An introduction to the Jones polynomial

Patrick Schmid

May 10, 2005

Abstract

We give an approach to the Jones polynomial via the Kauffman bracket and explain why it is such a good invariant for distinguishing knots. With the help of the Jones polynomial we will prove the Tait conjecture about the minimality of alternating diagrams and, as an outlook, talk briefly about Khovanov homology.

1 Basic definitions

Our main interests in this paper are knots. Intuitively everybody knows what is meant, but to deal with that topic formally some conventions are necessary. So to make clear what we are talking about, we start with some preliminary definitions.

Definition 1.1 *A* knot *is a subset of* $\mathbb{R}^3$ *which is homeomorphic to the circle* S^1 *and can be expressed as a disjoint of union finitely many points (vertices) and open arc segments (edges).*

Here it is important to insist on the finiteness because otherwise there might occur unexpected complications. So our notion of a knot is a polygon in $\mathbb{R}^3$ that does not intersect itself. As one would expect, there should be an equivalence relation between knots that can be obtained from each other by "wiggling around" (we let the equivalent knots inherit the name of the original one):

Definition 1.2 *Suppose that only one closed edge* e *of a triangle* Δ *in* $\mathbb{R}^3$ *is incident with a knot* K*. Then the process of replacing* K *by the knot* $K' = (K \cup \Delta) \setminus int(e)$ *is called a* Δ*-move. Two knots* K*,* J *are called* equivalent *or* isotopic *if there exists a sequence of knots* $K = K_0, K_1, ..., K_n = J$ *such that* K_{i+1} *is obtained from* K_i *by a* Δ*-move.*

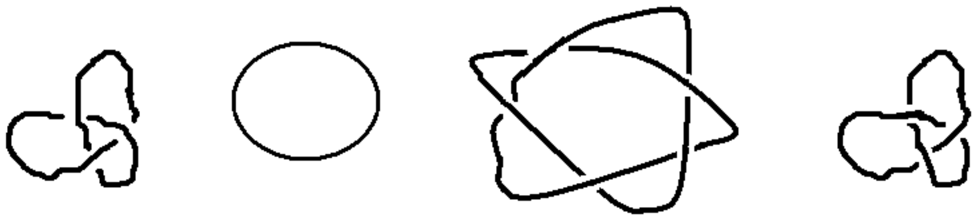
The next thing we have to consider is how we actually represent knots. One might already have heard of *knot diagrams* which can be defined rigorously in the following way:

Definition 1.3 *If K is a knot in $\mathbb{R}^3$, its* projection *is $\pi(K) \subseteq \mathbb{R}^2$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is simply the projection along the z-axis onto the plane $z = 0$. This projection is called* regular *if the preimage of every point consists either of one point or two points with none of them a vertex.*

By considering only regular projections, we disregard projections which are somehow ambiguous. With the following definition we are able reconstruct the 3-dimensional knots from their 2-dimensional representations:

Definition 1.4 *A* knot diagram *corresponding to a regular projection is a picture obtained by redrawing the projection with a "broken arc" at each* crossing *(a point with two preimages) to signify "over- and under-crossing".*

Rather than defining "picture" and "over-/under-crossing" formally (which is certainly not interesting), we give examples to make things clear. Note that we draw knots smoothly in spite of being polygonal since every smooth curve can be approximated by a polygon:

