Euler's formula

The purpose of this handout is to give a geometric explanation of Euler's formula, which states that if θ is a real number, then

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1}$$

We assume basic knowledge of calculus and of complex numbers.

We first review the definition of the exponential function and some of its more basic properties. If z is a complex number, define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots . \tag{2}$$

It follows from the ratio test that for every complex number z, this series is absolutely convergent. The reason is that the ratio between the absolute values of consecutive terms is

$$\frac{|z^{n+1}/(n+1)!|}{|z^n/n!|} = \frac{|z|}{n+1},$$

and for any given z, this ratio is less than 1 whenever n is large enough that $n+1 > |z|$.

We have

$$e^0 = 1, \tag{3}$$

because when $z = 0$, all the terms in (2) with positive powers of z vanish.

If z is a differentiable function of a real variable t, then

$$\frac{d}{dt} e^{z(t)} = z'(t) e^{z(t)}. \tag{4}$$

The reason is that because the series (2) is absolutely convergent, a theorem about power series allows us to differentiate it term by term to find that

$$\begin{aligned}\frac{d}{dt} e^{z(t)} &= \sum_{n=0}^{\infty} \frac{d}{dt} \frac{z(t)^n}{n!} = \sum_{n=1}^{\infty} \frac{n z(t)^{n-1} z'(t)}{n!} \\ &= z'(t) \sum_{n=1}^{\infty} \frac{z(t)^{n-1}}{(n-1)!} = z'(t) \sum_{m=0}^{\infty} \frac{z(t)^m}{m!} = z'(t) e^{z(t)}.\end{aligned}$$

The next basic property of the exponential function is the "law of exponents", which states that for any complex numbers z and w,

$$e^{z+w} = e^z e^w. \tag{5}$$

One can prove this directly from the power series (2) and the binomial theorem. A more elegant proof can be given using differential equations as follows. For each real number t, define

$$f(t) := e^{z+tw}.$$

Then by (4), the function f satisfies the ordinary differential equation

$$\frac{d}{dt}f(t) = wf(t), \qquad f(0) = e^z. \tag{6}$$

On the other hand, if we redefine $f(t)$ by the formula

$$f(t) := e^z e^{tw},$$

then by (4) again, this f also satisfies the ordinary differential equation (6). By the general theory of ODE's, the solution to (6) is unique. Thus the two solutions we have presented to (6) must be the same, that is

$$e^{z+tw} = e^z e^{tw}$$

for all t. Plugging in $t = 1$, we obtain the law of exponents (5).

Another basic property is that if $\overline{z}$ denotes the complex conjugate of z, then

$$e^{\overline{z}} = \overline{e^z}. \tag{7}$$

The reason is that to compute $\overline{e^z}$, we can conjugate all the terms in the power series (2), and this gives the power series for $e^{\overline{z}}$.

We now have all the ingredients in place to prove Euler's formula. For θ real, define

$$f(\theta) := e^{i\theta}.$$

The function f defines a curve in the complex plane parametrized by θ. By the various properties above, the absolute value of $f(\theta)$ is the square root of

$$|f(\theta)|^2 = f(\theta)\overline{f(\theta)} = e^{i\theta}e^{-i\theta} = e^0 = 1.$$

Thus the curve stays on the unit circle. By (4), the velocity vector of this parametrized curve is

$$\frac{d}{d\theta}f(\theta) = if(\theta). \tag{8}$$

The velocity vector has length

$$\left|\frac{d}{d\theta}f(\theta)\right| = |if(\theta)| = |f(\theta)| = 1.$$

Thus the parametrized curve moves at unit speed. Finally, equation (8) implies that the velocity vector $df(\theta)/d\theta$ points ninety degrees to the left of the vector $f(\theta)$. It follows that the parametrized curve moves counterclockwise around the unit circle. Since $f(0) = 1$, we conclude that *the point $e^{i\theta}$ in the complex plane is obtained by starting at $1 + 0i = (1, 0)$, and moving counterclockwise around the unit circle for distance θ.* This point is, by definition, $(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$. This proves Euler's formula (1).

Here are two quick corollaries of Euler's formula. First of all, if we plug it into (8), we obtain

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = i(\cos\theta + i\sin\theta).$$

Taking the real and imaginary parts of this equation, we recover the familiar facts

$$\frac{d}{d\theta}\cos\theta = -\sin\theta, \qquad \frac{d}{d\theta}\sin\theta = \cos\theta.$$

Second, Euler's formula for α, β, and $\alpha + \beta$, together with the formula

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$$

(which follows from the law of exponents), gives

$$\cos(\alpha + \beta) + i\sin(\alpha + \beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta).$$

Multiplying out the right hand side, and taking real and imaginary parts of the resulting equation, gives the angle addition formulas

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$