric on $\mathbb{R}^2$, see the next section. We will prove this proposition shortly. First let us consider one more example.

Let n be a positive integer and consider the equation

$$z^n = 1.$$

To solve this, by taking absolute values we find that $|z| = 1$, so we can write

$$z = \cos\theta + i\sin\theta.$$

Then the equation to solve is

$$\cos(n\theta) + i\sin(n\theta) = 1.$$

Equating the real parts of both sides, we find that $\cos(n\theta) = 1$, so $n\theta$ is a multiple of 2π. The solutions to this are

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}.$$

The resulting numbers z are evenly spaced around the unit circle; they are called n^{th} **roots of unity**. For example, when $n = 3$, we find that the three cube roots of unity are

$$1, \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i.$$

One can also obtain these by factoring $z^3 - 1 = (z-1)(z^2+z+1)$; one can then find the two nontrivial cube roots by solving $z^2 + z + 1 = 0$ using the quadratic formula.

A similar argument shows that any nonzero complex number z has n different n^{th} roots, which are evenly spaced around the circle of radius $|z|^{1/n}$ centered at the origin.

Now let us prove Proposition 1.3. Suppose there exists a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = z$ for all $z \in \mathbb{C}$. This is actually already impossible if we restrict attention to the unit circle. To see this, given $\theta \in [0, 2\pi)$, note that we can write

$$f(\cos\theta + i\sin\theta) = \cos(\psi(\theta)) + i\sin(\psi(\theta)) \tag{1.5}$$

for a unique $\psi(\theta) \in \{\theta/2, \theta/2 + \pi\}$. Since f is continuous, it follows that $\psi : [0, 2\pi) \to [0, 2\pi)$ is continuous. Then $\theta \mapsto \psi(\theta) - \theta/2$ is a continuous function which takes values in $\{0, \pi\}$ and hence must be constant. This constant must equal $\psi(0)$, so we have $\psi(\theta) = \psi(0) + \theta/2$. Thus $\lim_{\theta\to 2\pi}\psi(\theta) = \psi(0) + \pi$. By (1.5) we get

$$\lim_{\theta\to 2\pi} f(\cos\theta + i\sin\theta) = -f(1).$$

Since f is continuous, we obtain $f(1) = -f(1)$, which is impossible since $f(1) \neq 0$.

It turns out that there is a way to change the setup so that a continuous square root function exists, by "thinking outside of the box". We will discuss this later when we talk about Riemann surfaces.

1.3 $\mathbb{C}$ as a metric space

Recall that if (x_1, y_1) and (x_2, y_2) are two points in $\mathbb{R}^2$, the Euclidean distance between them is defined by

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

If we denote (x_k, y_k) by z_k, then we can write this more concisely as

$$d(z_1, z_2) = |z_1 - z_2|.$$

This distance function makes $\mathbb{C}$ into a metric space (review what this means if necessary). In particular, we have the triangle inequality

$$|z_1 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|.$$

To prove this, writing $z = z_1 - z_2$ and $w = z_2 - z_3$ reduces to the inequality

$$|z + w| \leq |z| + |w|. \tag{1.6}$$

To prove (1.6), observe that

$$|z + w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\overline{w}), \tag{1.7}$$

so the inequality (1.6) is equivalent to

$$\mathrm{Re}(z\overline{w}) \leq |z||w|. \tag{1.8}$$

To prove this last inequality, we observe that

$$\mathrm{Re}(z\overline{w}) \leq |z\overline{w}| = |z||\overline{w}| = |z||w|.$$

Note that equality holds in this last step if and only if $z\overline{w}$ is real and non-negative. Thus *equality holds in the triangle inequality* (1.6) *if and only if one of z, w is a positive real multiple of the other*. (Recall that the triangle inequality (1.6) holds more generally in any inner product space; in the proof, $\mathrm{Re}(z\overline{w})$ in (1.7) and (1.8) is replaced by the inner product of z and w, and (1.8) is replaced by the Cauchy-Schwarz inequality.)

Since $\mathbb{C}$ is a metric space, some basic definitions from real analysis carry over directly to this setting. Let us state these precisely.

- If $a \in \mathbb{C}$ and $r > 0$, define

  $$B(a, r) := \{z \in \mathbb{C} \mid |z - a| < r\},$$

  the open disk of radius r centered at a. A subset $U \subset C$ is **open** if for every $a \in U$, there exists $r > 0$ such that $B(a, r) \subset U$. Of course, $B(a, r)$ itself is open.

- If $(z_1, z_2, \ldots)$ is a sequence of complex numbers and $L \in \mathbb{C}$, we say that $\lim_{n\to\infty} z_n = L$ if for all $\varepsilon > 0$ there exists an integer N such that if $n > N$ then $|z_n - L| < \varepsilon$.

