ON THE NUMBER OF SMOOTH CONICS TANGENT TO FIVE FIXED CONICS

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ABSTRACT. In section 1, we define the abelian groups $A_k(X)$ in a manner analagous to the homology groups $H_k(X)$ by considering cycles of subvarieties modulo rational equivalence. In section 2, we construct, for any Cartier divisor D on X and subvariety $V \subset X$, an intersection class $D \cdot [V] \in A_{k-1}(|D| \cap |V|)$ and state its basic properties. Bezout's theorem follows easily. In section 3, we discuss the centerpiece problem of the paper, namely computing the number of smooth plane conics tangent to five fixed smooth conics. We will see that Bezout's theorem is inadequate to solve the problem. In section 4, we give intersection-theoretic definitions of the Chern Classes and Segre Classes of an algebraic vector bundle, and use them to complete the computation begun in section 3.

Advice to Reader: Sections 1 and 2 are primarily a resume and definitions of facts, following chapters one and two of [1]. Section 3 loosely follows pages 749-756 of [2], though we adapt the material there to Fulton's algebraic framework. Finally, section 4 follows chapters three and four of [1]. While proofs are given, they can probably be skimmed without losing much. The most important parts of the paper are section 3 and the final pages of section 4, in which concrete calculations are made.

1. Algebraic Cycles

In this paper, all schemes will be of finite type over a fixed field k, and a variety will simply mean a reduced, irreducible scheme over k. In general, the theory does not require k to be algebraically closed, but when we work examples in $\mathbb{A}^n$ or $\mathbb{P}^n$, we will tacitly make this assumption. X, Y, and Z will typically denote arbitrary schemes over k, while V and W will always denote varieties. For any variety V, we denote its fraction field by $K(V)$, and if $V \subset X$, then $\mathscr{O}_{X,V}$ denotes the local ring of X at the generic point of V. For any ring R, we let R^* denote the multiplicative group of units in R, and we let $l(R)$ denote the length of R as an R-module (i.e. the length of a maximal filtration of R by ideals with simple quotients, provided this length is finite).

Definition 1. The group of *k-cycles* on X, denoted Z_kX, is the free abelian group generated by k-dimensional subvarieties of X. If $V \subset X$ is a k-dimensional subvariety, $[V]$ will denote the corresponding element of Z_kX. Also, Z_*X will denote the graded abelian group $\oplus_{k \geq 0} Z_kX$.

As with singular homology, the groups Z_kX are too large to be useful, so we will quotient by a relation similar to homological or homotopic equivalence. Recall that if W is a $(k+1)$-dimensional variety and $f \in K(W)$, then f defines a dominant map $W \to \mathbb{P}^1$. The fibers of this map can be thought of as cycles in Z_kW, and they move in a continuous family inside W. Thus, it is reasonable to insist that all the fibers of

f be considered 'homotopically equivalent.' To formalize this idea, we need to know how to associate cycles to the fibers of a map.

In fact, if X is any scheme with irreducible components X_i, we can define a cycle $[X]$ as follows. Let V_i be the variety obtained by giving X_i the reduced-induced structure. Then set $[X] = \sum_i n_i[V_i]$, where $n_i = l(\mathscr{O}_{X,X_i})$. This length is finite because $\mathscr{O}_{X,X_i}$ is the localization of a ring at a minimal prime ideal, and is therefore Artinian. Continuing with the idea above, we can now associate to f the cycle $[f^{-1}(0)] - [f^{-1}(\infty)]$, which we denote $[\text{div}(f)]$. Now we define the relation of rational equivalence by simply demanding all cycles of the form $[\text{div}(f)]$ be equivalent to zero.

Definition 2. We say a k-cycle $\sum_i [V_i] \in Z_k X$ is *rationally equivalent to zero* if there exists a collection (W_j, f_j), with each W_j a $(k+1)$-dimensional subvariety of X and each $f_j \in K(W_j)^*$, such that $\sum_j [\text{div}(f_j)] = \sum_i [V_i]$. We denote the set of k-cycles rationally equivalent to zero by $\text{Rat}_k(X)$.

Definition 3. We define the group of *cycle classes*, denoted $A_k(X)$, as the quotient $Z_k(X)/\text{Rat}_k(X)$.

Remark. In our definition of cycles and rational equivalence, we only made reference to reduced subvarieties of X. Thus, we can canonically identify $A_k(X)$ with $A_k(X_{red})$. When defining intersection products, we will often make implicit use of this observation by failing to specify the scheme structure on the argument of $A_k(\cdot)$.

Example 1. Let X be the parabola in $\mathbb{A}^2$ defined by $y - x^2$. We will compute the divisor $[\text{div}(f)]$ when $f = y$. We have $f^{-1}(0) = \text{Spec}\, \mathscr{O}_X/(f)$, and the only irreducible component of $f^{-1}(0)$ is the origin O. We claim $l_{\mathscr{O}_{X,O}/(f)}(\mathscr{O}_{X,O}/(f)) = 2$. To see this note that $f = x^2$ in $\mathscr{O}_{X,O}$ and that the maximal ideal $m_O \subset \mathscr{O}_{X,O}$ is generated by x. Thus, $\mathscr{O}_{X,O}/(f) = \mathscr{O}_{X,O}/m_O^2$. This ring has length two since $\mathscr{O}_{X,O} \supset m_O \supset m_O^2$ is a maximal filtration, or, equivalently, since it is a k-vector space of dimension two with basis $\{1, x\}$. Thus, $[\text{div}(f)] = 2O$. Since f was tangent to the parabola at O, the coefficient of O can be interpreted as the order of vanishing of f along O.

Example 2.
(a) If X is any n-dimensional variety, it is immediate from the definition that $A_n X = Z_n X = \mathbb{Z}$, generated by $[X]$, and $A_k X = 0$ for $k > n$. In particular, this determines $A_* X$ when X is a point.
(b) If $X = \mathbb{A}^n$, then any hypersurface $V \subset X$ is defined by the vanishing of a single regular function $f \in k[x_1, \ldots, x_n]$, so $V = \text{div}(f)$. We conclude that $A_{n-1} X = 0$.
(c) If $X = \mathbb{P}^n$, any rational function on X has the form f/g, where $f, g \in k[x_0, \ldots, x_n]$ are homogenous polynomials . Since any hypersurface $V \subset X$ is defined by the vanishing of a single homogenous polynomial, we have $[V] \sim d[H]$, where $H \subset X$ is a hyperplane, and d is the degree of V. Furthermore, $m[H] \sim 0$ implies $m[H] = [\text{div}(f/g)]$ for some $f, g \in k[x_0, \ldots, x_n]$. This implies $\deg(g) = 0$, which implies $m = 0$. We conclude $A_{n-1}(\mathbb{P}^n) = \mathbb{Z}$, generated by the cycle class of a hyperplane.

These elementary observations determine the cycle class groups for $\mathbb{A}^n$ and $\mathbb{P}^n$ when $n \leq 2$, but break down for larger n. For example, if one tries to compute $A_1(\mathbb{A}^3)$ following the idea of example (b), the existence of curves which are not global complete intersections (i.e. whose ideal is not defined by two equations) poses problems. Thus, as with the singular homology groups of topology, we must develop some formal properties before anything is computable. As this development would take us too far afield, we will simply take the following results as given.

$$A_k\mathbb{A}^n = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{otherwise,} \end{cases}$$

with A_nX generated by $[X]$.

$$A_k\mathbb{P}^n = \mathbb{Z} \text{ for } k = 0, \dots, n,$$

with $A_k\mathbb{P}^n$ generated by the cycle class of a k-dimensional linear subspace $[L^k]$ for each k.

Algebraic cycles and cycle classes push-forward and pull-back under proper and flat morphisms respectively. For definitions and properties of proper and flat morphisms, the reader may check II.4 and III.9 in [3]. Unfortunately, the definitions presented there are somewhat technical, and for our purposes the reader may rely on the following intuitions. Proper morphisms are, as in topology, maps taking compact sets to compact sets. Flat morphisms are maps whose fibers do not vary too wildly, rather like fibrations in topology. For example, the fibers of a flat morphism to an irreducible base scheme all have the same dimension. In fact, we will always assume that all the fibers of a flat morphism have the same dimension, regardless of the base scheme (this extra condition is usually expressed by saying a morphism is flat of relative dimension n for some n).