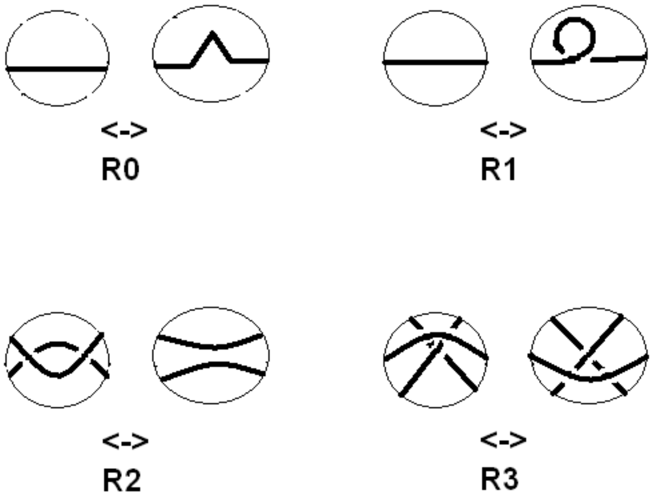


The first diagram is a knot diagram of what is known the be the *left trefoil*. The next diagrams represent the *unknot* (also known as the trivial knot), the 5_1 (the first knot in the classification of those with five crossings) and then the *right trefoil*. One can easily imagine these knots in three dimensions by interpreting broken arcs as being hidden because of the point of view. Certainly the question arises whether the first and last diagram are diagrams of equivalent knots (more on that later).
Easy to prove consequences of our approach are:
a) Every knot has a diagram (in case of a non regular diagram we just have to choose a perturbation of the knot).
b) A knot can be reconstructed from a diagram up to equivalence (not uniquely because of the ambiguity of the z coordinate).
c) Any knot having a diagram with no crossings is the unknot (by the *Jordan curve theorem*, see [2]).

Remark 1.5 *The definitions above can without effort be generalized to* links *which are disjoint unions of knots. Therefore, a knot is just a one component link.*

These diagrams will be a convenient way to describe equivalence of knots since the next theorem guarantees that equivalence of knots can be deduced from their diagrams with a finite number of different moves:

Theorem 1.6 *(Reidemeister's Theorem)*
*Two knots K, K' with corresponding diagrams D, D' are equivalent if and only if their diagrams can be related by a finite sequence $D = D_0, D_1, ..., D_n = D'$ of intermediate diagrams such that each differs from its preceding diagram by one of the following four moves (*Reidemeister moves*):*



(Please note that the disks indicate the part of the diagram which is altered, the rest of the diagram remains unchanged.)

Proof. (Sketch) The "if" direction is trivial (think of what happens to the 3-dimensional knot). For the "only if" direction the problem is that regions that show projections of Δ-moves might not resemble regions that are valid for Reidemeister moves, i.e. the projection of the triangle involves more of the diagram than we wish. But a suitable subdivisions into smaller triangles such that each triangle either contains a single crossing or single arc-segment will do (then only Reidemeister moves occur). We have used the fact that each Δ-move can be regarded as the composition of Δ-moves of smaller triangles. $\square$

Remark 1.7 *The theorem applies without modification to links.*

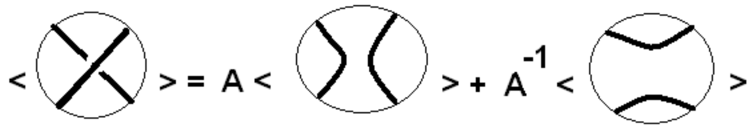
Remark 1.8 *Note that one can deduce from these axioms that either producing a kink as in $R1$ or producing the reverse kink are allowed moves. We will denote the latter as $R1'$.*

This theorem allows us to consider knots as equivalence classes of diagrams under Reidemeister moves. As for Algebraic Topology which is interested in producing invariants of knots (they only depend on the equivalence class), the procedure will be the following: we define functions that can be computed from knot diagrams. To check whether it is a knot invariant, we simply have to check if the function attains the same value on diagrams that differ by a single Reidemeister move.

2 The Kauffman bracket and the Jones polynomial

After these preliminaries we start our deduction of the *Jones polynomial* which was invented by Vaughan Jones in 1984. It is a combinatorially defined invariant of links which is extremely good at distinguishing equivalence classes. Rather than choosing his original way, our approach uses the *Kauffman bracket* which is almost(!) an invariant of links.

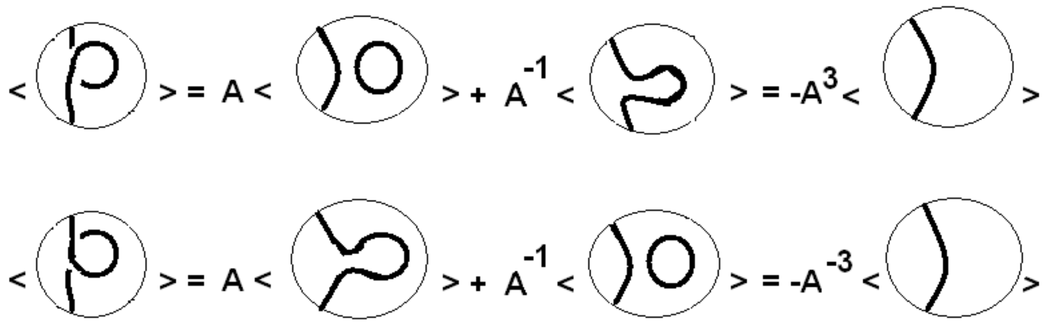
Definition 2.1 *The* Kauffman bracket *(KB) of a link diagram D is a Laurent polynomial $\langle D \rangle \in \mathbb{Z}[A^{\pm 1}]$ defined by*
(0) *It is invariant under planar isotopy of diagrams (i.e. $R0$).*
(1) *It satisfies the so called* Skein relation