- Let $A \subset \mathbb{C}$. The **closure** of A, denoted by $\overline{A}$, is the set of $z \in \mathbb{C}$ such that there exists a sequence $(z_1, z_2, \ldots)$ of points in A with $\lim_{n\to\infty} z_n = z$. Note that $A \subset \overline{A}$. For example, the closure of $B(a, r)$ is the closed disk

$$\overline{B(a,r)} = \{z \in \mathbb{C} \mid |z - a| \le r\}.$$

- The set A is **closed** if $\overline{A} = A$. Note that A is closed if and only if $\mathbb{C} \setminus A$ is open (prove it).

- Let $U \subset \mathbb{C}$ and $f : U \to \mathbb{C}$. If $a \in \overline{U}$ and $L \in \mathbb{C}$, we say that "$\lim_{z\to a} f(z) = L$" if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |z - a| < \delta$ and $z \in U$ then $|f(z) - L| < \varepsilon$. The function f is **continuous** if $\lim_{z\to a} f(z) = f(a)$ for all $a \in U$.

1.4 The Riemann sphere

There is one slight difference between the way limits work in real and complex analysis. Recall that in real analysis, if $f : \mathbb{R} \to \mathbb{R}$ and $L \in \mathbb{R}$, one says that "$\lim_{x\to+\infty} f(x) = L$" if for all $\varepsilon > 0$ there exists $M \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$. Likewise one says that "$\lim_{x\to-\infty} f(x) = L$ if for all $\varepsilon > 0$ there exists $M \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ whenever $x < M$.

In complex analysis there is no such distinction between "positive infinity" and "negative infinity". Rather, if $f : \mathbb{C} \to \mathbb{C}$ and $L \in \mathbb{C}$, one simply defines "$\lim_{z\to\infty} f(z) = L$" to mean that for all $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that $|f(z) - L| < \varepsilon$ whenever $|z| > M$. This definition also makes sense if f is only defined on a subset $U \subset \mathbb{C}$, provided that U is unbounded (i.e. the absolute values of elements of U can be arbitrarily large).

In real analysis, it is sometimes convenient to define an "extended real line" consisting of $\mathbb{R}$ together with two additional points, called "$+\infty$" and "$-\infty$", so that when x "approaches" $\pm\infty$ in the definition of limit, there is actually something to approach.

Likewise, we can define an "extend complex plane" $\widehat{\mathbb{C}}$ by adding to $\mathbb{C}$ a single point "∞", i.e.

$$\widehat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}.$$

The extended complex plane is also called the "Riemann sphere". The reason is that we can naturally identify it with a sphere via "stereographic projection" as follows.

Consider the unit sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

(Here z denotes a real coordinate in $\mathbb{R}^3$, not a complex number as usual.) Let $n = (0, 0, 1)$ denote the north pole of S^2. We now define a map

$$\phi : S^2 \setminus \{n\} \to \mathbb{R}^2$$

as follows. If $p = (x, y, z) \in S^2 \setminus \{n\}$, let ℓ denote the line in $\mathbb{R}^3$ through n and p. Define $\phi(p)$ to be the intersection of this line with the x, y plane, which we identify with $\mathbb{R}^2$ in the obvious way. (This intersection exists since the line ℓ is not horizontal.) One can check that

$$\phi(x, y, z) = \left(\frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

Also $\phi : S^2 \setminus \{n\} \to \mathbb{R}^2$ is a bijection (prove this). Note that if $p = (x, y, z)$ is close to n, so that z is close to 1, then $|\phi(p)|$ is very large. Indeed one can check from the above formula that

$$|\phi(p)|^2 = \frac{2}{1 - z} - 1. \tag{1.9}$$

We now extend ϕ to a bijection

$$\widehat{\phi} : S^2 \to \widehat{\mathbb{C}}$$

by defining $\widehat{\phi}(p) = \phi(p)$ for $p \in S^2 \setminus \{n\}$ and $\widehat{\phi}(n) := \infty$. The bijection $\widehat{\phi}$ has the following nice property: Let $f : \mathbb{C} \to \mathbb{C}$. Then composing with $\widehat{\phi}$ determines a function $g : S^2 \setminus \{n\} \to \mathbb{C}$. And we have $\lim_{z\to\infty} f(z) = L$ if and only if $\lim_{p\to n} g(p) = L$, where the latter limit is defined using the Euclidean distance on S^2.

A cool, but not completely trivial fact, is that if C is a circle in S^2 (i.e. the intersection of S^2 with a plane that is not tangent to it), then $\widehat{\phi}(C)$ is either a circle in $\mathbb{C}$ (if $n \notin C$) or the union of a line in $\mathbb{C}$ with ∞ (if $n \in C$). Conversely, every line or circle in $\mathbb{C}$ is obtained in this way. We will talk more later about maps that respect lines and circles.