With this understanding, the reader may believe that closed immersions are proper and open inclusions are flat. In these cases, the proper push-forward and flat pull-back of cycles, as defined below, are just the obvious inclusion and restriction of subvarieties. The only other cases in which we shall need to use the push-forward and pull-back are with the projection from a vector bundle or $\mathbb{P}^n$-bundle to its base space (which is both flat and proper), and the projection of a blow-up to its base scheme (which is proper).

If $f : X \to Y$ is a proper morphism, then for any subvariety $V \subset X$, the image $W = f(V)$ is a subvariety of Y. If $\dim W = \dim V$ then the induced extension of function fields $K(V)/K(W)$ is finite. We define a *proper push-forward* homomorphism $f_* : Z_kX \to Z_kY$ by setting

$$f_*[V] = \begin{cases} [K(V) : K(W)][W] & \text{if } \dim W = \dim V \\ 0 & \text{if } \dim W < \dim V \end{cases}$$

and extending linearly. An algebraic lemma, which we will not prove here, shows that if $\alpha \in Z_kX$ is rationally equivalent to zero, then $f_*(\alpha)$ is rationally equivalent to zero. Thus, the push-forward map also induces a well-defined homomorphism $f_* : A_kX \to A_kY$. In general, we denote both the push-forward of cycles and of cycle classes by f_* but the intended meaning should be clear from context.

Suppose $f : X \to Y$ is flat of relative dimension n. We define a *flat pull-back* homomorphism $f^* : Z_kY \to Z_{k+n}X$ by setting $f^*[V] = [f^{-1}(V)]$ and extending linearly. A simple calculation shows that, with this definition, $f^*[Z] = [f^{-1}(Z)]$ for any closed subscheme $Z \subset Y$. An algebraic lemma, whose proof is omitted, shows that if $\alpha \in Z_kX$ is rationally equivalent to zero, then $f^*(\alpha) \in Z_{k+n}X$ is rationally equivalent to zero. Thus, there is an induced homomorphism $f^* : A_kX \to A_{k+n}Y$.

As with proper push-forward, we will let context determine whether we are pulling back cycles or cycle classes.

Flat and proper morphisms are preserved under base extension, so the following lemma makes sense.

Lemma 1. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

be a Cartesian diagram, with g flat and f proper. Then for all $\alpha \in Z_*X$, $f'_*g'^*\alpha = g^*f_*\alpha$ as cycles in Z_*Y'.

2. Intersection Products

Our goal in this section is to define a good notion of intersecting a cycle class with a Cartier divisor D. First, we must recall several basic definitions and facts about divisors (all this may be found in II.6 of [3]). A Cartier divisor on X consists of a collection $\{(U_i, f_i)\}$, where the U_i form an open cover of X, and the f_i are non-zero rational functions on X satisfying the condition that f_i/f_j is a unit on $U_i \cap U_j$. With this description, two collections $\{(U_i, f_i)\}$ and $\{(V_j, g_j)\}$ define the same Cartier divisor if there exists a common refinement of the two open covers, say W_k, such that $f_i|_{W_k} = u_{ijk}g_j|_{W_k}$ for all i, j, with the $u_{ijk} \in \mathscr{O}_X(W_k)$. Note that there is a natural group structure on the set of Cartier divisors, induced by multiplication of rational functions on each open set of the cover (given two divisors, one may assume they are defined by the same open cover after taking a common refinement). We say that a Cartier divisor is *principal* if it is defined by (X, f) for a single rational function on X. From now on, a *divisor* will always mean a Cartier divisor.

To any divisor, we can associate a cycle $[D] = \sum_i n_i[V_i]$, where the integers n_i are computed as follows: For each codimension one subvariety $[V_i] \in X$, choose some U_j meeting V_i, and write $f_j = g_j/h_j$ with both g_j and h_j regular functions on U_j. Then set $n_i = l(\mathscr{O}_{X,V_i}/(g_j)) - l(\mathscr{O}_{X,V_i}/(h_j))$. This is independent of the choice U_j precisely because f_i/f_j is a unit on $U_i \cap U_j$. Note that this map is actually a group homomorphism, and that it takes principal divisors to cycles that are rationally equivalent to zero. The map from Cartier divisors to cycles is, in general, neither surjective or injective, but it is actually a bijection in a large number of cases. For example, this holds whenever X is a nonsingular variety over $\mathbb{C}$.

A divisor is *effective* if each $n_i \geq 0$ in $[D]$. This is equivalent to saying D can be represented as $\{(U_i, f_i)\}$ with each f_i regular (not merely rational) on U_i, so that D is actually realized as a closed subscheme of X. Note that when D is effective, our definition of $[D]$ coincides with the cycle associated to the closed subscheme realized by D. For any k-cycle α, we will denote its *support* by $|\alpha|$ (the closed algebraic set consisting of the union of all the $[V_i]$ appearing with non-zero coefficient in α). For any divisor D, we will just write $|D|$ in place $|[D]|$.

We should emphasize that not all codimension-one closed subschemes of a scheme X are effective divisors (although this will be true on a nonsingular variety in light of our previous remark). To be a divisor, it must be locally defined by the vanishing of a single regular function. In defining intersection products for divisors rather than for arbitrary subschemes, we are completely by-passing all the pathologies that have given intersection theory such a chequered history (these arise when intersecting schemes which are not locally defined by the correct - i.e. codimension-many - number of equations). Later, we will see an example of codimension-one subvariety which is not a divisor, and this example will at least suggest why intersecting arbitrary subschemes is problematic.

Recall that to any divisor $\{(U_i, f_i)\}$ on X, there is an associated line bundle $\mathscr{O}_X(D)$ defined by gluing together trivializations over the various U_i, using the f_i/f_j as the transition functions on $U_i \cap U_j$. In fact, on a variety V, all line bundles come from some divisor via this construction. We say two divisors are *linearly equivalent* if their difference is principal, and it is not hard to check that two divisors determine isomorphic line bundles if and only if they are linearly equivalent. Note that if D and D' are linearly equivalent, then $[D]$ and $[D']$ are certainly rationally equivalent, but the converse need not hold in general.

Finally, we note that if D is a divisor on X and $j : V \to X$ is a morphism such that $j(V) \subsetneq |D|$, then one obtains a divisor $j^{-1}D$ on V by pulling back the local equations for D to get local equations over the open cover $j^{-1}U_i$. (The hypothesis $j(V) \subsetneq |D|$ is necessary to ensure that no rational function pulls back to zero.) By contrast, if $\mathscr{L}$ is a line bundle on X, then the pull-back bundle $j^*\mathscr{L}$ is always defined. This makes possible the following notion of 'intersecting with a divisor D.'

Definition 4. Let D be a divisor on X, and $j : V \to X$ the closed inclusion of a k-dimensional subvariety. The *intersection product* $D \cdot [V]$ is a cycle class in $A_{k-1}(|D| \cap |V|)$ defined as follows:

Case (1) If $[V] \subsetneq |D|$, then let $D \cdot [V] = [j^{-1}D]$

Case (2) If $[V] \subseteq |D|$, then let $D \cdot [V] = [D']$, for any divisor D' such that $j^*\mathscr{O}_X(D) = O_V(D')$.

If $\alpha = \sum_i [V_i]$ is any cycle in Z_*X, we define $D \cdot \alpha = \sum_i D \cdot [V_i]$ as a cycle class in A_*X.

Note that in Case (2), we use the fact that all line bundles on a variety come from some divisor, and that any two divisors determining the same line bundle are linearly equivalent (and therefore rationally equivalent), so we get a well-defined cycle class.

If D is effective, then the prescription of case (1) simply amounts to taking the scheme theoretic intersection $D \cap V$, and then taking the associated cycle $[D \cap V]$. Of course, when $V \subset |D|$, then the scheme-theoretic intersection $D \cap V$ has the wrong dimension so we must use a different prescription. When D is effective, the recipe given in Case (2) may be interpreted geometrically as follows: $O_X(D)|_D$ is the normal bundle $N_{D/X}$. The process of restricting this bundle to V, and then taking the associated divisor D', is the algebraic analogue of taking a generic section s of $N_{D/X}$ and letting D' be the subscheme defined by the vanishing of s on V. In other words, one perturbs D to a section of its normal bundle, and defines $D.[V]$ by intersecting this section with the copy of V embedded in the zero-section.