which is a linear relation between the brackets of the diagrams which differ only locally by the indicated disks.
(2) *It satisfies $\langle D \amalg U \rangle = (-A^2 - A^{-2})\langle D \rangle$ where U is a crossingless loop (this describes how the bracket changes under deleting a loop from the diagram).*
(3) $\langle U \rangle = 1$ *(Normalization).*

In order to obtain a feeling for typical computations of this polynomial, we look at some examples: We examine how the KB changes if a kink is removed from a diagram (we also do this for the reverse kink, see the next page).
Since the diagrams yield different polynomials, the KB can not be an invariant of links because of Reidemeister's Theorem.

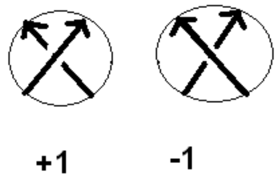
Similar calculations show that the KB is invariant under $R2$ and $R3$ (check for yourself!). So as we have already mentioned, the KB is almost an invariant of



links, it just lacks to satisfy invariance under $R1$ (even then it seems only to multiply by a power of $-A^{\pm 3}$). But there is a way to correct this by equipping links with an *orientation*.

Definition 2.2 *Think of a knot as a path in* $\mathbb{R}^3$ *traversed in time. Obviously there a two choices how to traverse the knot. Either of those is called an* orientation *of the knot. Orientations of links consist of the respective orientations of their component knots. These orientations give induced orientations in the diagrams which are indicated by arrows.*

In order to define the correction (which we will call the *writhe*), we have to attribute a sign to each crossing in a diagram of an oriented link by the following convention:



Note that without an orientation, no signs can be attributed in an intrinsic way to crossings in a diagram. But now the definition:

Definition 2.3 *If D is an oriented link diagram, then the writhe $w(D)$ is the sum of the signs of all crossings.*

The salient property of the writhe is characterized in the following lemma:

Lemma 2.4 *The writhe of an oriented link diagram is invariant under $R0$, $R2$ and $R3$ but changes by ± 1 under $R1$ and $R1'$.*

Proof. Trivial for $R0$, $R2$, $R3$ since the writhe does not change locally. Applying $R1$ or $R1'$, the writhe can either increase or decrease by one depending on whether we remove or produce a kink. Note that regardless of the orientation, a certain kink always leads to the same change. □

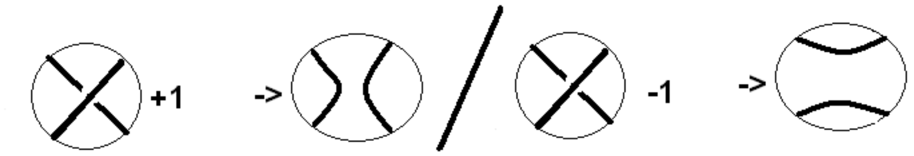
So we can formulate a first invariant:

Theorem 2.5 *If D is an oriented link diagram, then the polynomial $f_D(A) = (-A^3)^{-w(D)}\langle D\rangle$ is an invariant of oriented links.*

Proof. It suffices to show that the polynomial is invariant under the Reidemeister moves. For $R0$, $R2$ and $R3$ it is clear since the bracket and the writhe are invariant under them. Producing a kink somewhere leads to multiplication by $-A^3$ or $-A^{-3}$. But as one verifies, this is compensated by the writhe. □

That is pretty much the famous *Jones polynomial*. But before we move on, we give an alternate expression for the Kauffman Bracket which leads to more insight how it is actually computed. In order to do so, we need to introduce some terminology.