2 Complex functions

2.1 Recollections from real analysis

Let U be an open subset of $\mathbb{R}$ and let $f : U \to \mathbb{R}$. Recall that if $a \in U$, then the **derivative** of f at a, denoted by $f'(a)$, is defined by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists; otherwise $f'(a)$ is undefined. If $f'(a)$ is defined then f is continuous at a, because

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}(x-a)\frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a}(x-a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= 0\cdot f'(a) \\
&= 0.
\end{aligned}$$

If n is a nonnegative integer, then the n^{th} derivative of f at a, denoted by $f^{(n)}(a)$, is defined inductively by $f^{(0)}(a) = f(a)$ and $f^{(n)}(a) = (f^{(n-1)})'(a)$ for $n > 0$.

Now consider the following conditions on our function $f : U \to \mathbb{R}$:

- f is **differentiable** if $f'(a)$ is defined for all $a \in U$.
- f is **continuously differentiable** or C^1 if f is differentiable and the function f' is continuous.
- f is **infinitely differentiable** or C^∞ if the n^{th} derivative $f^{(n)}$ is defined on all of U for all positive integers n.
- f is **real analytic** if for all $a \in U$ there exists $r > 0$ with $B(a, r) \subset U$ and a sequence of real numbers $(a_0, a_1, \ldots)$ such that for all $x \in B(a, r)$ we have
$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{2.1}$$
with the right hand side absolutely convergent.

Note that each of the conditions "differentiable", "continuously differentiable", "infinitely differentiable", and "real analytic" is stronger than the previous one. It is immediate that "continuously differentiable" implies differentiable. Also "infinitely differentiable" implies "continuously differentiable" because if the second derivative is defined then the first derivative must be continuous. Finally, "real analytic" implies "infinitely differentiable" because one can differentiate a power series term by term, i.e. if (2.1) holds on $B(a, r)$ with the right hand side absolutely convergent, then for all $x \in B(a, r)$ we also have

$$f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \tag{2.2}$$

with the right hand side absolutely convergent. (We will prove this later in the complex analysis setting, and the same proof works here.)

Moreover, each of the above four conditions is *strictly* stronger than the previous one. For example, there exists a function which is differentiable but not continuously differentiable: define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then f is differentiable. The only nontrivial part to check is that f is differentiable at 0; here we have

$$f'(0) = \lim_{x \to 0} x \sin(1/x) = 0$$

by the squeeze theorem. However f' is not continuous at 0 because for $x \neq 0$ we have

$$f'(x) = 2x \sin(1/x) - \cos(1/x).$$

It is easy to find a function which is continuously differentiable but not infinitely differentiable.

Finally, a standard example of a function which is infinitely differentiable but not real analytic is

$$f(x) := \begin{cases} e^{-1/x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

To prove that this is infinitely differentiable, the only nontrivial part is to check that $f^{(n)}(0)$ is defined (and of course equal to 0) for all $n > 0$. To prove this, use induction and suppose we know that $f^{(n)}(0) = 0$. Note that $f^{(n)}(x)$ equals $e^{-1/x}$ times some rational function of x when $x > 0$; it does not matter exactly what rational function this is. Then $x^{-1} f^{(n)}(x)$ is also $e^{-1/x}$ times some rational function of x, and one can use l'Hospital's rule to prove that the limit of this is 0 as $x \to 0^+$ (i.e. $e^{-1/x}$ goes to 0 much faster than any power of x^{-1} as $x \to 0^+$), so $f^{(n+1)}(0) = 0$. (Functions like f are very important in analysis for the construction of "cutoff functions".)

Now why is f not real analytic? Suppose f is real analytic. Then f has a power series expansion around 0, i.e. there exist $r > 0$ and $(a_0, a_1, \ldots)$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

whenever $|x| < r$. Now the coefficients $a_0, a_1, \ldots$ cannot all equal zero, or else $f(x)$ would equal zero for all $x \in (-r, r)$, a contradiction. So let n be the smallest nonnegative integer such that $a_n \neq 0$. It then follows from (2.2) that

$$f^{(n)}(0) = n! a_n.$$

This contradicts our previous observation that $f^{(n)}(0) = 0$.

Incidentally a similar argument shows that if U is a connected open set, if $f : U \to R$ is any real analytic function, and if f vanishes on some open subset of U, then f vanishes on all of U. This is called "unique continuation", because it implies that if f and g are two real analytic functions on a connected open set U, and if they agree on some open subset of U, then they are equal on all of U. That is, when a real analytic function can be extended to larger domain, the extension is unique. The complex analysis version of this will be important later.

2.2 Complex differentiation

Now suppose U is an open subset of $\mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. What is the appropriate notion of the derivative of f at a point $a \in U$?

First of all, the multivariable real analysis approach is to regard f as a map from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 2.1. We say that f is differentiable (over $\mathbb{R}$) at $a = (s, t)$ if there exists a 2×2 matrix A such that

$$\lim_{(x,y)\to(s,t)} \frac{f(x, y) - f(s, t) - A(x - s, y - t)}{|(x - s, y - t)|} = 0.$$

The matrix A is called the Jacobian of f at a and denoted by df_a.