The reason we do not define $D \cdot [V]$ by the prescription of case (2) in general is that we want to obtain a cycle class in $|D| \cap |V|$ not simply on $|V|$. For example, if D and V are two distinct lines in $\mathbb{P}^2$, meeting at a point P, computing $D \cdot V$ by the method of case (2) would give an arbitrary point on the line V, while the method of case (1) gives precisely the cycle class $[P] \in A_0P$. This will be extremely important for us in section 3, when we need to keep track of the different connected components of $|D| \cap |V|$ on which $D \cdot [V]$ is supported. In section 4, however, we will be less concerned with precise support of a given cycle class, and will therefore be able to think of the intersection product as universally given by case (2).

Proposition 2. (Basic Properties of Intersection Classes)

(a) *Let D be a divisor on X, and let α, α' be k-cycles on X. Then*
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha'$$
in $A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|))$.

(b) *Let D, D' be divisors on X, and let α be a k-cycle on X. Then*
$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha$$
in $A_{k-1}((|D| \cup |D'|) \cap |\alpha|)$.

(c) (Flat Pull-Back) *Let D be a divisor on X, and let α be a k-cycle on X. If $f : X' \to X$ is flat of relative dimension n, and g is the restriction of f to $f^{-1}(|D| \cap |\alpha|)$, then*
$$f^{-1}D \cdot f^*\alpha = g^*(D \cdot \alpha)$$
in $A_{k+n-1}(f^{-1}(|D| \cap |\alpha|))$

(d) (Projection Formula) *Let D be a divisor on X, and let α be a k-cycle on X'. If $f : X' \to X$ is proper, g is the restriction of f to $f^{-1}(|D|) \cap |\alpha|$, then*
$$g_*(f^{-1}D \cdot \alpha) = D \cdot f_*(\alpha)$$
as cycle classes in $A_{k-1}(|D| \cap f(|\alpha|))$.

(e) *(*Commutativity) *If D and D' are divisors on X, then $D \cdot [D'] = D' \cdot [D]$.*

(f) *If D and D' are linearly equivalent divisors on X, and α is a k-cycle on X, then $D \cdot \alpha$ and $D' \cdot \alpha$ represent the same cycle class in $A_{k-1}(\alpha)$.*

Remark. Parts (c) and (d) of the proposition do not make sense as stated because the pull-back of a divisor is not defined for arbitrary morphisms $f : X' \to X$. The correct amendment is to define a generalized notion of divisor which still determines intersection products, but also has a universally defined pull-back. Since we do not have space to spell out this generalization, these parts may be assumed to carry the extra hypothesis that X' and X are varieties and $f(X') \subsetneq |\alpha|$ (in which case $f^{-1}D$ is defined). This extra hypothesis will in fact hold in the concrete cases where we make appeal to this lemma. The general framework of characteristic classes established in section 4, however, also relies on this generalized notion of pull-back so we recommend the reader simply accept this ambiguity in the statements of lemmas and propositions for the time being.

Proof. (a) is obvious from the definition, and in consequence of (a) other parts of the proposition may be checked in the case $\alpha = [V]$. Then (b) and (f) boil down to the fact that the map from divisors to cycles is a homomorphism and the fact that linearly equivalent divisors determine the same line bundle. Parts (c) and (d) require some technical algebraic lemmas to show that the multiplicities on both sides of the equation are correct, but the reader may verify that the stated equalities at least make sense set-theoretically.

Curiously, (e) turns out to be the key property of the intersection product requiring genuine proof. Note that it is obvious if, say, D and D' are both effective with no irreducible components in common (for then both sides are simply the cycle associated to the scheme-theoretic intersection $[D \cap D']$) or if D and D' are equal. In general, one must blow-up X along $D \cap D'$ and use (c) to reduce to these special cases. Details may be found in section 2.3 of [1]. □

An importance consequence of (c) is that intersection products may be computed locally on each connected component of $|D| \cap |\alpha|$. To be more precise, suppose $Z_1, \ldots, Z_n$ are the connected components of $|D| \cap |\alpha|$. Then there is a natural decomposition $A_*(|D| \cap |\alpha|) = A_*(Z_1) \oplus \ldots \oplus A_*(Z_n)$. When we form the intersection product $D \cdot \alpha \in A_*(|D| \cap |\alpha|)$, it is natural to expect that the piece of $D \cdot \alpha$ supported on Z_i should simply be the intersection product $D|_{U_i} \cdot \alpha|_{U_i}$, where U_i is an open neighborhood containing Z_i and none of the other connected components. If f is the inclusion of the open set U_i into X, then (c) says exactly that.

Example 3. Let X be singular cone in $\mathbb{A}^3$ defined by the equation $z^2 - xy$. Note that X contains two lines l and l', defined by $x = z = 0$ and $y = z = 0$ respectively. Let D be the principal divisor on X defined by the regular function x. One easily computes $[\mathrm{div}(x)] = 2l$. (The zero-locus of x is l, and in the local ring $\mathscr{O}_{X,l}$ the maximal ideal m_l is generated by z, so $\mathscr{O}_{X,l}/(x) = \mathscr{O}_{X,l}/m_l^2$). Now we compute $D \cdot l'$.

In this case $l' \subsetneq |D|$, so we simply pull-back the equation defining D to l', and take the cycle class of the corresponding divisor on l'. If we let P be the origin in $\mathbb{A}^3$, then we see at once that x vanishes on $P \in l'$ with multiplicity one, so $D \cdot l' = [P] \in A_0 P$.

In fact, we can use this result to show that there does not exist a divisor D' with $[D'] = [l']$. For if such D' existed, we must have $D' \cdot [l] = [P]$, since the scheme-theoretic intersection of l and l' is just P with multiplicity one. Then, using proposition 1 parts (e) and (a), we would have

$$[P] = D \cdot [D'] = D' \cdot [D] = D' \cdot 2[l] = 2D' \cdot [l] = 2[P],$$

a contradiction. This example may be taken as a first indication of the difficulty in constructing more general intersection products between subschemes which are not complete intersections (i.e. not cut out by a succession of divisors).

We are ready to consider the specific case of intersection theory in $\mathbb{P}^n$. Note that by proposition 1(e), the expression $P(D_1, \ldots, D_n) \cdot \alpha$, where P is a polynomial in n variables, gives a well-defined element of $A_*(|D_1| \cap \ldots \cap |D_n| \cap |\alpha|)$. When $D_1, \ldots, D_n$ are divisors on X, we will also use the notation $P(D_1, \ldots, D_n)$ as a short-hand for the cycle class $P(D_1, \ldots, D_n) \cdot [X]$.

Definition 5. Given a collection of effective divisors $D_1, \ldots, D_k$ on an n-dimensional variety X, we say that they intersect *properly* if every irreducible component of their scheme-theoretic intersection $D_1 \cap \ldots \cap D_n$ has dimension $n - k$. If, in addition, each irreducible component of $D_1 \cap \ldots \cap D_n$ appears with multiplicity one in $[D_1 \cap \ldots \cap D_n]$, then we say that they intersect *transversely*.

Definition 6. The *degree* of k-cycle $\alpha \in Z_k \mathbb{P}^n$ is the unique integer d, such that α is rationally equivalent to $d[L^k]$, where $[L^k]$ is the class of a k-dimensional linear subspace.

Theorem 3 (Bezout's Theorem)**.** *Let $D_1, \ldots, D_k$ be effective divisors (i.e. hypersurfaces) on $\mathbb{P}^n$. Then*

$$\deg(D_1 \cdot \ldots \cdot D_n) = \deg(D_1) \cdot \ldots \cdot \deg(D_n).$$

Proof. Using proposition 1(f), we see that both sides of the equation depend only on the linear equivalence classes of the $[D_i]$, so we may replace each D_i by $\deg(D_i)L_i$, where the L_i are hyperplanes in $\mathbb{P}^n$ which intersect transversely.