Definition 2.6 *If we consider unoriented diagrams, then a* state *s of a diagram is an assignment (arbitrary) of $+1$ or -1 to each crossing. A state s induces a new diagram sD by replacing crossings by the following rule depending on the sign:*



This is called splitting *or* resolving. *Then $|sD|$ is the number of disjoint loops in the resolved diagram and $\sum s$ is the sum of the* values *of a state (i.e. all signs attributed to crossings).*

Proposition 2.7 *An equivalent expression for the Kauffman bracket is the* state-sum formula

$$\langle D\rangle = \sum_s \langle D|s\rangle$$

where s runs over all states of D and $\langle D|s\rangle$ is the contribution of the state s

$$\langle D|s\rangle = A^{\sum s}(-A^2 - A^{-2})^{|sD|-1}$$

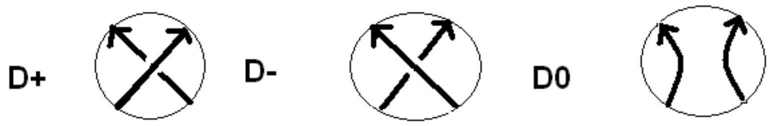
Proof. This formula expresses the computation of the bracket by applying the Skein relation at every crossing and evaluating the brackets of the crossingless diagrams. In more detail: Resolving a crossing by $+1$ multiplies the bracket of the resolved diagram by A, whereas resolving by -1 multiplies by A^{-1}. That is why we obtain the first factor. The second factor just counts all loops in the completely resolved diagram. Since the Skein relation describes a bracket as the sum of two other brackets, we have to sum up the brackets of all possible complete resolution with their respective factor in powers of A to get the bracket of the original diagram. □

Notice that by this formula, we see that the Kauffman Bracket was really well-defined as it does not matter in which order we apply the Skein relations. Now we are in good position to consider our main object of interest:

Definition 2.8 *The* Jones Polynomial $V_L(t)$ *of an oriented link* L *is obtained by computing* $f_D(A)$ *for any oriented diagram* D *of* L *and then substituting* $A = t^{-1/4}$ *(for traditional reasons). Therefore* $V_L(t) \in \mathbb{Z}[t^{\pm 1/4}]$.

From this definition, we deduce three important properties:

Theorem 2.9 *The Jones Polynomial*
a) is an invariant of oriented links lying in $\mathbb{Z}[t^{\pm 1/2}]$.
b) of the unknot is 1.
And furthermore:
c) There is a Skein relation $t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$ *where* L_+, L_- *and* L_0 *are oriented links differing only locally in their diagrams by the disks indicated*



Proof. a) It is useful to look at the state-sum formula. We see that the KB is a sum of powers of $(-A^2 - A^{-2})$ times $A^{\sum s}$. Furthermore, we realize that $\sum s$ is even or odd depending on whether the number of crossings in the diagram is even or odd. Therefore the KB consists only of even or odd powers of A depending on the number of crossings k. But since we multiply by $(-A^3)^{-w(D)}$, we obtain only even powers of A because k and $-w(D)$ (similar to $\sum s$) have the same parity. b) Trivial by construction. c) It is convenient to introduce the splitting D_∞ in the horizontal direction (see picture).

However a natural orientation in this case is not possible, so we use the KB and the corresponding Skein relations:

$$\langle D_+ \rangle = A\langle D_0 \rangle + A^{-1}\langle D_\infty \rangle, \langle D_- \rangle = A\langle D_\infty \rangle + A^{-1}\langle D_0 \rangle$$

By a linear combination of these two equations we eliminate D_∞ to obtain

$$A\langle D_+ \rangle - A^{-1}\langle D_- \rangle = (A^2 - A^{-2})\langle D_0 \rangle$$

Now we are able to write this relation in terms of $f_D(A)$ since D_∞ no longer occurs. Note that the writhes of D_+, D_- differ by one from $w(D_0)$, hence

$$-A^4 f_{D_+}(A) + A^{-4} f_{D_-}(A) = (A^2 - A^{-2}) f_{D_0}(A)$$

A final substitution $A = t^{-1/4}$ and a change of sign completes the proof. $\square$

Remark 2.10 *Alternatively, one could take these three properties as axioms to define the Jones polynomial (JP) and prove that there exists a unique polynomial satisfying the properties. However this approach seems to be "artificial".*

Next comes a property of the JP (which is almost trivial to prove by definition) that makes it very good at distinguishing knots.