If f is differentiable at a, then the partial derivatives of the components of f are defined at a, and these are the components of the Jacobian. That is, if we write $f = (u, v)$ then

$$df_a = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Bigg|_{(x,y)=a}$$

where u_x denotes $\partial u/\partial x$ and so forth. This follows directly from the definitions. For example, any polynomial in the coordinates x, y is differentiable on all of $\mathbb{R}^2$.

A less obvious fact is that if u and v have continuous partial derivatives near a, then f is differentiable at a. (This is less obvious, and might not hold if the partial derivatives are defined but not continuous, because the partial derivatives just describe how u and v change as one moves in the x direction or the y direction, while the definition of differentiability describes how u and v change as one moves in all directions.)

In any case, the above definition of differentiability is not what we want for complex analysis. Instead we want the following:

Definition 2.2. The (complex) **derivative** of f at a is defined by

$$f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a} \in \mathbb{C}$$

provided that this limit exists, in which case f is (complex) **differentiable**, or **holomorphic**, at a; otherwise $f'(a)$ is undefined. We inductively define $f^{(n)}(a)$ as before.

With this definition, the derivative of a constant is 0; the derivative of the identity map $f(z) = z$ is 1; and the usual rules for differentiating a sum or product hold (with the same proofs from real analysis). For example, if $f(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial, then f is differentiable on all of $\mathbb{C}$ with $f'(z) = n a_n z^{n-1} + \cdots + a_1$.

However a polynomial in the coordinates x, y will usually not be complex differentiable. Complex conjugation and absolute value are also not complex differentiable. The condition of complex differentiability in Definition 2.2 is much stronger than the condition of real differentiability in Definition 2.1. Let us clarify this.

Proposition 2.3. *f is complex differentiable at a point if and only if, at this point, $f = u + iv$ is differentiable over $\mathbb{R}$ and the partial derivatives of u and v satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x. \tag{2.3}$$

At such a point,

$$f' = u_x + i v_x = \frac{u_y + i v_y}{i}. \tag{2.4}$$

Proof. ($\Rightarrow$) Suppose f is complex differentiable at a. Then by definition,

$$\lim_{z \to a} \frac{f(z) - f(a) - f'(a)(z-a)}{z-a} = 0.$$

It follows that

$$\lim_{z \to a} \frac{f(z) - f(a) - f'(a)(z-a)}{|z-a|} = 0,$$

because we have not changed the absolute value of the function whose limit we are taking. Then by definition, f is differentiable over $\mathbb{R}$ at a, and its Jacobian at a is multiplication by $f'(a)$, regarded as a 2×2 matrix. Now if we write $f'(a) = \alpha + \beta i$, then multiplication by this complex number acting on $\mathbb{C}$ corresponds to the matrix $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ acting on $\mathbb{R}^2$. Equations (2.3) and (2.4) follow from this.

($\Leftarrow$) The proof of this is similar. $\square$

We can also write the Cauchy-Riemann equations (2.3) a bit more succintly as

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}. \tag{2.5}$$

When these equations are satisfied, we can write (2.4) more simply as

$$f' = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}. \tag{2.6}$$

From now on, unless otherwise stated, we always consider differentiation in the complex sense of Definition 2.2. To see how restrictive this is, note for example:

Proposition 2.4. *Let $U \subset \mathbb{C}$ be a connected open set and suppose $f : U \to \mathbb{R}$ is (complex) differentiable. Then f is constant.*

Proof. Since f is real, it follows from the Cauchy-Riemann equations (2.3) or (2.5) that $\partial f/\partial x = \partial f/\partial y \equiv 0$. One can then use line integrals to conclude that f is constant (see also Corollary 2.8). □

The chain rule for real functions carries over to the complex case:

Proposition 2.5. *Let f and g be complex functions defined on open subsets $U, V \subset \mathbb{C}$, let $a \in U$, and suppose $f(a) \in V$. If $f'(a)$ and $g'(f(a))$ are defined, then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. The proof from the real case carries over directly, but let us do this anyway as a review.

First note that since f is continuous at a, it follows that $g \circ f$ is defined in a neighborhood of a, so that it makes sense to consider $(g \circ f)'(a)$. Now the statement that f is differentiable at a with derivative $f'(a)$ is equivalent to the statement that if $a + h \in U$ then

$$f(a + h) = f(a) + f'(a)h + \varphi(h)$$

where $\lim_{h\to 0} \varphi(h)/h = 0$. Likewise, since g is differentiable at $f(a)$, if $f(a) + j \in V$ then

$$g(f(a) + j) = g(f(a)) + g'(f(a))j + \psi(j)$$

where $\lim_{j\to 0} \psi(j)/j = 0$. Now if $a + h$ is in the domain of $g \circ f$, let $j = f'(a)h + \varphi(h)$. Then by the above two equations, we have

$$g(f(a+h)) = g(f(a)) + g'(f(a))f'(a)h + \left[g'(f(a))\varphi(h) + \psi(f'(a)h + \varphi(h))\right].$$