Now, as we compute the intersection $D_1 \cdot \ldots \cdot D_k$, we are simply taking scheme-theoretic intersections at every step. Clearly, $L_1 \cdot \ldots \cdot L_k = [\mathbb{P}^{n-k}]$, the class of $(n-k)$-dimensional linear subspace. Since the intersection product is a homomorphism, we conclude

$$\deg(D_1)[L_1] \cdot \ldots \cdot \deg(D_k)[L_k] = \deg(D_1) \cdot \ldots \cdot \deg(D_n)[\mathbb{P}^{n-k}].$$

Thus,

$$D_1 \cdot \ldots \cdot D_n \sim \deg(D_1) \cdot \ldots \cdot \deg(D_n)[\mathbb{P}^{n-k}],$$

which says $\deg(D_1 \cdot \ldots \cdot D_n) = \deg(D_1) \cdot \ldots \cdot \deg(D_n)$ as desired. □

3. The Problem of Counting Conics

In this section, we explain a problem which was a great stimulus to the development of intersection theory, precisely because it exposed the inadequacy of Bezout's theorem. Fix five smooth conics $C_1, \ldots, C_5$ in the projective plane $\mathbb{P}^2$. How many smooth conics will be tangent to all five? In the next few paragraphs, we will formalize this problem. We reduce the problem to a few key calculations, which will be carried out at the end of section 4 using characteristic classes. In fact, with generous hindsight, it is possible to see these difficult calculations as motivation for establishing intersection theory within the framework of characteristic classes.

Plane conics are naturally parametrized by $\mathbb{P}^5$ via the correspondence

$$a_0x^2 + a_1y^2 + a_2xy + a_4xz + a_4yz + a_5z^2 \to [a_0 : a_1 : a_2 : a_3 : a_4 : a_5].$$

The locus of singular conics is parametrized by a hypersurface $Y \subset \mathbb{P}^5$. In fact Y is just the image of the closed immersion $j : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ defined by

$$[a : b : c] \times [a' : b' : c'] \to [aa' : bb' : ab' + ba' : ac' + ca' : bc' + cb' : cc'],$$

since a conic is singular if and only if it degenerates into two lines. The locus of double lines $Z \subset \mathbb{P}^5$ is the image of the embedding $i : \mathbb{P}^2 \to \mathbb{P}^5$ defined by

$$[a : b : c] \to [a^2 : b^2 : 2ab : 2ac : 2bc : c^2].$$

We shall say two conics are *tangent* if they intersect at some point with multiplicity greater than one, where the multiplicity of intersection of two conics f and g at a point P is defined as the $\dim_k(\mathscr{O}_{\mathbb{P}^2,P}/(f,g))$. Observe that, with this definition, any double-line f is tangent to every other conic g. Indeed, after a change of coordinates we may assume $f = x^2$ and $P = (0,0)$. Then the dimension of $\mathscr{O}_{\mathbb{P}^2,P}/(f,g) > 1$ since the $k[x,y]$-span of f and g can contain at most a one-dimensional subspace of the k-vector space spanned by $\{1, x, y\}$. This fact will be the major sticking point in the problem we wish to solve.

Let $V_C \subset \mathbb{P}^5$ be the locus of conics which are tangent to a fixed smooth conic C. Then the set of smooth conics tangent to $C_1, \ldots, C_5$ is simply the set $V_{C_1} \cap \ldots V_{C_5} - Y$. In general, this set will not be finite. Is it, however, reasonable to expect that the set will be a fixed finite integer for generic $C_1, \ldots, C_5$? (We use the adjective *generic* to mean that the locus of conics $(C_1, \ldots, C_5) \in \mathbb{P}^5_5$ satisfying our stated condition is open in $\mathbb{P}^5_5$.) Suppose we could prove that for generic $C_1, \ldots, C_5$,

(1) Each V_{C_i} is hypersurface of degree six in $\mathbb{P}^5$,

(2) The hypersurfaces V_{C_i} meet transversely at every component in the nonsingular locus, and
(3) The hypersurfaces V_{C_i} contain no singular conics in common.

Taken together, (2) and (3) imply that every irreducible component of the scheme-theoretic intersection $V_{C_1} \cap \ldots \cap V_{C_5}$ is a point corresponding to a nonsingular conic, and that each point occurs with multiplicity one in the intersection product $V_{C_1} \cdot \ldots \cdot V_{C_5}$. (Recall the notation $D_1 \cdot \ldots \cdot D_n$ just means the cycle class $D_1 \cdot \ldots \cdot D_n \cdot [X]$, where the D_i are divisors on X.) Thus, the number of nonsingular conics tangent to $C_1, \ldots, C_5$ would be the degree of this intersection product. By Bezout's theorem and (1), this degree must be 6^5.

Alas, we have already seen that (3) fails because every V_C must contain the surface Z of double-lines. Thus, the best we can hope for is that for $C_1, \ldots, C_5$ generic:

(3†) The hypersurfaces V_{C_i} contain no singular conics in common, except the surface of double-lines.

In fact, (1), (2), and (3†) hold (generically). The arguments are ad hoc, elementary, and will be omitted. For the rest of this section, we assume our choice of $C_1, \ldots, C_5$ satisfies these assumptions.

Now our situation is this: the scheme-theoretic intersection $V_{C_1} \cap \ldots \cap V_{C_5}$ consists of Z and a residual finite set of nonsingular conics tangent to $C_1, \ldots, C_5$. By the discussion following proposition 1, the intersection product $V_{C_1} \cdot \ldots \cdot V_{C_5}$ can be computed on each connected component separately. Thus, $V_{C_1} \cdot \ldots \cdot V_{C_5}$ will be the sum of some class $\alpha \in A_0 Z$ and the classes $[P]$ (in $A_0 P$) for each point P in the residual set. Thus, the number we seek is $6^5 - \deg(\alpha)$.

Now our problem is reduced to computing $\deg(\alpha)$, but this is hardly straightforward. The key idea is to blow-up X along Z, and use the compatibility of intersection products with proper push-forward. Blowing-up is a general method for reducing problems about arbitrary subvarieties $Z \subset X$ to problems about divisors. We shall use the following facts about blow-ups (for the definition of blowing-up and references for the following statements, please see II.7 in [3]).

Proposition 4. Properties of Blow-Ups

1 *When X is a variety, and $Z \subset X$ is a nonsingular subvariety, the blow-up $\tilde{X}$ of X along Z is a variety.*
2 *There is a proper morphism $\pi : \tilde{X} \to X$, which is an isomorphism when restricted to the open set $\pi^{-1}(X - Z)$.*
3 *The closed subscheme $\tilde{Z} = \pi^{-1}(Z)$ is an effective divisor on $\tilde{X}$, called the exceptional divisor.*
4 *For any effective divisor D on X, $\pi^{-1} D = \tilde{D} + mE$, where $\tilde{D}$ is the blow-up of D along $D \cap Z$ and m is the multiplicity of Z in D. (This multiplicity may be defined as the greatest power of the maximal ideal of $\mathscr{O}_{X,Z}$ containing a local equation for D in a neighborhood of Z.)*

Since $\pi : \tilde{X} \to X$ is proper and $\pi_*([\tilde{X}]) = [X]$, proposition 2(d) says

$$V_{C_1} \cdot \ldots \cdot V_{C_5} = \pi_*(\pi^{-1}(V_{C_1}) \cdot \ldots \cdot \pi^{-1}(V_{C_5}))$$

By blow-up property (4) above, we have $\pi^{-1}(V_{C_i}) = \tilde{V}_{C_i} + m\tilde{Z}$, where m is the multiplicity of Z in V_{C_i}. (Actually, this multiplicity is two, but this computation is also omitted.) Now, using basic properties of the intersection product, we compute

$$\begin{aligned}
\tilde{V}_{C_1}\cdot\ldots\cdot\tilde{V}_{C_5} &= (\pi^{-1}(V_{C_1})-m\tilde{Z})\cdot\ldots\cdot(\pi^{-1}(V_{C_5})-m\tilde{Z})\\
&= \pi^{-1}(V_{C_1})\cdot\ldots\cdot\pi^{-1}(V_{C_5})\\
&- \sum_{i<j<k<l} m\tilde{Z}\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j})\cdot\pi^{-1}(V_{C_k})\cdot\pi^{-1}(V_{C_l})\\
&+ \sum_{i<j<k} m^2\tilde{Z}^2\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j})\cdot\pi^{-1}(V_{C_k})\\
&- \sum_{i<j} m^3\tilde{Z}^3\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j}) + \sum_i m^4\tilde{Z}^4\cdot\pi^{-1}(V_{C_i}) - m^5\tilde{Z}^5,
\end{aligned}$$

as cycle classes in $A_0\tilde{X}$. This significance of this messy equation is illuminated once we realize that for $C_1,\ldots,C_5$ generically chosen,

(4) The set-theoretic intersection $\tilde{V}_{C_1}\cap\ldots\cap\tilde{V}_{C_5}$ is disjoint from $\tilde{Z}$.