Theorem 2.11 *The Jones Polynomial $V_{\bar{L}}(t)$ of the* mirror-image $\bar{L}$ *of an oriented link L (i.e. exchanging all over-/under- crossings in a diagram) is equal to $V_L(t^{-1})$. Therefore, any knot K whose JP is not* palindromic *($V_K(t) \neq V_K(t^{-1})$) is* chiral *(i.e. distinct from its mirror-image).*

Remark 2.12 *In fact, reflecting a knot across different planes in $\mathbb{R}^3$ yields different knots, but they are all equivalent as one checks. Hence, one can choose a disjoint plane that is parallel to the plane $z = 0$ to obtain a representative of the mirror-image equivalence class.*

Proof. It is easy to see from the state-sum that the effect of mirror-imaging on the Kauffman bracket is to replace A by A^{-1}. Furthermore, mirror-imaging negates the writhe of any oriented diagram, since positive and negative crossings are exchanged. $\square$

There are other knot invariants that can not make this distinction, e.g. the *crossing number* which counts the minimal number of crossings in diagram (see [2]).
To illustrate the computation of the JP by its Skein relation rather than using the bracket, we examine the right trefoil. We let L_+ be the right trefoil, L_- be the unknot and L_0 be the *positive Hopf link* (the one with positive writhe):

Therefore to compute V_{L_+}, we need the JP's of the unknot and the positive Hopf link which are more simple. We use again the Skein relation but with L'_+ the positive Hopf link, L'_0 the unknot and L'_- a two component unlink (two non-entangled loops). Hence we obtain (note that the JP of the unlink is $-t^{1/2} - t^{-1/2}$):

$$t^{-1}V_{L'_+}(t) - t(-t^{1/2} - t^{-1/2}) = (t^{1/2} - t^{-1/2})$$

from which we deduce that $V_{L'_+}(t) = -t^{5/2} - t^{1/2}$. We apply it immediately to our first Skein relation:

$$t^{-1}V_{L_+}(t) - t = (t^{1/2} - t^{-1/2})(-t^{5/2} - t^{1/2})$$

From that we see that the JP of the right trefoil is $V_{L_+}(t) = -t^4 + t^3 + t$. Since the left trefoil is the mirror-image of the right trefoil and the computed JP is certainly not palindromic, we can conclude that these two knots are not the same which answers the question from the beginning.

Remark 2.13 *As mentioned before, the JP is very powerful and distinguishes inequivalent knots in most cases that occur. Nonetheless, one can construct (with some effort) inequivalent knots that have the* same *JP.*

3 The Tait conjecture

After becoming familiar with the JP and its computation, we turn towards one of the Tait conjectures. If one has some experience in dealing with knot diagrams, one might come up with the conjecture that *alternating knot diagrams* (if one transverses the projection of the knot in time, then one alternately passes over- and under-crossings) that satisfy a mild technical requirement are minimal in the sense that there exists no diagram of the knot with fewer crossings. It is quite surprising that this conjecture which has been made approximately 100 years ago could not be proved until the invention of the JP. As usual we introduce terminology which helps us to tackle the problem:

Definition 3.1 *For any knot, $\mathbb{R}^2 \backslash \{\pi(K)\}$ is the disjoint union of its connected components. These induce* regions *in the corresponding diagram.*
A diagram is called reduced *if there are more than three regions incident at every crossing.*

In the following picture, we see an example of a crossing where only three regions are incident. The fact that there are only three regions incident implies

that one can move between two opposite regions at a crossing (which are in fact the same) without hitting the diagram. Hence we see that an unreduced diagram can easily be made reduced by "disentangling" the knot at the critical position such that the critical crossing vanishes in the diagram. We also observe that the number of regions equals the number of crossings plus two. This holds in fact for any knot diagram (for a proof, see [2]).
Just one more algebraic notion is needed before we start:

Definition 3.2 *The* span *or* breadth *of a Laurent polynomial in an indeterminate A is the difference (nonnegative) between the highest and the lowest exponent of A.*

We are now able to state the main theorem from which the proof of the Tait conjecture follows as an easy corollary:

Theorem 3.3 *The span of the Kauffman bracket of a c-crossing reduced alternating knot diagram is $4c$, whereas the span of the Kauffman Bracket of* any *c-crossing knot diagram is less than or equal to $4c$.*

To prove the theorem we need some corollaries which we accept with out respective proofs (which turn out to be rather technical, see [2]). With their help, we can identify the highest and lowest power in the Kauffman bracket of a c-crossing diagram. Since we will deal with the state-sum, we let s_+ be the state consisting entirely of pluses (analogously for s_-).