It follows from basic properties of limits (review these if necessary) that the limit as $h \to 0$ of $1/h$ times the expression in square brackets is 0. Thus we have

$$(g \circ f)(a + h) = (g \circ f)(a) + g'(f(a))f'(a)h + \xi(h)$$

where $\lim_{h\to 0} \xi(h)/h = 0$. This is exactly what it means for $g \circ f$ to be differentiable at a with derivative $g'(f(a))f'(a)$. □

There is also another chain rule, for the derivative of a complex derivative function along a path. To prepare for the statement, if $[a, b]$ is an interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{C}$ is a complex-valued function, then we define its derivative (when defined) exactly as with real-valued functions $f : [a, b] \to \mathbb{R}$, i.e.

$$\frac{d}{dt} f(t) = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \in \mathbb{C}$$

where the limit is now over real numbers h (and if $t = a$ or $t = b$ we only take the right or left limit). Equivalently, if $f(t) = u(t) + iv(t)$, then

$$\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}.$$

Proposition 2.6. *Let $f : U \to \mathbb{C}$ be a (complex) differentiable function, and let $\gamma : [a, b] \to \mathbb{C}$ be a (real) differentiable path. Then*

$$\frac{d}{dt}(f(\gamma(t))) = f'(\gamma(t))\frac{d\gamma(t)}{dt}.$$

Proof. Write $\gamma(t) = u(t) + iv(t)$. By the usual chain rule from multivariable calculus (using complex-valued functions instead of real-valued functions, which makes no difference, e.g. because you can apply the real chain rule separately to the real and imaginary parts of f),

$$\begin{aligned}
\frac{d}{dt}(f(\gamma(t))) &= u'(t)\frac{\partial f}{\partial x}(\gamma(t)) + v'(t)\frac{\partial f}{\partial y}(\gamma(t)) \\
&= (u'(t) + iv'(t))\frac{\partial f}{\partial x}(\gamma(t)) \\
&= \frac{d\gamma(t)}{dt} f'(\gamma(t)).
\end{aligned}$$

Here in the second and third lines we have used equations (2.5) and (2.6) respectively. □

Remark 2.7. The two versions of the chain rule in Propositions 2.5 and 2.6 are special cases of the real chain rule in multiple variables: If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^l$ are differentiable, then $g \circ f$ is differentiable and $d(g \circ f)_a = dg_{f(a)} \circ df_a$, i.e. the Jacobian of the composition is the product of the Jacobians. This is proved the same ways as Proposition 2.5. To deduce Proposition 2.5 or 2.6, one just has to interpret the corresponding matrix multiplication as multiplication of complex numbers in the complex differentiable case.

We can deduce from Proposition 2.6 the following important fact:

Corollary 2.8. *Suppose $U \subset \mathbb{C}$ is open and connected (review the definition) and $f : U \to \mathbb{C}$ is differentiable and satisfies $f' \equiv 0$. Then f is constant.*

Proof. Since $f' \equiv 0$, it follows from the Cauchy-Riemann equations that $\partial f/\partial x \equiv \partial f/\partial y \equiv 0$. One can then use line integrals (which we will review later) to deduce that f is constant.

To give another proof without using line integrals, let $z, w, \in U$. Since U is open and connected, there is a differentiable path $\gamma : [a, b] \to U$ with $\gamma(a) = z$ and $\gamma(b) = w$ (review why). By Proposition 2.6, the function $f \circ \gamma : [a, b] \to \mathbb{C}$ has derivative 0 on all of $[a, b]$. We know from the mean value theorem that a real-valued function on an interval with derivative 0 is constant. Applying this theorem separately to the real and imaginary parts of $f \circ \gamma$ shows that $f \circ \gamma$ is constant. Hence $f(\gamma(a)) = f(\gamma(b))$, or equivalently $f(z) = f(w)$. Since $z, w \in U$ were arbitrary, this proves that f is constant on U. $\square$

Note that if U is not connected, then one can only conclude that f is constant on each connected component of U.

2.3 Complex analytic functions

Let $a \in \mathbb{C}$. A **power series** centered at a is a function of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \tag{2.7}$$

where $a_0, a_1, \ldots \in \mathbb{C}$ are constants. We regard this as a function of z whose domain is the set of z such that the series (2.7) converges absolutely. (Recall the definitions of absolute convergence, conditional convergence, and divergence.)

Lemma 2.9. *Let $r = \liminf |a_n|^{-1/n}$.*

(a) If $|z - a| < r$ then the series (2.7) is absolutely convergent.

(b) If $|z - a| > r$ then the series (2.7) is divergent.