Now we see that the left side of the equation is a cycle class supported off $\tilde{Z}$. On the right side of the equation, every term past the first is supported on $\tilde{Z}$. As for $\pi^{-1}(V_{C_1})\cdot\ldots\cdot\pi^{-1}(V_{C_5})$, it must decompose into a sum of cycle classes supported on $\tilde{Z}$ and the inverse images of the residual points. Let the piece supported on $\tilde{Z}$ be denoted $(\pi^{-1}(V_{C_1})\cdot\ldots\cdot\pi^{-1}(V_{C_5}))^{\tilde{Z}}$. Now if we restrict our attention to what the equation says regarding cycle classes supported on $\tilde{Z}$, we have

$$\begin{aligned}
0 &= (\pi^{-1}(V_{C_1})\cdot\ldots\cdot\pi^{-1}(V_{C_5}))^{\tilde{Z}}\\
&- \sum_{i<j<k<l} m\tilde{Z}\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j})\cdot\pi^{-1}(V_{C_k})\cdot\pi^{-1}(V_{C_l})\\
&+ \sum_{i<j<k} m^2\tilde{Z}^2\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j})\cdot\pi^{-1}(V_{C_k})\\
&- \sum_{i<j} m^3\tilde{Z}^3\cdot\pi^{-1}(V_{C_i})\cdot\pi^{-1}(V_{C_j}) + \sum_i m^4\tilde{Z}^4\cdot\pi^{-1}(V_{C_i}) - m^5\tilde{Z}^5
\end{aligned}$$

Let p be the restriction of π to $\tilde{Z}$. Applying p_* to the equation above, and using proposition 1(c), we obtain

$$p_*(\pi^{-1}(V_{C_1})\cdot\ldots\cdot\pi^{-1}(V_{C_5}))^{\tilde{Z}} = \sum_{i<j} m^3 V_{Ci}\cdot V_{Cj}\cdot p_*([\tilde{Z}^3]) - \sum_i m^4 V_{Ci}\cdot p_*([\tilde{Z}^4]) + m^5 p_*([\tilde{Z}^5]),$$

as cycle classes in A_0Z. Note that the terms from the second and third line vanished because $p_*([\tilde{Z}]) = p_*([\tilde{Z}^2]) = 0$ (we are pushing forward to a two-dimensional space, while $[\tilde{Z}]$ and $[\tilde{Z}^2]$ are cycles in dimensions four and three respectively).

The expression on the left of this equation is just the piece of $V_{C_1}\cdot\ldots\cdot V_{C_5}$ supported on Z. Though it is not obvious, the expression on the right is intrinsically computable. It may help to think of it as the zero-dimensional component of

$$(1+6H)^5\cdot(m^5\pi_*(\tilde{Z}^5) - m^4\pi_*(\tilde{Z}^4) + m^3\pi_*(\tilde{Z}^3)),$$

where H is a hyperplane divisor in $\mathbb{P}^5$. Now we see the problem boils down to computing $(m^5\pi_*(\tilde{Z}^5) - m^4\pi_*(\tilde{Z}^4) + m^3\pi_*(\tilde{Z}^3)) \in A_*Z$. In the next section, we shall see that this is essentially the total Segre class of the normal bundle $N_{Z/\mathbb{P}^5}$.

In summary: the following five assertions, whose proofs all rely on relatively elementary geometric arguments and are omitted, all hold for generic $C_1, \ldots C_5$.

(1) $V_C \subset \mathbb{P}^5$ is a hypersurface of degree 6.
(2) The hypersurfaces V_{C_i} meet transversely at every component in the nonsingular locus,
(3†) The hypersurfaces V_{C_i} contain no singular conics in common, except the double lines,
(4) The set-theoretic intersection $\tilde{V}_{C_1}, \ldots, \tilde{V}_{C_5}$ is disjoint from $\tilde{Z}$, and
(5) The multiplicity m of Z in V_{C_i} is 2.

Given these assertions, we may conclude that the number of nonsingular conics tangent to give generic conics $C_1, \ldots, C_5$ is precisely $6^5 - d$, where d is the degree of the zero-dimensional component of

$$(1 + 6H)^5 \cdot (2^5\pi_*(\tilde{Z}^5) - 2^4\pi_*(\tilde{Z}^4) + 2^3\pi_*(\tilde{Z}))$$

We now turn to the problem of building the machinery which makes this computation trivial.

4. Algebraic Characteristic Classes

In this section, we will inductively build a definition of Chern Classes and Segre Classes entirely in terms of algebraic intersection theory. Recall from section 1 that for any divisor D, we have an intersection homomorphism $Z_kX \to A_{k-1}|D|$, given by $\alpha \to D \cdot \alpha$. There is a canonical inclusion $A_{k-1}|D| \to A_{k-1}X$ given by proper push-forward, so we can easily view this as a homomorphism $Z_kX \to A_{k-1}X$. If $\alpha \in Z_kX$ is rationally equivalent to zero, it's not hard to see that $D \cdot \alpha = 0$. Using proposition 1(e), we have

$$D \cdot (\sum_i [\mathrm{div}(f_i)]) = \sum_i \mathrm{div}(f_i) \cdot [D] = 0 \in A_{k-1}(|D|),$$

because $\mathrm{div}(f_i) \cdot [V]$ is, by the definition of the intersection product, the cycle associated to a principal divisor on V. Thus, there is an induced homomorphism

$$i_D : A_kX \to A_{k-1}X$$

.

By proposition 1(f), i_D does not depend on the linear equivalence class of D. Therefore, we can sensibly view the homomorphism as depending only on the line bundle $\mathscr{O}_X(D)$, rather than the divisor D. The homomorphism i_D has the form of a cap product by a cohomology class, which motivates the following definition.

Definition 7. Let L be a line bundle on X. For any variety $V \subset X$, we have $L|_V = \mathscr{O}_X(D_V)$ for some divisor D_V on V. Define the *first chern class* of L, denoted $c_1(L)$, as the intersection homomorphism

$$c_1(L) : A_kX \to A_{k-1}X,$$

which sends

$$[V] \to D_V \cdot [V] = [D_V]$$

.

Observe that if X is a variety, so that $L = \mathscr{O}_X(D)$ for some divisor D on X, this definition coincides with the homomorphism i_D defined above. The awkwardness in this definition simply results from the fact that a line bundle L over an arbitrary scheme need not come from any globally-defined divisor on X. Given $\alpha \in A_kX$, we will denote its image under $c_1(L)$ by $c_1(L) \cap \alpha$. The basic properties of the first chern class follow easily from the basic properties of the intersection product.

Proposition 5. (a) *If L, L' are line bundles on X, α a k-cycle on X, then*

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha)$$

in $A_{k-2}X$.

(b) (Additivity)*If L, L' are line bundles on X, α a k-cycle on X, then*

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L') \cap \alpha$$

in $A_{k-1}X$.

(c) (Flat pull-back) *If $f : X' \to X$ is flat of relative dimension n, L a line bundle on X, then*

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha)$$

in $A_{k+n-1}(X')$

(d) (Projection formula) *If $f : X' \to X$ is proper, L a line bundle on X, α a k-cycle on X', then*

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha)$$

in $A_{k-1}X$.