Lemma 3.4 *For a reduced alternating knot diagram, $|s_+D| > |s_1D|$ for each state s_1 that has exactly one minus.*

Lemma 3.5 *Given* any *c-crossing knot diagram and a state s with i minuses. If $s_+ = s_0, s_1, ..., s_i = s$ is a sequence of states with each s_j having j minuses, then the maximal power in $\langle D|s_{j+1}\rangle$ is less than or equal to that in $\langle D|s_j\rangle$.*

Lemma 3.6 *(DSL-Dual State Lemma)*
If we define for a state s the dual state $\hat{s}$ to be obtained by exchanging pluses and minuses, then for any knot diagram D with c crossings and state s we have $|sD| + |\hat{s}D| \leq c + 2$. Equality holds for reduced alternating diagrams.

Proof. (of theorem 3.3): For now we only consider reduced alternating diagrams. Obviously the highest power in $\langle D|s_+\rangle$ equals $c+2(|s_+D|-1)$. Looking at a state with one minus, we realize that the highest occurring power is less than or equal to $(c-2)+2(|s_+D|-2)$ which is strictly less than the highest power in $\langle D|s_+\rangle$. Therefore, no other state contributes a higher power. By similar reasoning, we conclude that the lowest power in the Kauffman bracket is

$-c-2(|s_-D|-1)$ (just exchange the pluses and minuses). Therefore the span equals $2c+2(|s_+D|+|s_-D|)-4$ which is the same as $2c+2(c+2)-4=4c$. This proves the first assertion.
Now we consider arbitrary diagrams. If we consider non-alternating diagrams, lemma 3.4 fails which means that the power $c+2(|s_+D|-1)$ could be canceled out. But at least we can say that the span is less than or equal to $2c+2(|s_+D|+|s_-D|)-4$. Applying the DSL proves the second statement. □

Corollary 3.7 *(Tait conjecture)*
Any reduced alternating knot diagram is minimal in the sense described above.

Proof. Assume that the given reduced alternating knot diagram has c crossings so that the span equals $4c$. If there would be a diagram with a smaller number of crossings, this would cause a contradiction since the span of the Kauffman bracket is an invariant of knots. □

Remark 3.8 *Dealing only with reduced alternating knot diagrams, we enjoy the following privileges:*
a) All diagrams of the same knot have the same number of crossings.
b) Any non-trivial diagram represents a non-trivial knot.

4 Khovanov homology

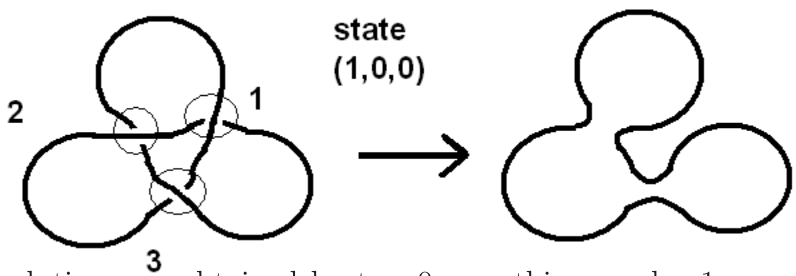
After having introduced the JP in a rigorous way, we turn to a more recent topic which somehow generalizes the JP: Khovanov homology.
Roughly, Khovanov's idea was to associate a complex to an oriented link whose homology yields the JP. It is conjectured that this could give a quantum field theoretic explanation of why the coefficients of the invariants computed by the Chern-Simons-Witten theory, a three-dimensional quantum field theory which yields them in form of integrals over an infinite dimensional space, are integers.
Let us have a look at the construction (not in every detail but by highlighting the relevant steps and sketching the main idea):
We first have to mention that Khovanov uses slightly altered axioms for the Kauffman bracket: $\langle\varnothing\rangle = 1$ (here $\varnothing$ denotes the diagram where all loops of the resolved diagram have been removed), $\langle \bigcirc \amalg L\rangle = (q+q^{-1})\langle L\rangle$, $\langle\times\rangle = \langle\asymp\rangle - q\langle)(\rangle$ where the crossing is supposed to depict the mirror-image of the crossing in definition 2.6, i.e. the northwest-southeast line passes over the other.
Let n denote the number of crossings in an oriented link diagram (think of them as being numbered from 1 to n) and n_+ the number of $+1$-crossings (respectively for n_-). This leads to what he calls *unnormalized JP*: $\hat{J}(L) =$