Proof. (a) The idea is to compare with a geometric series. If $|z - a| < r$, choose s with $|z - a| < s < r$. Then for n sufficiently large, $|a_n|^{-1/n} > s$, so $|a_n| < s^{-n}$, so $|a_n(z-a)^n| < |(z-a)/s|^n$. Since $|(z-a)/s| < 1$, the geometric series $\sum_{n=0}^{\infty} |(z-a)/s|^n$ is absolutely convergent. Since all but finitely many terms of (2.7) are less than the terms of an absolutely convergent series, it follows that (2.7) is absolutely convergent.

(b) If $|z - a| > r$, then there are infinitely many nonnegative integers n such that $|a_n|^{-1/n} < |z - a|$, so $|a_n| > |z - a|^{-n}$, so $|a_n(z - a)^n| > 1$. Since the series (2.7) contains infinitely many terms with absolute value greater than one, it cannot be absolutely convergent. $\square$

The number r is called the **radius of convergence** of the series (2.7). If $|z-a| = r$, the series may or may not converge. (Examples...)

Definition 2.10. Let U be an open set in $\mathbb{C}$ and let $f : U \to \mathbb{C}$. The function f is (complex) **analytic** if for every $a \in U$, there exist $r > 0$ and $a_0, a_1, \ldots \in \mathbb{C}$ such that the power series expansion (2.7) is valid (with the right side absolutely convergent) whenever $|z-a| < r$.

Exercise 2.11. If f is defined by (2.7) on $B(a,r)$, where r is less than or equal to the radius of convergence, then f is analytic on $B(a,r)$ (i.e. f can be expanded as a power series around every point in $B(a,r)$, not just a).

Proposition 2.12. *If $f : U \to \mathbb{C}$ is analytic, then f is infinitely differentiable. Moreover, the coefficients of the power series expansion* (2.7) *are given by*

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

Proof. This follows from the following lemma and induction. □

Lemma 2.13. *Define $f : B(a,r) \to \mathbb{C}$ by the power series* (2.7) *and assume that the radius of convergence is at least $r > 0$. Then f is differentiable on $B(a,r)$, and its derivative is given by differentiating term by term:*

$$f'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(z-a)^n. \tag{2.8}$$

The series in (2.8) *also has radius of convergence at least r.*

One approach to proving (2.8), which appears in many complex analysis textbooks, is to write

$$f'(z) = \lim_{h \to 0} h^{-1} \sum_{n=0}^{\infty} a_n((z+h)^n - z^n),$$

expand the expression whose limit we are taking using the binomial theorem, and (using absolute convergence to justify reordering the sums) write the result as the right hand side of (2.8) plus h times something, which via some estimates can be shown to converge to 0. This is a good exercise and does not take too long. However it is not very satisfying because it leaves us without any general understanding of when one can differentiate an infinite sum of functions term by term. So here is a more systematic approach, using some basic facts about line integrals which we will review later.

Lemma 2.14. *Let $f_k : B(a,r) \to \mathbb{C}$ be functions for $k = 0, 1, \ldots$ such that f_k converges uniformly on closed balls to $f : B(a,r) \to \mathbb{C}$, and f_k' converges uniformly on closed balls to $g : B(a,r) \to \mathbb{C}$. Then f is differentiable and $f' = g$.*

Proof. Since f'_k is the derivative of a function for each k, it follows that g is also the derivative of a function, call it h. (This is because there is a criterion in terms of line integrals for when a function is a derivative, and this criterion is preserved under uniform limits of functions.) By adding a constant we may assume that $h(a) = f(a)$. To complete the proof we just need to check that $h(z) = f(z)$ for all $z \in B(a, r)$.

To so, let $z \in B(a, r)$ be given. It is enough to show that $|f(z) - h(z)| < \varepsilon$ for all $\varepsilon > 0$. Given $\varepsilon > 0$, choose k sufficiently large so that $|f_k(z) - f(z)| < \varepsilon/2$ and $|f'_k(w) - g(w)| < \varepsilon/2r$ for all w on the line segment from a to w. Then it follows (using line integrals again) that $|f_k(z) - h(z)| < \varepsilon/2$. So by the triangle inequality, $|f(z) - h(z)| < \varepsilon$ as desired. $\square$

(A similar argument in real analysis shows that the space of C^1 functions on a closed interval is a complete metric space. In our complex setting, we will see later that one can actually drop the assumption that f'_k converges uniformly and still conclude that f is differentiable. This certainly does not work in real analysis.)