Proof. Parts (a),(b),(c), and (d) follow immediately from parts of (e),(b),(c),(d) of proposition 2. □

In order to generalize the first Chern class, we must recall some facts about projective bundles in algebraic geometry. If E is a vector bundle on X of rank $e+1$, then the sheaf of sections of E is a locally-free sheaf of rank $e+1$ on X, usually denoted $\mathscr{E}$, and we construct an associated projective space bundle as follows. Let $\{\operatorname{Spec} A_i\}$ be an open affine cover of X over which $\mathscr{E}$ is trivial. Then one forms the projective bundle $\mathbb{P}(E)$ by gluing together the open subschemes $\operatorname{Proj} A_i[x_0, \ldots x_e]$ according to the linear transition functions of $\mathscr{E}$. The natural maps $p_i : \operatorname{Proj} A_i[x_0, \ldots, x_e] \to \operatorname{Spec} A_i$ glue to define a morphism $p : \mathbb{P}(E) \to X$ which is both proper and flat. Also, the line bundles $\mathscr{O}_i(1)$ on each $\operatorname{Proj} A_i[x_0, \ldots, x_e]$ (these are, by definition, the line bundles corresponding to the divisor class of a hyperplane) glue to give a line bundle $\mathscr{O}_{P(E)}(1)$, often called the canonical bundle.

Given any morphism $f : X' \to X$, there is a natural identification of $P(E) \times_X X'$ with $P(f^*E)$. Thus, there is a natural map $f' : P(f^*E) \to P(E)$, such that $f'^*\mathscr{O}_{P(E)}(1) = \mathscr{O}_{P(f^*E)}(1)$. These facts may be found in Chapter II, Section 7 of [3].

Definition 8. Let E be a vector bundle of rank $e+1$ on X, P the associated projective space bundle with projection $p : P \to X$. Let $\mathscr{O}(1)$ be the canonical line bundle on P. Then the *Segre classes* $s_i(E)$ are homomorphisms $A_*X \to A_{*-i}X$ given by:

$$\alpha \to p_*(c_1(\mathscr{O}(1))^{e+i} \cap p^*\alpha)$$

Since, the action of $c_1(\mathscr{O}(1))$ on A_*P is given by intersecting with a (twisted) generic hyperplane section, this formula has a transparent geometric interpretation: Given a k-cycle in X, take it's inverse image to get a $k+e$ cycle in P, then cut with $e+i$ generic hyperplane sections, to get something $k-i$ dimensional in P, and finally project this down to X. And since the definition is built up from the definition of the first Chern class, it comes equipped with the same desirable formal properties.

Proposition 6. (a) (Commutativity)*If E, F are vector bundles on X, α a k-cycle on X, then*

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha)$$

in $A_{k-i-j}X$.

(b) *If E is a vector bundle on X, α a k-cycle on X, then*

$$s_i(E) \cap \alpha = 0 \text{ for } i < 0, \text{ and}$$

$$s_0(E) \cap \alpha = \alpha.$$

(c) (Flat pull-back) *If $f : X' \to X$ is flat of relative dimension n, E a vector bundle on X, and α a k-cycle on X, then*

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha)$$

in $A_{k+n-i}(X')$

(d) (Projection formula) *If $f : X' \to X$ is proper, E a line bundle on X, α a k-cycle on X', then*

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha)$$

in $A_{k-i}X$.

Proof. We prove (c) and (d) first. For (c), we have the following fibre square with f and f' flat:

$$\begin{array}{ccc} P(f^*E) & \xrightarrow{f'} & P(E) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{f} & X \end{array}$$

$$\begin{aligned} f^*(s_i(E) \cap \alpha) &= f^*p_*(c_1(\mathscr{O}_E(1)^{e+i} \cap p^*\alpha) \\ &= p'_*f'^*(c_1(\mathscr{O}_E(1))^{e+i} \cap p^*\alpha) && \text{by lemma 1} \\ &= p'_*(c_1(f^*\mathscr{O}_E(1))^{e+i} \cap f'^*p^*\alpha) && \text{by proposition 5(c)} \\ &= p'_*(c_1(f^*\mathscr{O}_E(1))^{e+i} \cap p'^*f^*\alpha) && \text{by functoriality of flat pull-back} \\ &= s_i(f^*E) \cap f^*\alpha \end{aligned}$$

The proof of (d) is a nearly identical unravelling of definitions, using proposition 5 (d) at the appropriate juncture.

For (b), it suffices to prove the statement when $\alpha = [V]$, a variety. Making the proper base change to V, part (c) says we may assume $X = V$. Then $A_{k-i}X = 0$ for $i < 0$, so the first statement holds. For the second statement, we must have $s_0(E) \cap [V] = m[V]$ for some integer m (simply because $A_kV = \mathbb{Z}[V]$). The integer

m is locally determined, so, after making a flat base change to an open set U over which E is trivial, we may assume $P(E) = X \times \mathbb{P}^e$. But then

$$s_0(E) \cap [X] = c_1(\mathscr{O}(1))^e \cap p^*[X] = c_1(\mathscr{O}(1))^e \cap [X \times P^e] = [X],$$

since the action of $c_1(\mathscr{O}(1))$ is simply to cut by a hyperplane section of the bundle. Thus, $m = 1$ as desired.

For (a), we consider the fibre square

$$\begin{array}{ccc} Q & \xrightarrow{p'} & P(F) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ P(E) & \xrightarrow{p} & X \end{array}$$

If $e+1$ and $f+1$ are the ranks of E and F, then

$$\begin{aligned} s_i(E) \cap (s_j(F) \cap \alpha) &= p_*(c_1(\mathscr{O}_E(1)^{e+i}) \cap p^* q_*(c_1(\mathscr{O}_F(1))^{f+j} \cap q^*\alpha)) \\ &= p_*(c_1(\mathscr{O}_E(1)^{e+i}) \cap q'^* p'^*(c_1(\mathscr{O}_F(1))^{f+j} \cap q^*\alpha)) \qquad \text{by lemma 1} \\ &= p_* q'_*(c_1(q'^*\mathscr{O}_E(1))^{e+i}) \cap (c_1(p'^*\mathscr{O}_F(1))^{f+j} \cap p' * q^*\alpha)))\text{by proposition 5 (c),(d)} \\ &= q_* p'_*(c_1(p'^*\mathscr{O}_F(1))^{f+j} \cap (c_1(q'^*\mathscr{O}_E(1))^{e+i} \cap q'^* p^*\alpha)) \end{aligned}$$

by proposition 5 (a) and functoriality of push-forward/pull-back. Reversing the steps, the last expression is seen to equal $s_j(F) \cap (s_i(E) \cap \alpha)$. □

Now we shall see that the Chern classes of a vector bundle are determined by the Segre Classes in a straightforward manner. While the Segre classes have a straightforward geometric interpretation, the Chern classes have, in some sense, more natural combinatorial properties.

The Segre classes naturally form a commutative ring with identity, with multiplication given by composition of homomorphisms. Thus, for any vector bundle E on X, we may define a formal power series

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i.$$

Definition 9. We define the *Chern classes* $c_i(E) : A_k X \to A_{k-i} X$ by the relation

$$s_t(E)^{-1} = \sum_{i=0}^{\infty} c_i(E) t^i.$$

For example, $c_0(E) = 1$, $c_1(E) = -s_1(E)$, and $c_2(E) = s_1(E)^2 - s_2(E)$.

We should check immediately that if E is a line bundle on X, then our new definition of c_1 is compatible with our original definition for line bundles. (Equivalently, $c_1(E) \cap [X] = [D]$ where D is the divisor corresponding to E). Let us temporarily denote our new definition of the first Chern class by c_1' and the old definition by c_1. Then we have

$$c_1'(E) \cap [X] = -s_1(E) \cap [X] = -c_1(\mathscr{O}_E(1)) \cap [P(E)] = -c_1(E^\vee) \cap [X] = c_1(E) \cap [X],$$

where the last equality holds by additivity of the original first Chern class (proposition 5), and the second to last equality holds because the natural identification of $P(E)$ with X takes $\mathscr{O}_E(1)$ to $E^\vee$. (Here $E^\vee$ is the dual $\mathscr{H}om(E, \mathscr{O}_X)$.)

The following formula is the key both to proving basic properties of Chern classes, and for many practical computations.

Proposition 7. *Suppose E has a filtration by line bundles L_i. This means that there exists a sequence of subbundles $E = E_1 \supset E_2 \supset \ldots \supset E_n \supset E_{n+1} = 0$ such that E_i/E_{i+1} is a line bundle L_i, where rank$(E) = n$. Then*

$$c_t(E) = \prod_{i=1}^{n}(1 + c_1(L_i)t)$$

The proof requires the following lemma.