$(-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$. Multiplication with $1/(q + q^{-1})$ yields his JP (in fact to obtain the JP as we defined it, we just need a substitution $q = -t^{1/2}$). But that does not matter since by some renaming and regrading we could have used our original axioms, so we give the results according to his notation. Before we proceed, we need to introduce some terminology for graded vector spaces and complexes:

Definition 4.1 *Let $W = \oplus_m W_m$ be a graded vector space with homogeneous components $\{W_m\}$, then the graded dimension of W is the power series $qdim(W) := \sum_m q^m dim(W_m)$. Furthermore let $\cdot\{l\}$ be the degree shift operation on graded vector spaces, i.e. we set $W\{l\}_m := W_{m+l}$.*

Definition 4.2 *Analogously let $\cdot[s]$ be the "height shift" operation on chain complexes, i.e. for a chain complex $\bar{C} : ... \to \bar{C}^r \to \bar{C}^{r+1} \to ...$ of vector spaces (possibly graded) $C := \bar{C}[s]$ is defined to be the complex with vector spaces $C^r := \bar{C}^{r+s}$ and respectively shifted differentials.*

The following categorification turns the JP in a homological object by replacing the polynomials in the state-sum of the Kauffman bracket by graded vector spaces.
So let V be the graded vector space with the two basis elements $v_\pm$ which have respective degrees ± 1. We have seen that there are two possible ways to resolve a crossing $\langle \times \rangle$ (specified above) in a diagram of a knot K; let us call the horizontal resolution by $\langle \asymp \rangle$ 0-smoothing and the vertical one by $\langle)(\rangle$ 1-smoothing. Then every element α of the n-dimensional cube $\{0,1\}^n$ corresponds to a complete resolution of the diagram which is just a union of loops. Instead of associating a polynomial to each α as for the state-sum of the Kauffman bracket, we associate to each α the graded vector space $V_\alpha(L) := V^{\otimes k}\{r\}$, where k is the number of loops in α and $r := |\alpha| := \sum_i \alpha_i$ the "height" of α. The main point is now that $qdim(V_\alpha(L))$ equals up to sign the polynomial which would be attributed by the KB. Let us look at an example of a complete smoothing for the right trefoil:



This resolution was obtained by two 0-smoothings and a 1-smoothing. Thus the bracket corresponding to this state equals $-q(q+q^{-1})$. The corresponding graded vector space would be $V\{1\}$ with graded dimension $qdim(V\{1\}) =$

$q(q+q^{-1})$.
We then set the r^{th} chain group $[[L]]^r$ $(0 \leq r \leq n)$ to be the sum of all vector spaces at height r, i.e. $[[L]]^r := \oplus_{\{\alpha | r=|\alpha|\}} V_\alpha(L)$ and, to shift everything in the right place, we define $C(L) := [[L]][-n_-]\{n_+ - 2n_-\}$ (this is in fact determined so that we can obtain the JP as stated in the theorem below).
This is not yet a complex, but it can be endowed with degree zero differentials where these are alternating sums of maps corresponding to cobordisms between the vertices of the n-dimensional cube that are connected by an edge (see [1] or [3] for details). With these differentials, we obtain that $C(L)$ is a chain complex.
If we denote the r^{th} cohomology group of $C(L)$ as $\mathcal{H}^r(L)$, we let $Kh(L)$ denote graded *Poincaré polynomial* of the complex $C(L)$ in the variable l:

$$Kh(L) := \sum_r l^r qdim(\mathcal{H}^r(L))$$

Finally we can state our main theorem :

Theorem 4.3 *(Khovanov)*
The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are link invariants, and hence $Kh(L)$, a polynomial in the variables l and q, is a link invariant that specializes to the unnormalized Jones polynomial at $l = -1$.

It is worth mentioning that $Kh(L)$ is strictly stronger than the Jones polynomial. The fact that $Kh(L)$ specializes to the unnormalized JP turns out to be trivial by construction. But for the cohomology groups one has as usual to check invariance under the Reidemeister moves which is non-trivial (see [1] for details).

References

[1] D. Bar-Natan. On Khovanov's Categorification of the Jones polynomial. Alg. Geom. Top., 2:337-370, 2002.

[2] J. D. Roberts, Knots Knotes, http://www.math.ucsd.edu/~justin/papers.html

[3] M. Khovanov. A categorification of the Jones polynomial. math.QA/9908171, 1999.