Proof of Lemma 2.13. First observe that because $\lim_{n\to\infty} |n + 1|^{1/n} = 1$ (why?), the series in (2.8) has the same radius of convergence as the series in (2.7). In particular, the series (2.8) converges absolutely on $B(a, r)$. Now consider the k^{th} partial sum

$$f_k(z) = \sum_{n=0}^{k} a_n (z - a)^n.$$

Then the sequence of functions f_k converges uniformly to f on any closed ball in $B(a, r)$. (Review this.) And by what was just said, f'_k converges uniformly on closed balls to the right hand side of (2.8). Now invoke Lemma 2.14. $\square$

This completes the proof that a complex analytic function is differentiable, in fact infinitely differentiable. Now the miracle of complex analysis is that conversely, if a function is complex differentiable, then it is complex analytic:

Theorem 2.15. *Let U be an open subset of $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ is differentiable. Then f is analytic. Moreover, for any $a \in U$, if $B(a, r) \subset U$, then the power series expansion of f around a is valid on $B(a, r)$.*

Contrast this with real analysis, where a differentiable function need not even have a continuous derivative! We will prove Theorem 2.15 later using complex integration.

2.4 The exponential function, sine and cosine

Recall that for x real one has $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We would like to extend the definition to allow taking the exponential of a complex number. It is natural to simply use the same power series and for $z \in \mathbb{C}$ define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2.9}$$

This is sometimes also denoted by $\exp(z)$. This power series has radius of convergence ∞ (why?), so by Exercise 2.11 it defines an analytic function on all of $\mathbb{C}$. Of course it agrees with the usual exponential function when restricted to the real line.

It follows immediately from (2.9) that

$$e^0 = 1. \tag{2.10}$$

Also, Lemma 2.13 implies that

$$(e^z)' = e^z. \tag{2.11}$$

In fact, the general theory of ODE's implies that the function e^z satisfying (2.10) and (2.11) is unique[2], so one could take this as an alternate definition of e^z. (To give a more elementary proof of the uniqueness, if f and g are two functions satisfying $f' = f$ and $g' = g$, and $f(0) = g(0) = 1$, then applying Corollary 2.8 to $h(z) = f(z)g(-z)$ shows that $f(z)g(-z)$ is a constant, so $f(z)g(-z) = 1$ for all z. The same argument shows that $g(z)g(-z) = 1$, so $f(z) = g(z)$.)

As in the real case, we have the law of exponents

$$e^{z+w} = e^z e^w. \tag{2.12}$$

One way to prove this is to expand the left hand side using the binomial theorem, and rearrange the sum to obtain the right hand side (using absolute convergence to justify reordering the sums). But one can also prove this without doing that calculation as follows. Fix $w \in \mathbb{C}$ and define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^{z+w}$. Also define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z e^w$. Observe that

$$f(0) = e^w, \qquad f'(z) = f(z),$$

where the second equation uses the chain rule (Proposition 2.5). On the other hand we also have

$$g(0) = e^w, \qquad g'(z) = g(z).$$

[2] The relevant theorem about ODE's asserts that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (we are interested in the case $n = 2$), and if $p \in \mathbb{R}^n$, then there exists $\varepsilon > 0$ and a differentiable path $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $\gamma'(t) = f(\gamma(t))$ and $\gamma(0) = p$. Moreover, for any $\varepsilon > 0$, if a path γ as above exists then it is unique.

Since f and g both satisfy this differential equation, they must be equal, for the same reason that the exponential function is the unique function satisfying (2.10) and (2.11).

If $z = x + iy$ with x, y real, then by (2.12) we have

$$e^z = e^x e^{iy}.$$

So to understand e^z, we just need to understand e^{iy}. This is given by the famous formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.13}$$

for θ real.

To prove (2.13), we first need a rigorous definition of sin and cos. The geometric idea that we would like to capture is that if one starts at the point $(1, 0)$ in the plane and transverses the unit circle counterclockwise at unit speed for time t, then the coordinates of the point where one ends up are $(\cos t, \sin t)$. To turn this idea into equations we can work with, note that if one is traversing the unit circle counterclockwise at unit speed, and if one is at the point (x, y), then the velocity vector is obtained by rotating the vector (x, y) a quarter turn to the left, that is the velocity vector is $(-y, x)$. (Explain.) Thus the functions $\cos t$ and $\sin t$ should satisfy the system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}, \qquad \begin{pmatrix}\cos 0\\ \sin 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}. \tag{2.14}$$

Proposition 2.16. *There exist unique functions* $\sin, \cos : \mathbb{R} \to \mathbb{R}$ *satisfying* (2.14)*. These also satisfy* (2.13)*.*

Proof. To prove existence, note that by the chain rule (Proposition 2.6), we have

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}.$$

It follows from this that if we define $\cos\theta$ and $\sin\theta$ to be the real and imaginary parts of $e^{i\theta}$, then they satisfy (2.14) (and of course (2.13) also).