Lemma 8. *If E satisfies the hypotheses of the above proposition, and E has a nowhere vanishing section s then $\prod_{i=1}^{r} c_1(L_i) = 0$.*

Proof. (Lemma 8) It suffices to prove the following stronger statement by induction on r: If s is a section of E vanishing on the closed subset $Z \subset X$, α a k-cycle on X, then $\prod_{i=1}^{r} c_1(L_i) \cap \alpha$ is rationally equivalent to a cycle supported on Z.

The section s determines a section $\bar{s}$ of the quotient bundle L_1, and $\bar{s}$ vanishes on some closed subset $Y \supset Z$. Now $c_1(L_1) \cap \alpha$ is rationally equivalent to a cycle supported on Y since L_r corresponds to a divisor with support Y. Thus, $c_1(L_1) \cap \alpha = j_*\beta \in A_{k-1}X$ for some $\beta \in A_{k-1}Y$, where $j : Y \to X$ is the inclusion. By the projection formula (proposition 5 (d)),

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = j_*\left(\prod_{i=2}^{r} c_1(j^*L_i) \cap \beta\right)$$

Now s induces a section of j^*E_2 which vanishes on Z. By induction, $\prod_{i=2}^{r} c_1(j^*L_i)$ is rationally equivalent to a cycle supported on Z, hence $\prod_{i=2}^{r} c_1(j^*L_i) \cap \beta$ is as well. □

We return to the proof of proposition 7.

Proof. Let $p : P(E) \to X$ be the associated projective bundle.The bundle $p^*E \otimes \mathscr{O}(1)$ has a filtration with line bundle quotients $p^*L_i \otimes \mathscr{O}(1)$, and the tautological subbundle $\mathscr{O}(-1)$ of p^*E induces a trivial line subbundle of $p^*E \otimes \mathscr{O}(1)$. Thus, the lemma implies

$$\prod_{i=1}^{n} c_1(p^*L_i \otimes \mathscr{O}(1)) = 0.$$

Let $\zeta = c_1(\mathscr{O}(1))$, σ_k the k^{th} elementary symmetric function of the $c_1(L_i)$, and $\tilde{\sigma}_k$ the k^{th} elementary symmetric function of the $c_1(p^*L_i)$. By additivity (proposition 5 (b)), this equation may be expanded out to obtain

$$\zeta^n + \tilde{\sigma}_1\zeta^{n-1} + \ldots \tilde{\sigma}_n = 0.$$

Thus, for all $i > 0$ and $\alpha \in A_*X$,

$$p_*(\zeta^{n+i-1} \cap p^*\alpha) + p_*(\tilde{\sigma}_1\zeta^{n+i-2} \cap p^*\alpha) + \ldots p_*(\tilde{\sigma}_n\zeta^{i-1} \cap p^*\alpha) = 0.$$

By the projection formula (proposition 6 (d)), this gives

$$s_i(E) \cap \alpha + \sigma_1 s_{i-1}(E) \cap \alpha + \ldots \sigma_r s_{i-r}(E) \cap \alpha = 0.$$

Thus,

$$(1 + \sigma_1 t + \ldots \sigma_r t^r)s_t(E) = 1,$$

which implies the statement of the proposition. □

Proposition 9. (a) (Commutativity)*If E, F are vector bundles on X, α a k-cycle on X, then*

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$$

in $A_{k-i-j}X$.

(b) (Flat pull-back) *If $f : X' \to X$ is flat of relative dimension n, E a vector bundle on X, and α a k-cycle on X, then*

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

in $A_{k+n-i}(X')$

(c) (Projection formula) *If $f : X' \to X$ is proper, E a line bundle on X, α a k-cycle on X', then*

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$

in $A_{k-i}X$.

(d) (Vanishing) *If E is a vector bundle on X, then for all $i > rank(E)$, $c_i(E) = 0$.*

(e) (Whitney Sum Formula) *Given an exact sequence of vector bundles on X:*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

We have $c_t(E) = c_t(E')c_t(E'')$.

Proof. Since each Chern class is a polynomial in the Segre Classes, statements (a),(b),(c) each follow from the corresponding statement for Segre classes proved in proposition 6.

For (d), the statement holds when E has a filtration by line bundles by proposition 7. Also, (e) holds when E' and E'' have filtrations with line bundle quotients L_i and L_j. For then there is an induced filtration on E with quotients L_i and L_j, and again proposition 7 gives the desired result. By the *Splitting principle*, below, these special cases imply the formulas in general. □

The *Splitting principle* says that to prove any formula involving the Chern classes of a finite number of vector bundles, it is sufficient to establish the formula in the case where all the bundles involved have filtrations with line bundle quotients. This principal is justified by the naturality with respect to flat pull-back (proposition 9(b)), in conjunction with the following construction.

Given a finite collection $\mathscr{S}$ of vector bundles on X, we construct a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective, and
(2) For each E in the collection, f^*E has a filtration with line bundle quotients.

First, fix one bundle E in the collection, let $P = P(E)$, and consider the projection $p : P(E) \to E$. Note that the homomorphism $p^* : A_k \to A_{k+n-1}$ is injective for each k by proposition 6 (b), since $p_*(c_1(\mathscr{O}_E(1))^{n-1} \cap (\cdot))$ is inverse to it. Also, p^*E has the tautological subbundle $\mathscr{O}_E(-1)$ (this is just the bundle whose fiber over a point $p \in P(E)$ is the line corresponding to p). Now E'=$p^*E/\mathscr{O}_E(-1)$ has rank one less than E; repeating this process with E' and continuing inductively, we get a finite sequence of maps, whose composition satisifies (1) and satisfies (2) for the fixed bundle E. To finish, we simply repeat the construction for each bundle in the collection.

The great virtue of characteristic classes is their intrinsic computability. We will compute a few examples, before proving the crucial proposition linking these characteristic classes to the intersection numbers we were attempting to calculate in the problem of the five conics. The following definition is a convenient way of packaging the information encoded in these characteristic classes.

Definition 10. Let E be a rank n vector bundle on X. The *total Chern class* of E is an element $c(E) \in A_*X$ defined by

$$c(E) = (1 + c_1(E) + \ldots + c_n(E)) \cap [X].$$

Similarly, the *total Segre class* of E is an element $s(E) \in A_*X$ defined by

$$s(E) = (1 + s_1(E) + s_2(E) \ldots) \cap [X].$$

This sum is finite since $s_i(E) \cap [X] = 0$ for all $i > \dim X$.

Note that the Whitney sum formula implies $c(E) = c(E')c(E'')$ when there is an exact sequence of bundles

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

Example 4. Recall the fundamental exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

From this sequence, we obtain

$$c(T_{\mathbb{P}^n}) = c(\mathscr{O}_{\mathbb{P}^n}(1)^{n+1})c(\mathscr{O}_{\mathbb{P}^n}) = (1 + H)^{n+1},$$

since $c(\mathscr{O}_{\mathbb{P}^n}) = 1$ and $c(\mathscr{O}_{\mathbb{P}^n}(1)) = (1 + H)$, where H is the class of a hyperplane in $\mathbb{P}^n$.