To prove uniqueness of the solution to (2.14), one can either invoke the uniqueness theorem for solutions to ODE's, or argue similarly to our proof of the uniqueness of the exponential function. □

Equation (2.13) has two useful corollaries. First, from (2.9) and (2.13) we obtain the familiar power series expansions

$$\begin{aligned}\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots,\\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots.\end{aligned} \tag{2.15}$$

Second, if α and β are any real numbers then by the law of exponents (2.12) we have

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}.$$

If we expand both sides using (2.13) and take the real and imaginary parts of both sides, then we obtain the angle addition formulas

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta. \end{aligned} \tag{2.16}$$

In fact, one can take (2.15) as a definition of $\sin\theta$ and $\cos\theta$ also for complex θ (although if θ is not real then $\sin\theta$ and $\cos\theta$ are usually not real either). Equivalently,

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

With this definition, a short calculation shows that the angle addition formulas (2.16) also hold when α and β are complex.

2.5 Logarithms

One would like to define a function log which is the inverse of e^z. Note that the domain of log cannot include 0 because the exponential function never vanishes (because $e^z e^{-z} = 1$). However the exponential function surjects onto $\mathbb{C} \setminus \{0\}$. This is because any complex number z can be written as $r(\cos\theta + i\sin\theta)$ with $r > 0$ and θ real, and then

$$z = e^{\log r + i\theta}$$

where $\log r$ for r a positive real number is understood to be the usual natural logarithm. However since θ is only defined modulo 2π, there are infinitely many complex numbers w such that $e^w = z$, which differ by integer multiples of 2π. If we want to define $\log z$, we need to pick one such w. Unfortunately, there is no way to do so continuously for all nonzero z. The proof is similar to that of Proposition 1.3 and we leave it as an exercise.

However we can define a continuous log function if we remove enough points from the domain. In particular the **principal branch** of the logarithm is a well-defined function

$$\log : \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \le 0\} \longrightarrow \mathbb{C}.$$

To define this, given z which is not a nonnegative real number, we can uniquely write $z = r(\cos\theta + i\sin\theta)$ with $r > 0$ and $\theta \in (-\pi, \pi)$, and we define $\log z = \log r + i\theta$. This function is differentiable and satisfies

$$(\log z)' = \frac{1}{z}. \tag{2.17}$$

To see why, observe that the restriction of the exponential function defines a bijection

$$\exp : \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in (-\pi, \pi)\} \xrightarrow{\simeq} \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}.$$

The principal branch of the logarithm is the inverse of this bijection. Now the inverse function theorem implies that if U and V are open subsets of $\mathbb{C}$ and if $f : U \to V$ is a holomorphic bijection with f' never vanishing[3], then $f^{-1} : V \to U$ is differentiable, and then by the chain rule we must have $(f^{-1})'(f(z)) = 1/f'(z)$. This implies (2.17).

2.6 Harmonic functions

Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be holomorphic. Write $f(z) = u(z) + iv(z)$. We will prove later that f is infinitely differentiable, so that all of the n^{th} partial derivatives of u and v are defined for all n. Accepting this for now, it follows from the Cauchy-Riemann equations (2.3) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

That is, $\Delta u = \Delta v = 0$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. A twice continuously differentiable function f satisfying $\Delta f = 0$ is called **harmonic**. We will discuss the geometric meaning of this condition later, after reviewing line integrals. For now, if u and v are harmonic functions such that $u + iv$ is holomorphic, we say that v is a **conjugate harmonic function** of u.

Proposition 2.17. *Let $U \subset \mathbb{C}$ be an open disc and let $u : U \to \mathbb{R}$ be a harmonic function. Then u has a conjugate harmonic function v, which is unique up to an additive constant.*

Proof. Without loss of generality the disc U is centered at the origin. We need to find v solving the equations $v_y = u_x$ and $v_x = -u_y$. We solve these equations by integration. The first equation holds if and only if

$$v(x, y) = v(x, 0) + \int_0^y u_x(x, t)dt.$$

The second equation holds when $y = 0$ if and only if

$$v(x, 0) = v(0, 0) - \int_0^x u_y(t, 0)dt.$$

[3]This last condition is actually redundant; we will see later that if $f : U \to V$ is a holomorphic bijection then f' can never vanish. This is another fact which has no analogue in real analysis, for example consider the bijection $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$

So we have no choice but to define

$$v(x,y) = v(0,0) - \int_0^x u_y(t,0)dt + \int_0^y u_x(x,t)dt,$$

which is unique up to the choice of the additive constant $v(0,0)$. To complete the proof, we have to check that this definition of v satisfies $v_x = -u_y$ for all $(x,y) \in U$, not just when $y = 0$. To do so, we can apply the fundamental theorem of calculus to the first integral, and differentiate under the integral sign (we will justify this a bit later) in the second integral, to get

$$\begin{aligned} v_x(x,y) &= -u_y(x,0) + \int_0^y u_{xx}(x,t)dt \\ &= -u_y(x,0) - \int_0^y u_{yy}(x,t)dt \\ &= -u_y(x,0) - (u_y(x,y) - u_y(x,0)) \\ &= -u_y(x,y). \end{aligned}$$

(Where did we use the assumption that U is a disk? For which other U will this work? We will see a more systematic approach to this later.) $\square$