Example 5. We have an exact sequence

$$0 \longrightarrow T_X \longrightarrow i^*T_Y \longrightarrow N_{X/Y} \longrightarrow 0$$

whenever $i : X \to Y$ is a closed immersion of nonsingular varieties. We obtain

$$c(i^*T_Y) = c(N_{X/Y})c(T_X).$$

For example, suppose $X \subset \mathbb{P}^m$ has codimension r and $X = D_1 \cdot \cdots \cdot D_r$ is the scheme-theoretic intersection of r hypersurfaces. Then the normal sheaf $N_{X/Y}$ has a filtration whose quotients are isomorphic $\phi_i^* N_{D_i/\mathbb{P}^n}$, where $\phi_i : X \to D_i$ is the natural inclusion. Thus,

$$\begin{aligned} c(N_{X/Y}) &= c(\phi_1^* N_{D_1/\mathbb{P}^n}) \ldots c(\phi_r^* N_{D_r/\mathbb{P}^n}) \\ &= \prod_{i=1}^r (1 + n_i h) \end{aligned}$$

where h is the divisor class of $i^*\mathscr{O}_{\mathbb{P}^m}(1)$ on X, and n_i is the degree of D_i (so that $\phi_i^* N_{D_i/\mathbb{P}^m} = i^*\mathscr{O}_{\mathbb{P}^m}(n_i)$). Using the previous example, we deduce

$$c(T_X) = (1 + h)^{m+1} / \prod_{i=1}^r (1 + n_i h)$$

Example 6. Consider the d-uple embedding $i : \mathbb{P}^n \to \mathbb{P}^m$, $m = \binom{n+d}{d} - 1$. From the exact sequence

$$0 \longrightarrow T_{\mathbb{P}^n} \longrightarrow i^*T_{P^m} \longrightarrow N_{\mathbb{P}^n/\mathbb{P}^m} \longrightarrow 0,$$

we have

$$c(N_{\mathbb{P}^n/\mathbb{P}^m}) = c(i^*T_{\mathbb{P}^m})/c(T_{\mathbb{P}^n}) = (1+dh)^{m+1}/(1+h)^{n+1},$$

where h is the class of a hyperplane in $\mathbb{P}^n$. Here, we are using the fact that $i^*\mathscr{O}_{\mathbb{P}^m}(1) = \mathscr{O}_{\mathbb{P}^n}(d)$.

For example, if $Z \subset \mathbb{P}^5$ is the surface of double lines discussed in section 3 (the image of $\mathbb{P}^2$ under the 2-uple embedding), then

$$\begin{aligned} c(N_{\mathbb{P}^2/\mathbb{P}^5}) &= (1+2h)^6/(1+h)^3 \\ &= (1+12h+60h^2)/(1+3h+3h^2) \\ &= (1+12h+60h^2)(1-3h+6h^2) \\ &= 1+9h+30h^2, \end{aligned}$$

using the fact that $h^3 = 0$ in $A_*\mathbb{P}^2$. Since $s(N_{\mathbb{P}^2/\mathbb{P}^5}) = c(N_{\mathbb{P}^2/\mathbb{P}^5})^{-1}$, we also have

$$s(N_{\mathbb{P}^2/\mathbb{P}^5}) = 1 - 9h + 51h^2$$

Finally, we are ready to pursue an explicit link between intersection theory and our characteristic classes. For notational convenience, when X is a nonsingular subvariety of Y, we will denote the total Segre class of the normal bundle $N_{X/Y}$ by $s(X,Y)$.

Proposition 10. *Suppose that Y is a variety, and that $X \subset Y$ is a nonsingular subvariety. Let $\tilde{Y}$ be the blow-up of Y along X, let $\tilde{X} = f^{-1}(X)$ be the exceptional divisor, and let $g : \tilde{X} \to X$ be the induced morphism. Then $g_*s(\tilde{X},\tilde{Y}) = s(X,Y) \in A_*X$.*

Proof. Let $\mathscr{O}(1) = \mathscr{O}_{N_{X/Y}}(1)$ and $\mathscr{O}'(1) = \mathscr{O}_{N_{\tilde{X}/\tilde{Y}}}(1)$. There exists a commutative diagram

$$\begin{array}{ccc} P(N_{\tilde{X}/\tilde{Y}}) & \xrightarrow{G} & P(N_{X/Y}) \\ \downarrow p' & & \downarrow p \\ \tilde{X} & \xrightarrow{g} & X \end{array}$$

with $G_*[P(N_{\tilde{X}/\tilde{Y}})] = [P(N_{X/Y})]$, and $G^*\mathscr{O}(1) = \mathscr{O}'(1)$. To see this, recall that $P(N_{X/Y})$ is naturally identified with $\tilde{X}$, the exceptional divisor, and under this identification $\mathscr{O}(1)$ is identified with $N_{\tilde{X}/\tilde{Y}}$. But since $N_{\tilde{X}/\tilde{Y}}$ is a a line bundle, $P(N_{\tilde{X}/\tilde{Y}})$ is also naturally identified with $N_{\tilde{X}/\tilde{Y}}$ in such a way that $\mathscr{O}'(1) = N_{\tilde{X}/\tilde{Y}}$. Thus, the desired isomorphism G exists. We have

$$\begin{aligned} g_*s(\tilde{X},\tilde{Y}) &= g_*p'_*\sum_i c_1(\mathscr{O}'(1))^i \cap [P(N_{\tilde{X}/\tilde{Y}})] \\ &= q_*G_*\sum_i c_1(\mathscr{O}'(1))^i \cap [P(N_{\tilde{X}/\tilde{Y}})] \qquad \text{by functoriality of push-forward} \\ &= q_*\sum_i c_1(\mathscr{O}(1))^i \cap [P(N_{X/Y})] \qquad \text{by projection formula (proposition 5)} \\ &= s(X,Y) \end{aligned}$$

□

An immediate corollary of this proposition is that

$$s(X,Y) = \sum_{k\geq 1}(-1)^{k-1} g_*(\tilde{X}^k).$$

(Simply observe that $s(\tilde{X},\tilde{Y}) = \sum_{k\geq 1}(-1)^{k-1}(\tilde{X}^k)$ since $c_1(\mathscr{O}'(1)) = -c_1(\mathscr{O}'(-1))$ and the action of $c_1(\mathscr{O}'(-1))$ corresponds to intersection with the exceptional divisor $\tilde{X}$, which has normal bundle $\mathscr{O}'(-1)$ in $\tilde{Y}$).

We are now in a position to pick up the computation we left off in section 3. Recall that we had $\mathbb{P}^5$ parametrizing the set of plane conics, and the locus of double-lines parametrized by a surface $Z \subset \mathbb{P}^5$ (the image of the 2-uple embedding of $\mathbb{P}^2$). We blew-up $\mathbb{P}^5$ along Z, and needed to compute the degree of

$$(1+6H)^5 \cdot (2^5\pi_*(\tilde{Z}^5) - 2^4\pi_*(\tilde{Z}^4) + 2^3\pi_*(\tilde{Z}))$$

where $\tilde{Z}$ is the exceptional divisor of the blow-up, and H is the cycle class of a hyperplane in $\mathbb{P}^5$.

We computed $s(\mathbb{P}^2,\mathbb{P}^5) = 1 - 9h + 51h^2$ in example 6, where h was the class of a hyperplane in $\mathbb{P}^2$. Therefore, by the above discussion

$$2^5\pi_*(\tilde{Z}^5) - 2^4\pi_*(\tilde{Z}^4) + 2^3\pi_*(\tilde{Z}^3) = 2^3 - 2^4\cdot 9h + 2^5\cdot 51h^2.$$

Also, the class of H restricted to $\mathbb{P}^2$ is $2h$, since the 2-uple embedding has degree two. Thus, we are left to compute the degree of

$$(2^3 - 2^4\cdot 9h + 2^5\cdot 51h^2)(1+12h)^5$$

Multiplying out, we get $8\cdot 1440 - 60\cdot 16\cdot 9 + 32\cdot 51 = 4512$. Thus, the number of nonsingular conics tangent to five generic conics is $6^5 - 4512 = 3264$.

Lest this calculation seem all too magical, let us pause and recap what has occurred. Our essential problem (very common in enumerative geometry) was to compute the multiplicity of an intersection of hypersurfaces along Z, the locus of degeneracy. Using the geometric technique of blowing-up and compatibility of intersection products with proper morphisms, this multiplicity was seen to be computable in terms of the push-forward of the self-intersection classes of the exceptional divisor of the blow-up. The key intuition which might lead one down this road, and which was finally proven rigorously as a consequence of proposition 10, is that these cycle classes $\pi_*(\tilde{Z}^i)$ are encoded naturally and functorially in the geometry of the vector bundle $N_{Z/\mathbb{P}^5}$. Once this is realized, it is a purely formal affair to express these classes as polynomial expressions in the Chern classes of the bundle, which are easily computed using the Whitney Sum formula. Laying aside for a moment the problem of the five conics, we would suggest that the real significance of the these results is they provide one (among many possible) geometric explanation of where characteristic classes 'come from.' While the interpretation only works for vector bundles which arise as normal bundles of closed embeddings, it is nevertheless very satisfying to think of them as arising concretely out of the functorial operations of blowing-up and computing self-intersections.

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